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FOLIATIONS, GROUPOIDS, AND THE BAUM–CONNES CONJECTURE

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The Baum–Connes conjecture establishes, for foliated manifolds, an analog of the well-known isomorphism between the topological K-theory of a locally compact space M and the analytic K-theory of the C^*-algebra of continuous functions on M vanishing at infinity. In this work, we describe the principal notions involved in the statement of the conjecture and indicate its contemporary status. Bibliography: 11 *titles.*

1. Approach to the Baum–Connes conjecture

In the "commutative world," the most immediate and powerful tools are homology and fundamental groups. But these tools have no obvious "noncommutative" generalizations. Nevertheless, the topological K-theory [1] is the mostly successful tool since it passes easily to the noncommutative world.

It is well known that, for any locally compact space M, the C^*-algebra $C_0(M)$ of continuous functions vanishing at infinity allows us to "reconstruct" M and that there is an isomorphism between the *topological K-theory* of M and the *analytic K-theory* of $C_0(M)$.

The *Baum–Connes conjecture*, independently of its meaning in the context of index theory, looks to establish an analog of this isomorphism for some "singular" spaces, i.e., for the leaf spaces of foliated manifolds. Precisely, if $\mathcal{F}$ is a C^∞-foliation on a manifold M, then $M/\mathcal{F}$ is a bad quotient in many cases. Thus, to obtain information concerning the transverse structure of the foliation, it is necessary to use other types of objects. Let us mention some of these objects.

(1) The dynamics of $\mathcal{F}$ is described by its holonomy groupoid G, which is a Lie groupoid. The groupoid G can be considered as a desingularization of the leaf space $M/\mathcal{F}$. As to any Lie groupoid, we can associate to G a C*-algebra $C^*_{\mathrm{red}}(G)$ of functions, which is interpreted as the "space of continuous functions vanishing at infinity" on $M/\mathcal{F}$. The *analytic K-theory of the leaf space*, $K_{\mathrm{an}}(M/\mathcal{F})$, is defined as the K-theory $K_*(C^*_{\mathrm{red}}(G))$ of this C*-algebra.

(2) Moreover, we can construct a *classifying space* BG for G. The groupoid G acts freely and properly on BG, which is not, in general, a manifold (and it does not even have the homotopy type of a manifold!) The space BG can be considered as the leaf space modulo homotopy. In [2], Baum and Connes introduce a generalized G-equivariant K-theory $K_{*,\tau}(BG)$ associated with this object, which is defined as the *topological K-theory* $K_{\mathrm{top}}(M/\mathcal{F})$ *of the leaf space.*

Intuitively, G, $C^*_{\mathrm{red}}(G)$, and BG are "items" completely determined by $\mathcal{F}$ and carrying the same information. Elliptic operators give us a map between the K-theory groups previously described,

$$\mu : K_{\mathrm{top}}(M/\mathcal{F}) \to K_{\mathrm{an}}(M/\mathcal{F}),$$

and the Baum–Connes conjecture asserts that μ is a group isomorphism if the holonomy groups are torsion-free.

The proof of the conjecture would furnish us with a relation between the information given by the transverse structure of the foliation (through G) and the geometry granted by the space BG. In other words, such a proof would "present" a geometric interpretation of the analytic object $K_*(C^*_{\mathrm{red}}(G))$. In this talk, we describe the principal notions involved in the statement of the conjecture and indicate its contemporary status.

2. Lie groupoids

2.1. Algebraic groupoids.

Definition 1. *An algebraic groupoid is defined by the following objects:*

(1) *a pair of sets (M, G), where $M \subset G$ is the unit space and G is the total space;*

(2) *two surjective maps $\alpha, \beta : G \to M$, called the projections of the source and range, respectively, such that $\alpha(x) = \beta(x) = x$ for $x \in M$;*

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(3) *a bijection* $i : G \to G$ *called the inversion and such that* $i = i^{-1}$ *(we note that* $i(\gamma) = \gamma^{-1}$*);*
(4) *a partial composition law,* $. : G^2 \to G$ *called the multiplication, where the set of composable pairs is* $G^2 = \{(\gamma_2, \gamma_1) \in G \times G : \alpha(\gamma_2) = \beta(\gamma_1)\}$*. The multiplication has the following properties.*
 - *Associativity: if* $\gamma_1, \gamma_2, \gamma_3 \in G$ *and either* $((\gamma_2, \gamma_1) \in G^2$ *and* $(\gamma_3, \gamma_2.\gamma_1) \in G^2)$ *or* $((\gamma_3, \gamma_2) \in G^2$ *and* $(\gamma_3.\gamma_2, \gamma_1) \in G^2)$*, then* $\gamma_3.(\gamma_2.\gamma_1) = (\gamma_3.\gamma_2).\gamma_1$.
 - *Units: for any* $\gamma \in G$*,* $(\gamma, \alpha(\gamma)) \in G^2$ *and* $(\beta(\gamma), \gamma) \in G^2$*, hence* $\gamma.\alpha(\gamma) = \gamma = \beta(\gamma).\gamma$.
 - *Inversion: for any* $\gamma \in G$*, we have the inclusions* $(\gamma, \gamma^{-1}) \in G^2$ *and* $(\gamma^{-1}, \gamma) \in G^2$ *and the identities* $\gamma.\gamma^{-1} = \beta(\gamma)$ *and* $\gamma^{-1}.\gamma = \alpha(\gamma)$.

