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Almost without Holonomy Foliations and Graphs of Groups

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In this poster, the holonomy groupoid of an almost without holonomy foliation is described as a graph of abelian groups ([HM] and [Ma2]).
As a consequence, and in a similar way to the one described by A. Connes for **R**-actions [C], we obtain in [HM] and [Ma2] the Baum-Connes conjecture for almost without holonomy foliations.

1 Almost without holonomy foliations

Let $(M, \mathcal{F}, \mathcal{N})$, where M^m is compact, $\mathcal{F}$ a C^r-foliation ($r \geq 1$) of codimension one, transversally oriented by a vector field Y, tangent to $\partial(M)$ (when $\partial(M) \neq \emptyset$) and $\mathcal{N}$ a transverse foliation, orientable, of dimension one, defined by the flow corresponding to Y.
$(M, \mathcal{F}, \mathcal{N})$ is an *almost without holonomy foliation*, when every non closed leaf $L \in \mathcal{F}$, has trivial holonomy.
The study of the almost without holonomy foliations can be reduced to the study of the so called models [He1]:
$(M, \mathcal{F}, \mathcal{N})$ is a *model of almost without holonomy foliation* of type 1 or 2, if the corresponding condition is verified:

(i) $M = L^{m-1} \times [0,1]$ (L is C^r-diffeomorphic to a component of $\partial(M)$) and $\mathcal{N}$ is tangent to $[0,1]$;

(ii) the foliation $\overset{\circ}{\mathcal{F}}$ induced on $\overset{\circ}{M}$ by $\mathcal{F}$ is a without holonomy foliation and:

 (a) $\overset{\circ}{\mathcal{F}}$ is a fibration of $\overset{\circ}{M}$ over $\mathbb{S}^1$, or

 (b) each $L \in \overset{\circ}{\mathcal{F}}$ is dense in $\overset{\circ}{M}$.

$(M,\mathcal{F},\mathcal{N})$ is <u>*of finite type*</u>, if $\mathcal{F}$ has only a finite number of closed leaves.
In the following, $(M,\mathcal{F},\mathcal{N})$ will be an almost without holonomy foliation of finite type.
We have the following essential properties ([He1], [He2] and [I]):

(i) a model of type 1 over $M = L \times [0,1]$, is described by an abelian representation of $\pi_1(L)$ in $\mathrm{Diff}^{\mathrm{r}}_{+}([0,1])$;

(ii) if $(M,\mathcal{F},\mathcal{N})$ is a model of type 2 and class C^r, $r \geq 2$, the leaves of $\overset{\circ}{M}$ are all diffeormorphic;

(iii) for each $L \in \mathcal{F}$, $hol(L)$ is an abelian group;

(iv) $\mathcal{F}$ has not exceptional leaves.

Let $F = C_1 \cup \cdots \cup C_n$ be the union of the closed leaves of $\mathcal{F}$ (which is closed in M). Then, $M - F = U_1 \cup \cdots \cup U_m$, where U_j is a connected component of $M - F$. If $\mathcal{F}_j$ is (respectively, $\overline{\mathcal{F}_j}$) the foliation induced by $\mathcal{F}$ over U_j (respectively, over $\overline{U_j}$), then $\mathcal{F}_j$ is a without holonomy foliation, and $(\overline{U_j}, \overline{\mathcal{F}_j})$ is a model of type 2.

2 $(M,\mathcal{F},\mathcal{N})$ gives a graph of groups $\Gamma(\mathcal{F})$

An <u>*oriented graph*</u> is $\Gamma = (\Gamma^0, \Gamma^1, s, r)$, where:

(i) Γ^0 is the set of *vertices*,

(ii) Γ^1 is the set of *edges*,

(iii) the maps $s, r\colon \Gamma^1 \longrightarrow \Gamma^0$ are the *incidence maps*, where for each $a \in \Gamma^1$, $s(a) \in \Gamma^0$ is *the source* of a and $r(a) \in \Gamma^0$ is *the end* of a (the orientation is included in the definition of r and s). $s(a)$ and $r(a)$ are the *extremities* of a.

An oriented graph is <u>*finite*</u>, when the sets Γ^0 and Γ^1 are finite.