Examples 1. The first examples of algebraic groupoids are as follows:
(1) if M is a point, then the groupoid is reduced to a *group*;
(2) if $M = G$, $\alpha(\gamma) = \beta(\gamma) = \gamma$, and $\gamma^{-1} = \gamma$, then G^2 is the diagonal of $G \times G$, and we obtain the *trivial groupoid*;
(3) given a set X, we consider $G = X \times X$, let M be the diagonal of G (identified with X), and define $\alpha(y, x) = x$, $\beta(y, x) = y$, and $(x, y)^{-1} = (y, x)$. The set of composable pairs is $G^2 = \{((y, x), (x, z)) : x, y, z \in X\}$, and the multiplication is given by the formula $(y, x).(x, z) = (y, z)$. This is the *coarse groupoid*;
(4) the *graph of an equivalence relation* R on a set X is a groupoid, where the unit space is the diagonal, $\alpha(x, y) = x$, $\beta(x, y) = y$, and $(x, y)^{-1} = (y, x)$. Then $G^2 = \{((x, y)(y, z)) \in G \times G : x, y, z \in X\}$ is the set of composable pairs, and $(x, y).(y, z) = (x, z)$.

Definition 2. *Given a groupoid* G *and* $x, y \in M$*, we define the* α*-fiber over* x*,* $G_x = \{\gamma \in G : \alpha(\gamma) = x\}$*, the* β*-fiber over* y*,* $G^y = \{\gamma \in G : \beta(\gamma) = y\}$*, and the set* $G_x^y = G_x \cap G^y$*. The set* G_x^y *can be empty. However, if* $x \in M$*, then* G_x^x *is a group (where the point* x *is the unit element), the isotropy group of* G *at* x*. The isotropy groupoid of* G *is* $\mathrm{Is}(G) = \bigcup_{x \in M} G_x^x$ *(with* M *as the unit space and with obvious operations).*

Definition 3. *A homomorphism of groupoids from* G_1 *to* G_2 *is a function* $f : G_1 \to G_2$ *such that if* $(\gamma_2, \gamma_1) \in G_1^2$*, then* $(f(\gamma_2), f(\gamma_1)) \in G_2^2$ *and* $f(\gamma_2.\gamma_1) = f(\gamma_2).f(\gamma_1)$.

2.2. Lie groupoids.

Definition 4. *We say that* G *is a Lie groupoid (below, "differentiable" means* C^∞*) if the following conditions hold:*
(1) G *and* M *are differentiable manifolds, and* M *is Hausdorff;*
(2) *the maps* α, β, i*, and . are differentiable,* α *and* β *are submersions, and* i *is a diffeomorphism.*

Examples 2. The first examples of Lie groupoids are as follows:
(1) the *action of a Lie group on a manifold.* We consider the action of a connected Lie group H on a manifold M: $\Phi : H \times M \to M$. We define in this way a Lie groupoid with the total space $G = H \times M$, with the unit space M, and with the operations $\alpha(g, x) = x$, $\beta(g, x) = \Phi(g, x)$, and $(g, x)^{-1} = (g^{-1}, \Phi(g, x))$ such that if $x_2 = \Phi(g_1, x_1)$, then the multiplication is $(g_2, x_2).(g_1, x_1) = (g_2 g_1, x_1)$;
(2) the *homotopy groupoid of a manifold.* Given a differentiable manifold M, we consider $\mathcal{P}(M)$, the set of paths on M provided with the compact-open topology. We consider the following open equivalence relation on $\mathcal{P}(M)$:

$$\gamma \sim \gamma' \text{ if } \gamma \text{ is homotopic to } \gamma' \text{ with fixed extremities.}$$

The quotient space, $\Pi_1(M) = \mathcal{P}(M)/\sim$, is a locally compact groupoid such that its unit space is the manifold M (identified with the class of constant paths), where the structure of a groupoid is defined through the relations $\alpha(\gamma) = \gamma(0)$ and $\beta(\gamma) = \gamma(1)$, and the multiplication and inversion of the groupoid are defined following the usual composition and inversion of paths, respectively. The continuous function $(\alpha, \beta) : \Pi_1(M) \to M \times M$ is a covering map, and thus we can lift the structure of a differentiable manifold to $\Pi_1(M)$. This structure is compatible with the quotient topology, hence $\Pi_1(M)$ is a Lie groupoid. If $x \in M$, then we have two universal coverings of M, $\alpha : \Pi_1(M)^x \to M$ and $\beta : \Pi_1(M)_x \to M$. The isotropy group at $x \in M$ is $\Pi_1(M)_x^x = \pi_1(M, x)$. The Lie groupoid $\Pi_1(M)$ is called the *fundamental groupoid* or *homotopy groupoid* of M.

Definition 5. *Given two Lie groupoids* G_1 *and* G_2*, a homomorphism (of Lie groupoids)* $f : G_1 \to G_2$ *is a differentiable map which, in addition, is a groupoid homomorphism.*

3. G-fiber bundles

3.1. G-actions. Let G be a Lie groupoid. Let Z be a non-Hausdorff, locally compact, and differentiable manifold. Given a differentiable map $\rho : Z \to M$, we consider the set $Z *_M G = \{(z, \gamma) \in Z \times G : \rho(z) = \beta(\gamma)\}$.

Definition 6. *A right differentiable action of G on Z is a differentiable map $\Phi : Z *_M G \to Z$ denoted by $\Phi(z, \gamma) = z.\gamma$ and having the following properties:*

1. *$\rho(z.\gamma) = \alpha(\gamma)$ for any $(z, \gamma) \in Z *_M G$;*
2. *if one of the expressions $(z.\gamma).\gamma'$ or $z.(\gamma.\gamma')$ is defined, then the other one is also defined, and they coincide;*
3. *$z.\rho(z) = z$ for any $z \in Z$.*

We say that Z is a right G-differentiable space if Z is a non-Hausdorff differentiable manifold provided with a differentiable right G-action.

Definition 7. *The action of G on Z is called*

1. *proper if the map $\Psi : Z *_M G \to Z \times Z$ defined by $\Psi(z, \gamma) = (z, z.\gamma)$ is proper, and*
2. *free if, given $(z, \gamma) \in Z *_M G$, the equality $\gamma.z = z$ holds if and only if $\gamma \in M$.*

Definition 8. *A G-space Z is called principal if the action is proper and free. In this case, the canonical projection $\pi : Z \to Z/G$ is a submersion, and the quotient space Z/G is locally compact and Hausdorff.*

Example 1. If G is a Lie groupoid, $Z = G$, and $\rho = \alpha$, then $Z *_M G = G^2$. The multiplication of the groupoid is a natural right G-action of G on itself. This is a free action, and G is a principal G-space. If $x \in M$, then the orbit of this point at the action is $G(x) = G^x$.