A <u>*finite graph of abelian groups of homeomorphisms of finite type*</u> of $\mathbb{R}$ ($GH\mathbb{R}$, for brevity), is defined by:

(i) a graph $\Gamma = (\Gamma^0, \Gamma^1, s, r)$ finite and oriented,

(ii) for each $s \in \Gamma^0$, $G_s \subset \mathrm{Homeo}(\mathbb{R})$ is an abelian group, of finite type and without fixed points,

(iii) for each $a \in \Gamma^1$, $H_a \subset \mathrm{Homeo}(\mathbb{R}, 0)$ is an abelian group, of finite type, and with 0 as unique fixed point,

(iv) for each $a \in \Gamma^1$, there are homomorphisms: $S_a\colon H_a \longrightarrow G_{s(a)}$ and $R_a\colon H_a \longrightarrow G_{r(a)}$, where $R_a(f) = log \circ f \circ exp$ and $S_a(f) = log \circ f \circ -exp$.

S_a and R_a are not, in general injective or surjective. Geometrically, the passage of H_a to $G_{r(a)}$ consists in to restrict f to $(0, \infty)$ and to reparametrize to obtain an homeomorfism of $\mathbb{R}$, which evidently has not yet fixed points.
A *morphism* $f\colon \Gamma = (\Gamma^0, \Gamma^1, s, r) \longrightarrow \widehat{\Gamma} = (\widehat{\Gamma^0}, \widehat{\Gamma^1}, \widehat{s}, \widehat{r})$ between two $GH\mathbb{R}$, is defined by:

(i) a couple of maps $f_0\colon \Gamma^0 \longrightarrow \widehat{\Gamma^0}$ and $f_1\colon \Gamma^1 \longrightarrow \widehat{\Gamma^1}$, which preserve the orientation and verifying $\widehat{s} \circ f_1 = f_0 \circ s$ and $\widehat{r} \circ f_1 = f_0 \circ r$,

(ii) two homomorphisms $h_a\colon H_a \longrightarrow \widehat{H}_{f_1(a)}$ and $g_s\colon G_s \longrightarrow \widehat{G}_{f_0(s)}$, such that $g_{s(a)} \circ S_a = \widehat{S}_{f_1(a)} \circ h_a$ and $g_{r(a)} \circ R_a = \widehat{R}_{f_1(a)} \circ h_a$.

If $F = C_1 \cup \cdots \cup C_n$, is the union of the closed leaves in $\mathcal{F}$ and $U = M - F = U_1 \cup \cdots \cup U_m$, then for C_i $(i \in \{1, \cdots, n\})$, we observe:

- there is exactly one connected component in U, $U_{r(i)}$, such that C_i is left-adherent to $U_{r(i)}$ (the transverse field is "attracting" in $U_{r(i)}$);
- there is exactly one connected component in U, $U_{s(i)}$, such that C_i is right-adherent to $U_{s(i)}$ (the transverse field is "expanding" in $U_{s(i)}$);
- we can have $U_{s(i)} = U_{r(i)}$.

The transverse foliation $\mathcal{N}$ is defined by a flow φ. If $\varphi\colon C_i \times \mathbb{R} \longrightarrow M$, $\mathcal{F}$ can be lifted on $C_i \times \mathbb{R}$, in a foliation $\varphi^*(\mathcal{F})$, such that:

- $\varphi^*(\mathcal{F})$ is almost without holonomy, but not of finite type,
- the closed leaves of $\varphi^*(\mathcal{F})$ are isolated and their union K is closed in $C_i \times \mathbb{R}$,
- $(C_i \times \mathbb{R}) - K = \bigcup_{n \in \mathbb{N}} W_n$; if C_i (identified with $C_i \times \{0\}$) is left-adherent to $W_{r(i)}$ and right-adherent to $W_{s(i)}$, then we have $W_{r(i)} \neq W_{s(i)}$.

If $W_i^+ = W_{r(i)} \cup C_i$ and $W_i^- = W_{s(i)} \cup C_i$, then $W_i = W_i^+ \cup W_i^-$ is a neighborhood of C_i in $C_i \times \mathbb{R}$, saturated for $\varphi^*(\mathcal{F})$. And $\varphi^*(\mathcal{F})|_{W_i}$, verifies:

- $\varphi^*(\mathcal{F})|_{W_i}$ is an almost without holonomy foliation,

- there is exactly one closed leaf C_i,
- W_i fibers over C_i with fiber $\mathbb{R}$ (after reparametrization of the leaves of $\varphi^*(\mathcal{N})|_{W_i}$ by $\mathbb{R}$),
- $\varphi^*(\mathcal{F})|_{W_i}$ is defined by the suspension of a group of abelian homeomorphisms of $\mathbb{R}$, which represents the holonomy of C_i: W_i is the *geometric realisation* of the holonomy group of C_i.