Definition 9. *Given two right G-differentiable spaces Z_1 and Z_2, we define a G-map, $f : Z_1 \to Z_2$, to be a differentiable and G-equivariant map, i.e., a map such that if $(z_1, \gamma) \in Z_1 *_M G$, then $(f(z_1), \gamma) \in Z_2 *_M G$ and $f(z_1.\gamma) = f(z_1).\gamma$.*

3.2. Morita-equivalence of groupoids.

Definition 10. *If Z is a G-space, then a G-bundle over Z is defined by a G-space E and a G-map $p : E \to Z$ called the projection so that the following statements hold:*

1. *$p : E \to Z$ is a complex bundle;*
2. *for any $(z, \gamma) \in Z *_M G$, the map $\phi : E_z \to E_{z.\gamma}$ given by $\phi(u) = u.\gamma$ is linear.*

For our purposes, a good notion of equivalence of groupoids is the following one.

Definition 11. *A Morita-equivalence between two Lie groupoids G_1 and G_2 is defined by*

1. *a non-Hausdorff locally compact manifold Z_f provided with two submersions $r : Z_f \to M_1$ and $s : Z_f \to M_2$ having the following properties:*
2. *Z_f is a left principal G_1-space and a right principal G_2-space, and the actions commute;*
3. *r induces a diffeomorphism between Z_f/G_2 and M_1, and s induces a diffeomorphism from $G_1\backslash Z_f$ onto M_2.*

4. From groupoids to foliations

4.1. Groupoids defining foliations.

Definition 12. *Let G be a Lie groupoid. For any $\gamma \in G$, the differentiable map $L_\gamma : G^{\alpha(\gamma)} \to G^{\beta(\gamma)}$ defined by the equality $L_\gamma(\lambda) = \gamma.\lambda$ is a G-bundle homomorphism of β-fibers with the inverse map $L_{i(\gamma)}$. The map L_γ is called the left translation by γ. We also define the right translation $R_\gamma : G_{\beta(\gamma)} \to G_{\alpha(\gamma)}$ by the equality $R_\gamma(\lambda) = \lambda.\gamma$. This map is a diffeomorphism of α-fibers. Obviously, the left and right translations commute.*

Thus, the projection of β-fibers by α (similarly, the projection of α-fibers by β) is a partition of M, and these partitions are equal; for any $x \in M$, $\Phi(x) = \beta(G_x) = \alpha(G^x)$ is the element of the partition containing the point x. In fact, connected components of elements of the partition defined in this way are leaves of a singular foliation $\mathcal{S}$ on M. This is a regular foliation if we impose some restrictions on the groupoid action. If intersections of α-fibers and β-fibers are of constant dimension (i.e., if the isotropy groupoid $\mathrm{Is}(G)$ is a manifold), the dimension of leaves is also constant, and thus $\mathcal{S}$ is a regular foliation.

Definition 13. *A Lie groupoid G is called regular if*

(1) *α-fibers are connected;*
(2) *the isotropy groups are discrete (thus, $\dim(G_x^x) = 0$ for any $x \in M$).*

The groupoid G is called oriented if the induced foliation is orientable.

The inversion i changes α-fibers and β-fibers. Thus, under the conditions of Definition 13, β-fibers are also connected and, following the previous discussion, M is provided with a regular foliation $\mathcal{S}$. We say that this foliation is *defined by the action of G on M*. For example, the homotopy groupoid of a manifold (inducing a foliation by a unique leaf) is regular.

Definition 14. *A regular groupoid G defines a foliation $\mathcal{F}$ on M if orbits of the groupoid action are leaves of $\mathcal{F}$.*

Reciprocally, every foliation can be defined via the action of a regular Lie groupoid. The most important are the homotopy and holonomy groupoids of the foliation. The remaining groupoids are "in the middle" (the holonomy groupoid is the largest one, the homotopy groupoid is the smallest one, and the remaining groupoids are constructed as quotients of the holonomy groupoid).

4.2. Homotopy groupoid of a foliation. Let $(M, \mathcal{F})$ be a foliation of dimension p and codimension q on a manifold M of dimension $n = p + q$. We consider the set $\mathcal{P}(\mathcal{F})$ of tangent paths of $\mathcal{F}$ (i.e., of paths such that their images are contained in leafs) provided with the compact-open topology. We define the following open equivalence relation:

$$\gamma \sim \gamma' \text{ if } \gamma \text{ is homotopic (with fixed extremities) to } \gamma' \text{ in the leaf.}$$

On the quotient $\Pi_1(\mathcal{F}) = \mathcal{P}(\mathcal{F})/\sim$ by this action, we define a structure of an algebraic groupoid as follows: the unit space is M (identified with the class of constant paths), the projections are $\alpha(\gamma) = \gamma(0)$ and $\beta(\gamma) = \gamma(1)$, and the groupoid multiplication and inversion are induced by the usual composition and inversion of paths, respectively. The topology of $\mathcal{P}(\mathcal{F})$ induces on $\Pi_1(\mathcal{F})$ a quotient topology. With respect to this topology, α, β, i, and . are continuous maps, α and β are open, and i is a homeomorphism. If $x \in M$, then $\beta : \Pi_1(\mathcal{F})_x \to L_x$ and $\alpha : \Pi_1(\mathcal{F})^x \to L_x$ are universal coverings of L_x, the leaf containing x, and the isotropy group at x is just $\Pi_1(\mathcal{F})_x^x = \pi_1(L_x, x)$.

The differentiable structure is developed in detail in [11]. Concisely, we can describe it in the following manner: if γ is a tangent path in $\mathcal{F}$, then we consider two distinguished cubes $U_i = P_i \times T_i$, where U_i is a neighborhood of $\gamma(i)$ $(i \in \{0, 1\})$, P_i is a plaque, and T_i is a transversal of $\mathcal{F}$. Modulo possible restrictions on T_0 and T_1, the local triviality of the foliation allows us to define a holonomy diffeomorphism $h_\gamma : T_0 \to T_1$ represented by a "tube of tangent paths of $\mathcal{F}$," $\widehat{h_\gamma} : T_0 \times [0,1] \to M$. This tube is parametrized by T_0, and the map h_γ is determined by passing from the origin to the ends of these paths. The tube $\widehat{h_\gamma}$ extends to a differentiable family of tangent paths $\mathcal{F}$ parametrized by $P_0 \times T_0 \times P_1$ and inducing a diffeomorphism from U_0 to U_1. Passing to homotopy classes, we obtain a local chart on $\Pi_1(\mathcal{F})$. The atlas constructed in this way defines on $\Pi_1(\mathcal{F})$ a structure of differentiable manifold of dimension $2p + q$. Since all the objects defining the groupoid structure on $\Pi_1(\mathcal{F})$ are differentiable, we obtain a Lie groupoid called the *homotopy* or *fundamental groupoid* of the foliation. This is a "foliated version" of the homotopy groupoid of a manifold.

4.3. Holonomy groupoid of a foliation. If we define on $\mathcal{P}(\mathcal{F})$ a finer equivalence relation "$\sim_h$" as follows:

$$\gamma \sim_h \gamma' \text{ if the holonomy germ of the path } (\gamma')^{-1}.\gamma \text{ is trivial,}$$

then we obtain a Lie groupoid $\mathrm{Hol}(\mathcal{F}) = \mathcal{P}(\mathcal{F})/\sim_h$ (of dimension $2p+q$), called the *holonomy groupoid* or *graph of the foliation*. The isotropy group of $x \in M$ is $\mathrm{Hol}(\mathcal{F})_x^x = h(\pi_1(L_x, x))$, where the map $h : \pi_1(L_x, x) \to \mathrm{Diff}(T_0, x)$ (defined by $h(\gamma) = h_\gamma$) is the holonomy representation of the leaf L_x through x. Fibers of this groupoid are the holonomy coverings of leaves of $\mathcal{F}$. Note that the quotient $\mathrm{Hol}(\mathcal{F})^x / \mathrm{Hol}(\mathcal{F})_x^x$ is diffeomorphic to L_x, hence $\mathrm{Hol}(\mathcal{F})$ is the natural desingularization of $M/\mathcal{F}$, and in fact it "unwraps" all leaves simultaneously!

The lifting property of paths allows us to construct a Lie groupoid surjective homomorphism $\phi : \Pi_1(\mathcal{F}) \to \mathrm{Hol}(\mathcal{F})$. Unlike $\Pi_1(\mathcal{F})$, $\mathrm{Hol}(\mathcal{F})$ "forgets" the leaf structure and carries only information about the transverse structure of $\mathcal{F}$. In general, $\Pi_1(\mathcal{F})$ and $\mathrm{Hol}(\mathcal{F})$ are not Hausdorff.

If we consider the foliation $\mathcal{F}_U$ restricted to a distinguished cube $U = P \times T$, where P is a plaque and T is a transversal, then the two groupoids previously defined coincide,

$$\Pi_1(\mathcal{F}_U) = \mathrm{Hol}(\mathcal{F}_U) = P \times P \times T.$$

In fact, we have a parametrized (by T) family of coarse groupoids (see item (3) of Example 1); this is the *local triviality property* of the homotopy and holonomy groupoids of a foliation.

Investigation of the leaf space is the study of transverse properties. It is useful to introduce the following notion.

Definition 15. *Let G be a regular groupoid defining a foliation $\mathcal{F}$ on M and let T be a transverse submanifold intersecting every leaf (a total transversal, eventually not connected). Set $G_T^T = \{\gamma \in G : \alpha(\gamma), \beta(\gamma) \in T\}$. Considering α_T and β_T, the restrictions of α and β to G_T^T (these restrictions are local diffeomorphisms), we get a Lie groupoid G_T^T, called the transverse groupoid associated with the transversal T.*

The transverse groupoid is simpler than the original one, and the following statement holds.

Proposition 1. *The natural immersion $i_T : T \to M$ of a transversal on the manifold induces a Morita-equivalence between the groupoids G and G_T^T.*

5. C*-algebras and foliations

5.1. What is a C*-algebra?

Definition 16. *A C*-algebra A is a complex Banach algebra provided with an involution $* : A \to A$ satisfying the relation $\|a^*a\| = \|a\|^2$ with respect to the Banach norm.*

Examples 3. The first examples of C*-algebras are as follows:

(1) if M is a locally compact Hausdorff space, then the algebra $C_0(M)$ of continuous complex-valued functions on M vanishing at infinity with the involution $f^*(x) = \overline{f(x)}$ (for $x \in M$) is a commutative C*-algebra;

(2) if $\mathcal{H}$ is a Hilbert space and $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators acting on $\mathcal{H}$ with the involution defined by the usual adjunction (T^* is the adjoint of $T \in \mathcal{B}(\mathcal{H})$ if $\langle Th, h\rangle = \langle h, T^*h\rangle$ for $h \in \mathcal{H}$), then $\mathcal{B}(\mathcal{H})$ is a C*-algebra. Every norm-closed auto-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is also a C*-algebra. For example, the subalgebra of compact operators $\mathcal{K}(\mathcal{H})$ is a C*-algebra.

We have a category whose objects are C*-algebras and whose morphisms are linear maps $f : A \to B$ with the following properties: they are multiplicative ($f(a_1a_2) = f(a_1)f(a_2)$) and preserve involution ($f(a^*) = f(a)^*$) (i.e., these linear maps are $*$-*homomorphisms*). In fact, every C*-algebra can be considered as an auto-adjoint and norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space.

The central theorem of this theory shows that the category of commutative C*-algebras and $*$-homomorphisms is dual to the category of locally compact spaces and proper continuous maps.