We have $\varphi: W_i^+ \longrightarrow V_i^+ = \varphi(W_i^+)$, $\varphi: W_i^- \longrightarrow V_i^- = \varphi(W_i^-)$, and we consider $\varphi(W_i) = V_i$, which is a neighborhood of C_i in M. We have two possibilities: $V_i = V_i^+ = V_i^-$ (when $U_{s(i)} = U_{r(i)}$) or $C_i = V_i^+ \cap V_i^-$.

For $j \in \{1, \cdots, m\}$, $\mathcal{F}|_{U_j} = \mathcal{F}_j$ is a without holonomy foliation. If we fix a base point $x_j \in U_j$, there is a unique transversal $N_j \in \mathcal{N}$ which contains this point. For a convenient election of x_j, N_j is homeomorphic to $\mathbb{R}$, and we consider a parametrisation $p_j: \mathbb{R} \longrightarrow N_j$. $\mathcal{F}_j$ is defined by a group of homeomorphisms without fixed points, of finite type (thus abelian and archimedian) of N_j (thus of $\mathbb{R}$, by reciprocal image): it is the global holonomy group G_j of $\mathcal{F}_j$. In the same manner, for $i \in \{1, \cdots, n\}$, the induced foliation $\mathcal{F}|_{V_i}$, is defined by an abelian group of homeomorphism of a transversal N_i^* (thus, of $\mathbb{R}$, under reparametrisation), of finite type, with 0 as fixed point: it is the holonomy group H_i of the leaf C_i. And we have an homomorphism $R_i: H_i \longrightarrow G_{r(i)}$, defined by $R_i = \lambda_i \circ \rho_i \circ \eta_i$, where:

- $\eta_i: H_i \longrightarrow H_i^+$, $H_i^+ = \{f|_{[0,\infty)} : f \in H_i\}$ is the right holonomy group of C_i and η_i is the restriction to $[0, \infty)$ (η_i is not, in general, injective. But, all the η_i are injective, by example, when we have a C^ω-foliation),
- $\rho_i: H_i^+ \longrightarrow H_{r(i)}$, where $H_{r(i)}$ is the holonomy of the foliation on $W_{r(i)}$, which is the reciprocal image by φ of the foliation over $U_{r(i)}$: we identify $H_{r(i)}$ with the holonomy group of $\mathcal{F}|_{V_{r(i)}}$. ρ_i is the restriction to $(0, \infty)$, composed with the exponential map,
- $\lambda_i: H_{r(i)} \longrightarrow G_{r(i)}$ is the natural inclusion.

We define $S_i: H_i \longrightarrow G_{s(i)}$ in a similar manner.

Proposition 1. *We have a GH*$\mathbb{R}$*, the* graph of $(M, \mathcal{F})$, $\Gamma(\mathcal{F})$*, defined by:*

(i) the finite and oriented graph $(\Gamma^1, \Gamma^0, s, r)$, $\Gamma^1 = \{V_1, \cdots V_n\}$, $\Gamma^0 = \{U_1, \cdots, U_m\}$, $s(V_i) \subset U_{s(i)}$ *and* $r(V_i) \subset U_{r(i)}$*, where* $r(V_i)$ *is an open set in* $U_{r(i)}$*, represented by the inclusion by* φ *of* $W_{r(i)}$ *in* $U_{r(i)}$ *(similarly for* $s(V_i)$*),*

(ii) *for* $j \in \{1, \cdots, m\}$*,* $G_{U_j} \subset \mathrm{Homeo}(\mathbb{R})$ *is the global holonomy group* G_j *of* $(U_j, \mathcal{F}_j)$*, which is abelian, of finite type and without fixed points,*

(iii) *for* $i \in \{1, \cdots, n\}$*,* $H_{V_i} \subset \mathrm{Homeo}(\mathbb{R}, 0)$ *is the holonomy group* H_i *of the closed leaf* C_i*, which is abelian, of finite type, and with 0 as unique fixed point,*

(iv) *for* $i \in \{1, \cdots, n\}$*,* $R_{V_i}: H_{V_i} \longrightarrow G_{r(V_i)}$ *is defined by* $R_{V_i} = R_i$*.* $r(V_i)$ *is open in* $U_{r(i)}$ *and* $G_{r(V_i)}$ *is a subgroup of* $G_{U_{r(i)}}$ *(similarly for* S_{V_i}*).*

Theorem 2. $T = (\bigsqcup_{j=1}^{m} N_j) \cup (\bigsqcup_{i=1}^{n} N_i^*)$ *is a total transversal to* $\mathcal{F}$*, naturally embedded in* M*.* $\Gamma(\mathcal{F})$ *is the transverse groupoid of* $\mathcal{F}$*, relatively to* T*.*