Theorem 1 (Gelfand–Naimark). *Every commutative C*-algebra is isomorphic to $C_0(M)$ for some Hausdorff locally compact space M.*

We intend to transpose these properties to "singular spaces."

Examples 4. We have more examples of C*-algebras:

(1) the universal C*-algebra generated by a unitary u with the relations $u^*u = uu^* = 1$ is $C^*(u) \simeq C(\mathbb{S}^1)$;

(2) the universal C*-algebra generated by $\{u_1, \dots, u_n\}$ with the relations $u_i^*u_i = u_iu_i^* = 1$ and $u_iu_j = u_ju_i$ is $C^*(u_1, \dots, u_n) \simeq C(\mathbb{T}^n)$;

(3) the universal C*-algebra generated by $\{h_0, \dots, h_n\}$ with the relations $h_i = h_i^*$, $h_ih_j = h_jh_i$, and $\sum_{i=0}^{n} h_i^2 = 1$ is $C^*(h_0, \dots, h_n) \simeq C(\mathbb{S}^n)$;

(4) the *C*-algebra of a locally compact group is defined as follows*: we complete the involutive algebra $C_c(G)$ of complex-valued continuous functions with compact support on G with respect to the following norms:

- the maximal norm of the C*-algebra (involving the unitary representations of $C_c(G)$) (this norm gives us $C^*_{\mathrm{max}}(G)$)
and
- the norm induced by the left regular representation of the group (this norm gives us $C^*_{\mathrm{red}}(G)$);

(5) the *C*-algebra of a group action is defined as follows*: it is a generalization of item (4) giving us the C*-algebra *crossed product* $A \rtimes_\alpha G$ (a "twisted" group C*-algebra with coefficients in A) and the C*-algebra *reduced crossed product*, $A \rtimes_{\alpha,\mathrm{red}} G$.

Some simple examples of the latter construction are as follows:

- if G acts by a homeomorphism on a locally compact space M (i.e., by $*$-automorphisms on $C_0(M)$), then $C_0(M) \rtimes G$ is the *transformation group* C*-algebra. If the action is free and proper, then $C_0(M) \rtimes G$ is isomorphic to $C_0(M/G) \otimes \mathcal{K}(L^2(G))$;
- if $A = \mathbb{C}$ and α is trivial, then the crossed product is $C^*(G)$, and the reduced crossed product is $C^*_{\mathrm{red}}(G)$;
- if A and G are arbitrary and α is trivial, the crossed product is the tensor maximal product $A \otimes_{\max} C^*(G)$.

5.2. Groupoid C*-algebras. The following construction is valid for many groupoids. To simplify our presentation, we assume that G is the holonomy groupoid of a regular and orientable foliated space $(M, \mathcal{F})$.

Since G is not, in general, Hausdorff, it is necessary to adapt to this situation the definition of the algebra of continuous functions with compact support on G.

Definition 17 [3]. *The algebra $C_c(G)$ is the vector space generated by finite sums of the form $f = \varphi \circ \chi$, where*

- *$\chi : U \to \mathbb{R}^k$ is a local chart for the manifold structure on G,*
- *φ is a continuous function with compact support on $\chi(U)$, i.e., $f = \varphi \circ \chi$ on U, and f vanishes on $G - U$.*

If G is Hausdorff, this definition coincides with the usual definition of $C_c(G)$.

The choice of a Riemannian metric on M defines an orthogonal decomposition of the tangent bundle: $T(M) = \nu(\mathcal{F}) \oplus T(\mathcal{F})$. We consider a pure form ω of type $(0, p)$ on M (p is the dimension of the foliation). By the restriction to leaves, this form defines a volume associated with the induced metric (the volume element ω changes differentiably on M). If $x \in M$ and if L_x is the leaf through x, then the restriction $\omega|_{L_x} = \omega_x$ is a volume form on L_x. In addition, $\alpha : G^x \to L_x$ is a covering map (corresponding to the kernel of the holonomy representation). Thus, it is possible to lift ω_x in λ^x on G^x. Globally, we lift ω by α in a volume form $\lambda = \alpha^*(\omega)$ on G. This defines a volume by the restriction to β-fibers.

Definition 18. *We say that $\{\lambda^x\}_{x \in M}$ is a left Haar system on G. Intuitively, a Haar system is an object of metric nature playing the role of the Haar measure for groups.*

This volume is invariant with respect to left translations.

Lemma 1. *If $f \in C_c(G^x)$ and $\gamma_0 \in G_x$, then*

$$\int_{G^x} f(\gamma) d\lambda^x(\gamma) = \int_{G^{\beta(\gamma_0)}} f(L_{\gamma_0}(\gamma)) d\lambda^{\beta(\gamma_0)}(\gamma).$$

We also have the continuity condition.

Lemma 2. *For $f \in C_c(G)$, the map $\lambda(f) : M \to \mathbb{C}$ defined by $\lambda(f)(x) = \int_{G^x} f(\gamma) d\lambda^x(\gamma)$ is continuous.*

The previous lemmas imply the following statement.

Proposition 2. *The algebra $C_c(G)$ is a $*$-algebra for the following operations for $f, g \in C_c(G)$:*

(1) involution, $f^*(\gamma) = \overline{f(\gamma^{-1})}$ *for $\gamma \in G$.*

(2) convolution, $f * g(\gamma) = \int_{G^{\beta(\gamma)}} f(\gamma_1) g(\gamma_1^{-1}.\gamma) d\lambda^{\beta(\gamma)}(\gamma_1)$ *for $\gamma \in G$.*

We consider the nondegenerate representation $R_x : C_c(G) \to \mathcal{L}(L^2(G^x))$ of $C_c(G)$ defined by the formula

$$R_x(f)(\xi)(\gamma) = \int_{G^x} f(\gamma^{-1}.\gamma_1) \xi(\gamma_1) d\lambda^x(\gamma_1)$$

for $x \in M$, $f \in C_c(G)$, $\xi \in L^2(G^x)$, and $\gamma \in G^x$.