Proposition 3. $(M, \mathcal{F}, \mathcal{N})$ *is in fact without holonomy iff* $\Gamma(\mathcal{F})$ *is reduced to a unique vertex* s*, whose group* G_s *is the holonomy group of the foliation.*

Theorem 4. *With the obvious notations,* $(M_1, \mathcal{F}_1, \mathcal{N}_1)$ *is Morita-equivalent to* $(M_2, \mathcal{F}_2, \mathcal{N}_2)$*, iff* $\Gamma(\mathcal{F}_1)$ *and* $\Gamma(\mathcal{F}_2)$ *are isomorphic.*

3 A $GH\mathbb{R}$ gives an almost without holonomy foliation

$(M, \mathcal{F})$ is a *classifying* foliation, if the holonomy covering of each leaf is contractible, that is, the fibers of the holonomy groupoid are contractible. In this case, the holonomy and homotopy groupoids of the foliation, are isomorphic.

Lemma 5. $(M, \mathcal{F}, \mathcal{N})$ *is classifying iff all the non closed leaves are contractible and the closed leaves are torus with injective holonomy.*

Let Γ be a $GH\mathbb{R}$, and consider the subgroups of H_a:

$$H_a^+ = \{f|_{[0,\infty)} : f \in H_a\} \quad \text{and} \quad H_a^- = \{f|_{(-\infty,0]} : f \in H_a\}.$$

If $r_a: H_a \longrightarrow H_a^+$ is the restriction map ($r_a(f) = f|_{[0,\infty)}$) and $j_a: H_a^+ \longrightarrow G_{r(a)}$ is defined by $j_a(h) = log \circ h \circ exp$, we have $R_a = j_a \circ r_a$. We make a similar decomposition for $S_a: H_a \longrightarrow G_{s(a)}$, through H_a^-. In this manner, we associate to each edge a, two semi-edges a^+ and a^-.
We realize geometrically every objet in the graph:

(V) Vertices We fix a vertex $s \in \Gamma^0$. Since $G_s \subset \mathrm{Homeo}(\mathbb{R})$ is abelian, of finite type and without fixed points, the Hölder theorem states that there is $n_s \in \mathbb{N}$ such that $G_s \simeq \mathbb{Z}^{n_s}$. We suspend G_s (with fibre $\mathbb{R}$) and we obtain a fibre bundle

$\mathbb{R} \to E_s \to \mathbb{T}^{n_s}$, with discrete structural group $\mathbb{Z}^{n_s}$, where E_s (diffeomorphic as manifold to $\mathbb{T}^{n_s} \times \mathbb{R}$), is provided of a foliation $\mathcal{F}_s$, by n_s-planes, which is tranverse to the fibers of the fibration.

(SE) Semi-edges If $a \in \Gamma^1$, $H_a^+ \subset \mathrm{Homeo}([0,\infty),0)$ is an abelian group, of finite type and having 0 as unique fixed point. Thus, there is $p_a^+ \in \mathbb{N}$, such that $H_a^+ \simeq \mathbb{Z}^{p_a^+}$. We suspend this group, and we obtain a fibre bundle $[0,\infty) \to E_a^+ \to \mathbb{T}^{p_a^+}$. E_a^+ (diffeomorphic as manifold to $\mathbb{T}^{p_a^+} \times [0,\infty)$) is provided of a transverse foliation, $\mathcal{F}_a^+$, by p_a^+-planes. We make the analogous construction for H_a^-.

(VSE) Vertices and open semi-edges Let $a \in \Gamma^1$. The injective homomorphism $j_a: H_a^+ \longrightarrow G_{r(a)}$, induce a morphism of foliated bundles $\widehat{j_a}: E_a^+ \longrightarrow E_{r(a)}$, which is an isomorphism between the foliated bundles $\widehat{j_a}: \overset{\circ}{E}{}_a^+ \longrightarrow j_a(\overset{\circ}{E}{}_a^+ \subset E_{r(a)}$. On the disjoint sum, $E_a^+ \sqcup E_{r(a)}$, we define the equivalence relation: $x \sim \widehat{j}_a(x)$ for $x \in \overset{\circ}{E}{}_a^+$: we identify in this manner two open sets in the bundles E_a^+ and $E_{r(a)}$. The quotient space $E_a^+ \sqcup E_{r(a)}/\sim$, is a space E (not yet a fibre bundle), provided of a foliation $\mathcal{F}_E$ induced by the quotient map: we have glued the foliated bundles E_a^+ and $E_{r(a)}$ together. The same construction is valid for E_a^- and $E_{s(a)}$.