Definition 19. *The reduced C*-algebra $C^*_{\text{red}}(G)$ of G is the completion of $C_c(G)$ with respect to the C*-algebra norm $\| f \| = \sup_{x \in M} \| R_x(f) \|$.*

The previous construction depends on the choice of a Riemannian metric on M. If λ and λ_1 are volumes on G obtained as above from different Riemannian metrics, these volumes are related by the identity $\lambda_1 = f.\lambda$, where the map $f : G \to \mathbb{R}$ is continuous, positive, and constant on fibers of the map $h : G \to M \times M$ defined by $h(\gamma) = (\beta(\gamma), \alpha(\gamma))$. The C*-algebras $C^*_{\lambda,\text{red}}(G)$ and $C^*_{\lambda_1,\text{red}}(G)$ coincide in the set-theoretic sense, i.e., the same functions are bounded for the two norms. In addition, for any $x \in M$, the measure λ^x is absolutely continuous with respect to λ_1^x, i.e., there is a unique λ-measurable function $f \geq 0$ (the Radon–Nikodym derivative) such that $\lambda_1(E) = \int_E f d\lambda$. Thus, the C*-algebra $C^*_{\text{red}}(G)$ does not depend on the choice of λ or λ_1. Thus, the previous construction is intrinsic.

5.3. Morita-equivalence of C*-algebras.

Definition 20. *Let A be a C*-algebra and let $\mathcal{E}$ be a right A-module. The module $\mathcal{E}$ is called a Hilbert A-module if it is provided with an A-valued inner product $\langle \, , \, \rangle_A : \mathcal{E} \times \mathcal{E} \to A$, linear with respect to the second variable ($\lambda\langle \xi, \eta\rangle_A = \langle \xi, \lambda\eta\rangle_A$), antilinear with respect to the first variable ($\lambda\langle \xi, \eta\rangle_A = \langle \overline{\lambda}\xi, \eta\rangle_A$), and such that for $\xi, \eta \in \mathcal{E}$ and $a \in A$, the following statements hold:*

1. $\langle \xi, \eta\rangle_A = \langle \eta, \xi\rangle_A^*$;
2. $\langle \xi, \eta a\rangle_A = \langle \xi, \eta\rangle_A a$ and $\langle \xi a, \eta\rangle_A = a^*\langle \xi, \eta\rangle_A$;
3. $\langle \xi, \xi\rangle_A \geq 0$ *(i.e., $\langle \xi, \xi\rangle_A$ is positive as an element of A) and $\langle \xi, \xi\rangle_A = 0$ if and only if $\xi = 0$;*
4. *the map $n : \mathcal{E} \to \mathbb{R}$ defined by $n(\xi) = \|\langle \xi, \xi\rangle_A\|^{1/2}$ is a complete space norm on $\mathcal{E}$.*

Definition 21. *A Hilbert A-module $\mathcal{E}$ is full if the A-valued inner product $\langle \, , \, \rangle_A$ on $\mathcal{E}$ generates A as a closed bilateral ideal.*

Definition 22. *Two C*-algebras A and B are Morita-equivalent if there is a full Hilbert B-module $\mathcal{E}$ such that A is isomorphic to $\mathcal{K}_B(\mathcal{E})$.*

The following statement holds.

Proposition 3. *If G_1 and G_2 are Morita-equivalent groupoids, then $C^*_{\text{red}}(G_1)$ and $C^*_{\text{red}}(G_2)$ are Morita-equivalent C*-algebras.*

5.4. C*-algebra of a foliation.

Definition 23. *The C*-algebra $C^*(M/\mathcal{F})$ of a foliated space $(M, \mathcal{F})$ is defined as the reduced C*-algebra $C^*_{\text{red}}(G)$ of its holonomy groupoid.*

The construction of $C^*(M/\mathcal{F})$ is *local* in the following sense.

Proposition 4. *If $U \subset M$ is open and $\mathcal{F}_U$ is the restriction of $\mathcal{F}$ to U, then G_U (the graph of $(U, \mathcal{F}_U)$) is an open subgroupoid of G, and the inclusion $C_c(G_U) \subset C_c(G)$ extends to a $*$-isometric isomorphism from $C^*(U/\mathcal{F}_U)$ to $C^*(M/\mathcal{F})$.*

Note that $C^*(M/\mathcal{F})$ is *stable* in the sense of the following statement.

Proposition 5. $C^*(M/\mathcal{F}) \otimes \mathcal{K} \cong C^*(M/\mathcal{F})$.

Propositions 1 and 3 imply that the C*-algebra of a foliation *depends only on its transverse structure.*

Theorem 2. *If $(M, \mathcal{F})$ is a foliated manifold and T is a total transversal, then the C*-algebras $C^*(M/\mathcal{F})$ and $C^*_{\text{red}}(G_T^T)$ are Morita-equivalent.*

We easily prove the following result.

Theorem 3. *If $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are topologically equivalent foliations, then*

$$C^*(M_1/\mathcal{F}_1) \otimes \mathcal{K} \cong C^*(M_2/\mathcal{F}_2) \otimes \mathcal{K}.$$

Some properties of a foliated manifold are reflected in the structure of the associated C*-algebra. For example, note the following result.

Proposition 6. *If the graph is Hausdorff, then*

1. *$C^*(M/\mathcal{F})$ is simple if and only if each leaf of $\mathcal{F}$ is dense;*
2. *$C^*(M/\mathcal{F})$ has an injective irreducible representation if and only if there is a dense leaf;*
3. *the foliation $\mathcal{F}$ is closed (i.e., its leaves are closed) if and only if $C^*(M/\mathcal{F})$ has compact operators as the quotient.*

In the non-Hausdorff case (for example, if the foliation is minimal), the C*-algebra is not necessarily simple.

Examples 5. Some examples of C*-algebras of foliations are constructed as follows:

1. if M is locally compact and is foliated by points, then $G = M$ and $C^*(M/\mathcal{F}) = C_0(M)$;
2. if M is locally compact and is foliated by a unique leaf, then $G = M \times M$. In this case, the Haar system is simply a measure λ on M supported by M. Obviously, elements of the dense subalgebra from the definition can be chosen as integral operators such that their kernels have compact support on $L^2(M, \lambda)$. Its completion is obviously the family of compact operators $\mathcal{K}$ on $L^2(M, \lambda)$;
3. if the foliation is defined by a fibration $F^p \to M \to B^q$, where F is connected, then M is foliated by the inverse images of points of B. Any leaf is closed and diffeomorphic to F. In this case, the holonomy groupoid is the graph of the equivalence relation corresponding to the partition of M by leaves and the C*-algebra $C^*(M/\mathcal{F})$ is isomorphic to $C_0(B, \mathcal{K}(L^2(F))$;
4. if the foliation comes from the action of a Lie group Γ so that the graph is $M \times \Gamma$ (in general, this is not true), then $C^*(M/\mathcal{F})$ is the reduced crossed product $C_0(M) \rtimes_{\text{red}} \Gamma$.

6. K-theory of a C*-algebra

6.1. K-theory of topological spaces. Let M be a compact space. Denote by $V(M)$ the set of all isomorphism classes of locally trivial complex bundles over M. It is well known that V is a contravariant functor from the category of compact spaces to the category of Abelian semigroups and is homotopy invariant. $K^0(M)$ is defined as the Grothendieck group of $V(M)$ and is also a contravariant functor from the category of compact spaces to the category of Abelian groups. This definition is generalized to the case of a locally compact M.

The *reduced suspension* $S^n(M)$ of order n of M is defined as the noncompact space $S^n(M) = M \times \mathbb{R}^n$. The *K-group* of order n of M is defined by the equality $K^n(M) = K^0(S^n(M))$.

Let M be a locally compact space and let E be a complex vector bundle over M. Then the following statement holds.

Proposition 7. *If $\Gamma(M, E)$ is the set of continuous sections on E, then*

1. *$\Gamma(M, E)$ is a module on the ring $C(M)$ of $\mathbb{C}$-valued continuous functions over M;*
2. *an isomorphism of complex vector bundles induces an isomorphism on the corresponding section modules;*
3. *if E is a trivial bundle of dimension n, then $\Gamma(M, E) \simeq C(M)^n$;*
4. *if M is compact, then $\Gamma(M, E)$ is a projective module of finite type.*

Thus, Γ is a covariant functor from the category of complex vector bundles over a compact Hausdorff space M to the category of projective modules of finite type on $C(M)$, and we can prove that Γ is bijective (this is Swan's theorem).

According to the previous arguments, if M is compact, then $K^0(M)$ can also be described as the group of formal differences $[P] - [Q]$ of isomorphism classes of projective modules of finite type on $C(M)$. This result has enormous importance since it has a natural generalization to the noncommutative case.

6.2. K-theory of C*-algebras. If A is a C*-algebra, then we define K_0 as a contravariant functor from the category of C*-algebras to the category of Abelian groups so that elements of $K_0(A)$ can be considered as the formal differences $[p] - [q]$, where p and q are projections on $\mathbb{M}_k(\widetilde{A})$ ($\widetilde{A}$ is the unitary algebra associated with the C*-algebra A in the usual manner) for a certain $k \in \mathbb{N}$ and $p - q \in \mathbb{M}_k(A)$.

If A is unitary, then it is also possible (and very useful) to consider $K_0(A)$ as the formal differences $[\mathcal{E}] - [\mathcal{F}]$ of equivalence classes of A-projective modules of finite type.

In addition, we can define the K-theory groups of higher dimension (introducing the notion of *suspension of a C*-algebra*), and Swan's theorem shows that if $A = C_0(M)$ (where M is a locally compact space), then its K-analytic group $K_j(A)$ is naturally isomorphic to the K-topological group $K^j(M)$.

The functor K_j is homotopy invariant, preserves inductive limits, and is stable (i.e., $K_j(A) \simeq K_j(A \otimes \mathcal{K})$, where $\mathcal{K}$ is the algebra of compact operators), and semi-exact (if J is an ideal in A and we have the short exact sequence $0 \to J \xrightarrow{i} A \xrightarrow{\pi} A/J \to 0$, then we have the short exact sequence of groups $K_j(J) \xrightarrow{i_*} K_j(A) \xrightarrow{\pi_*} K_j(A/J)$).

Let us state the main result for our purposes.

Proposition 8. *If A and B are Morita-equivalent C*-algebras, then the corresponding K-theory groups $K_*(A)$ and $K_*(B)$ are isomorphic.*

6.3. K-orientation. Consider a foliated manifold $(M, \mathcal{F})$. A theorem due to Buffet–Lor shows that there is a topological space BG and a continuous map $i : BG \to M/\mathcal{F}$ with the following universal property: for any space X and map $f : X \to M/\mathcal{F}$, there is a map $g : X \to BG$ (defined modulo homotopy) such that $f = i \circ g$. The space BG is a CW-complex, and there is a principal (contractible) G-bundle EG over BG such that for any space X and a principal G-bundle E over X, there is a map $f : X \to BG$ such that $E = f^*(EG)$. The space BG is unique up to a homotopy.

We can regard BG (which is a transverse object) as a foliated space $(W, \mathcal{F}_W)$ such that leaves of the foliation $\mathcal{F}_W$ are contractible and the holonomy transverse groupoids are locally isomorphic to the initial groupoid.

K-theory is a complex theory with compact support. We work with real bundles (tangent bundles, tangent bundles of foliated spaces, etc.), hence we need an additional structure for "complexifying" them. K-orientation defined through *spin*^{c}-structures is the natural orientation which guarantees the existence of index, duality, Thom isomorphism theorems, etc., for this cohomology theory.