(ESE) Edges and semi-edges If $a \in \Gamma^1$, the goal is to glue the foliated bundles E_a^+ and E_a^-. But $H_a \subset \mathrm{Homeo}(\mathbb{R})$, is an abelian group, of finite type, with 0 as unique fixed point; thus there is $q_a \in \mathbb{N}$, such that $H_a \simeq \mathbb{Z}^{q_a}$. $r_a: H_a \longrightarrow H_a^+$, can be extended in an homomorphism $r_a: \mathbb{T}^{q_a} \longrightarrow \mathbb{T}^{p_a^+}$, which define (since r_a is a restriction map) the trivial fibre bundle $\mathbb{T}^{q_a - p_a^+} \to \mathbb{T}^{q_a} \overset{r_a}{\to} \mathbb{T}^{p_a^+}$. If we consider the section $\epsilon_a: \mathbb{T}^{p_a^+} \longrightarrow \mathbb{T}^{q_a}$ of this fibre bundle, the compact leaf of the frontier of the fibre bundle E_a^+ has been embedded in $\mathbb{T}^{q_a}$. Finally, we glue $\mathbb{T}^{q_a}$ and E_a^+ together, by identifying over the disjoint sum $E_a^+ \sqcup \mathbb{T}^{q_a}$ the points $(x,0)$ and $\epsilon_a(x)$, for $x \in \mathbb{T}^{p_a^+}$ (that is, we have glued a and a^+ together, on the graph). We make the same construction for E_a^-. In this manner we have glued a^- and a^+ together on the graph.

Geometric realization One "model" in the language of graph theory, corresponds to a fixed vertex and all the edges that have it as extremity. Thus, we obtain a "geometric model" by a finite number of identifications of type **(VSE)**. In the language of graph theory, we glue two models together, when they are common vertices: we make this by identifications of type **(ESE)**, by retrieving an edge from the corresponding semi-edges. We obtain in this manner the *geometric realization* of the graph, that is, a "foliated" CW-complex $(M_0, \mathcal{F}_0)$, which is also classifying, by construction. $\mathcal{F}_0$ is a regular foliation of codimension one over a locally compact space. The dimension of the leaves depends of the different "models". We can obtain a constant dimension by uniformizing the dimension step by step in the construction of the geometric realization:

- a "model" corresponds to a vertex s of the graph. We consider the finite family $\{a_1, \cdots, a_k\}$ of edges having s as extermity. Thus, $n_s \geq p^{\pm}_{a_i}$ for each $i \in \{1, \cdots, k\}$. For the foliated bundles $(E^{\pm}_{a_i}, \mathcal{F}^{\pm}_{a_i})$, we complete the dimension multiplying the leaves by factors $\mathbb{R}^{n_s - p_{a_i}{}^{\pm}}$;

- when we glue different "models" together, there are two dimensions to uniformize. We obtain this, by multiplying the model of less dimension by a convenient factor $\mathbb{R}^l$ (we embed otherwise the torus $\mathbb{T}^{p^{\pm}_a}$ in $\mathbb{T}^{q_a}$: instead of embedding the torus, we embed a convenient nhood of $\mathbb{T}^{p^{\pm}_a}$ in $\mathbb{T}^{q_a}$).

Then, by induction, we prove that $(M_0, \mathcal{F}_0)$ is Morita-equivalent to $(M'_0, \mathcal{F}'_0)$, where M'_0 is a non compact manifold. Thus:

Theorem 6. *If Γ is a GH$\mathbb{R}$, its geometric realization $(M_0, \mathcal{F}_0)$ verifies:*

(i) $\mathcal{F}_0$ is an almost without holonomy and classifying foliation;

(ii) the graph of the foliation $(M_0, \mathcal{F}_0)$, $\Gamma(\mathcal{F}_0)$, is isomorphic to Γ.

The Morita-equivalence class of an almost without holonomy foliation $(M, \mathcal{F}, \mathcal{N})$ is an invariant of the graph of groups of the foliation:

Theorem 7. *The geometric realization of $\Gamma(\mathcal{F})$ is a foliated space Morita-equivalent to $(M, \mathcal{F}, \mathcal{N})$.*

The Baum-Connes conjecture for an almost without holonomy foliation $(M, \mathcal{F})$ is now easier to prove: we change $(M, \mathcal{F})$ by its classifying foliation $(M_0, \mathcal{F}_0)$ (Theorems 6 and 7).
For a classifying foliation, the statement of the Baum-Connes conjecture is:
"*The Thom-Connes homomorphism $\mu\colon K^*(M) \longrightarrow K_*(C^*((G))$ is a group isomorphism*"
(G is the holonomy groupoid of $\mathcal{F}$), and using a generalization of the techniques in [Ma1], we prove in [HM] and [Ma2] this result.

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