Definition 24. *Let* Ml_n^c *be the group* $\mathrm{Ml}_n(\mathbb{R}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{U}(1)$*, where* $\mathrm{Ml}_n(\mathbb{R})$ *is the metalinear group, i.e., the nontrivial twofold covering of the set of real matrices with positive determinant group* $\mathrm{Gl}_n^+(\mathbb{R})$*. The maximal compact of* Ml_n^c *is* $spin^c(n)$*.*

Definition 25. *The holonomy groupoid of $\mathcal{F}$ is called K-oriented if the structural groupoid of the tangent bundle to the foliation $T(\mathcal{F})$ is reduced to* $spin^c$*.*

Definition 26. *If X is a G-manifold and $\nu(\mathcal{F})$ is the normal bundle of the foliation, then a map $f : X \to BG$ is called K-orientable if the bundle $T(X) \oplus f^*(\nu(\mathcal{F}))$ has a* $spin^c$*-structure. When we choose one of these structures, we say that f is K-oriented.*

7. Statement of the Baum–Connes conjecture

Let $(M, \mathcal{F})$ be a foliated manifold and let G be its holonomy groupoid. A purely geometric manner of defining K-theory is the following.

Definition 27. *A K-cycle is a triple (X, E, f), where*

(1) *X is a closed manifold and E is a complex G-bundle over X;*
(2) *$f : X \to BG$ is a K-oriented map (i.e., a* $spin^c$*-structure over $T(X) \oplus f^*(\nu(\mathcal{F}))$ is fixed).*

We do not assume that X is connected, and E can have different dimensions on different connected components of X.

We define an equivalence relation "$\sim$" on the set of K-cycles via disjoint sum, bordism, and vector bundle modification conditions (relations connected with the multiplicativity of the index of elliptic operators on vector bundles).

Definition 28. *$K_{*,\tau}(BG)$ is the quotient of K-cycles by the equivalence relation "$\sim$," where the subindex τ means the "twist" by the transverse bundle of the foliation. This object is the topological K-theory $K_{\mathrm{top}}(M/\mathcal{F})$ of the leaf space.*

Examples 6. In the simplest cases, we have the following properties:

(1) if $\mathcal{F}$ is the foliation by a unique leaf, then $\nu(\mathcal{F}) = 0$ and $K_{*,\tau}(BG) \simeq K^*(BG)$;
(2) if M is foliated by points, then $\nu(\mathcal{F}) \simeq T(M)$, BG is homotopy equivalent to M, and $K_{*,\tau}(BG) \simeq K^*(M)$.

Definition 29. *The analytic K-theory $K_{\mathrm{an}}(M/\mathcal{F})$ of the leaf space is defined as the K-theory $K^*(C^*(M/\mathcal{F}))$.*

The longitudinal index theorem [5] defines the *K-index map*:

$$\mu : K_{\mathrm{top}}(M/\mathcal{F}) \to K_{\mathrm{an}}(M/\mathcal{F}).$$

Conjecture 1 (Baum–Connes). *The K-index map is an isomorphism if the holonomy groups are torsion-free.*

8. Our work in this field

A particularly "favorable" situation is the case of a foliation $(M, \mathcal{F})$ for which the holonomy groupoid G has contractible fibers (α- or β-fibers), i.e., the case of a *classifying foliation*. In this case, the space BG is identified with the manifold M and the topological K-theory of the leaf space is reduced to the K-theory of M (or of $C_0(M)$). In this simple case, we cannot give an immediate formulation of the Baum–Connes conjecture since the "foreseen" isomorphism is defined in terms of a *K-oriented map* between leaf spaces (or groupoids) which can be interpreted as a projection from M to $M/\mathcal{F}$.

Our study is centered on classifying foliations, or, more generally, foliations which can be transformed into classifying ones through simple topological manipulations (topological equivalences, Morita-equivalences, etc.). In these cases, we prove the following "reduced formulation" of the Baum–Connes conjecture.

Conjecture 2 (reduced form of the Baum–Connes conjecture)**.** *If the holonomy groupoid of the foliation is classifying and K-oriented, then the K-index map* $\mu : K^*(M) \to K_{\mathrm{an}}(M/\mathcal{F})$ *is an isomorphism.*

The Baum–Connes conjecture has already been verified for some cases of foliations. Let us list the following cases:

(1) fibrations $F \to M \to B$; in this case, $C^*(M/\mathcal{F}) \simeq C_0(B) \otimes \mathcal{K}$ and $K_{\mathrm{an}}(M/\mathcal{F}) \simeq K^*(B)$, where the leaf space is identified with the base space of the fibration B;

(2) foliations induced by free actions of $\mathbb{R}^n$; following the Thom isomorphism of Connes [3], we have $C^*(M/\mathcal{F}) \simeq C_0(M) \rtimes \mathbb{R}^n$, and the K-theory is $K^*(M)$ if n is odd and $K^{*+1}(M)$ otherwise. For foliations induced by free actions of solvable simply connected Lie groups Γ, we have $K_{\mathrm{an}}(M/\mathcal{F}) \simeq K^{*+dim(\Gamma)}(M)$;

(3) the Reeb foliations on $\mathbb{T}^2$ and $\mathbb{S}^3$ [10];

(4) foliations without holonomy ([9] in the C^∞-case and [7] in the topological case). In this case, BG is a torus $\mathbb{T}^n$ (n is the order of the holonomy group of the foliation) provided with a linear foliation, and we apply (2);

(5) foliations almost without holonomy [6]. Applying graphs of groups, we reduce this case to (4). The reasoning used in this case is also valid for another type of foliated spaces, where a certain "scheme" in closed and open sets works;

(6) some nontrivial examples: the Sacksteder foliation, the Hirsch foliation, $\mathbb{Z}$-periodic foliations, and so on.

In all these cases, the space BG has the homotopy type of a manifold, and we use "strong" arguments involving Morita-equivalences of groupoids and C*-algebras (see Propositions 1, 3, and 8).

At present, we are studying (in collaboration with O'uchi [8]) properties of *groupoid correspondences* to prove the conjecture in other situations.

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