Effective Homology of Hopf Algebras¹

Ainhoa Berciano-Alcaraz

Euskal Herriko Unibertsitatea
Dpto. Matemática Aplicada, Estadística e Investigación Operativa,
ainhoa.berciano@ehu.es

Abstract
In this paper we explain the importance of the notion of $A_\infty$-Hopf algebra in the Algebraic Topology framework. In particular, we focus our efforts to show how, given a Hopf algebra, it is possible to weaken the structure in the computation of the homology. Furthermore, we will see that this process induces a new sophisticated structure, an $A_\infty$-Hopf algebra.

Keywords: Homological perturbation theory, reduction, Basic Perturbation Lemma, Hopf algebras.

1 Introduction

Classification in Algebraic Topology frequently depends on homology groups, but these groups are most often difficult to reach. Various methods are available to facilitate and structure the calculation of these groups. In particular reductions between:

- Chain complexes which are richly structured (algebras, coalgebras, simplicial, cosimplicial. . . ) but not of finite type;
- Simpler chain complexes poorly structured but of finite type;

play an important role, see [6].

¹ Work supported by NEST-Adventure contract 5006 (GIFT).
In particular the $A_{\infty}$-algebra and $A_{\infty}$-coalgebra structures are defined in this way. A homology equivalence between a differential algebra $A$ and a simple chaincomplex $C$ defines an $A_{\infty}$-algebra structure over $C$ [10], the same for coalgebras. An $A_{\infty}$-algebra is a sort of “weakened” algebra where the standard requirements for an algebra are satisfied only \textit{up to homotopy.} The same for coalgebras.

The right definition for the notion of $A_{\infty}$-Hopf algebra is a challenge for a long time, see in particular Ron Umble and Samson Saneblidze’s papers [7, 8, 9]. We propose here an original point of view.

The notion of $A_{\infty}$-algebra (resp. $A_{\infty}$-coalgebra) greatly depends on the Bar (resp. Cobar) construction. A Hopf algebra is simultaneously an algebra and a coalgebra, so that it is tempting to define a “Bar-Cobar” construction for a Hopf algebra. It happens the Hopf relation explaining how algebra and coalgebra structures fit to each other is the key point allowing us to define the Bar-Cobar construction.

The Bar construction leads to the notion of $A_{\infty}$-algebra, and the Bar-Cobar construction gives by an analogous process which seems a natural notion of $A_{\infty}$-Hopf algebra.

2 Preliminaries and didactic examples

Here we will collect some basic definitions and results of homological algebra, as well as some simple examples.

Take a commutative ground ring with unit, $\Lambda$. A \textit{differential graded module or dg-module}, $(M, d)$ is a graded module, with a differential $d : M \to M$ (that is, a morphism of degree $-1$ such that $d d = 0$). A graded module $M$ is \textit{connected} whenever $M_0 = \Lambda$ and \textit{simply connected} if it is connected and $M_1 = 0$. In such a case, the graded module $\overline{M}$ is defined as $\overline{M}_n = M_n$ for $n > 1$ and $\overline{M}_0 = 0$.

The \textit{homology} of a dg-module $M$ is the graded module $H_*(M)$, where

$$H_n(M) = \text{Ker } d_n / \text{Im } d_{n+1}.$$
A dg-algebra, \((A, d_A, \mu_A)\), is a dg–module endowed with an associative product with unit, compatible with the differential. Analogously, a dg-coalgebra \((C, d_C, \Delta_C)\) is a dg–module provided with a compatible coproduct and counit.

Furthermore, if \((H, d_H, \mu_H, \Delta_H)\) is a dg-algebra, a dg-coalgebra and product and coproduct verify the Hopf relation, i.e., \(\Delta \mu = \mu \otimes 2(1 \otimes T \otimes 1) \Delta \otimes 2\), then it is a dg-Hopf algebra.

As simple examples of dg-Hopf algebras, let us mention

- The polynomial algebra \(P(v, 2n)\), generated by \(v\) of degree \(2n\), where \(n\) is a positive integer. The product is the usual one of monomials i.e., \(v^iv^j = v^{i+j}\). The coproduct is \(\Delta(v) = v \otimes 1 + 1 \otimes v\) and extended thanks to the Hopf relation.

- The truncated polynomial algebra \(Q_p(v, 2n)\) is the quotient algebra \(P(v, 2n) / (v^p)\), where \(p\) is a primer number.

- The exterior algebra \(E(u, 2n + 1)\), \(n \geq 0\), with algebra generator \(u\) of degree \(2n + 1\) and trivial product \(u^2 = 0\).

- The divided polynomial algebra \(\Gamma(w, 2n)\), \(n \geq 1\), generated by \(\gamma_1(w) = w\) with the product given by \(\gamma_k(w)\gamma_h(w) = (k+h)! \gamma_{k+h}(w)\). The coproduct is defined by \(\Delta(\gamma_k(w)) = \sum_{i+j=k} \gamma_i(w) \otimes \gamma_j(w)\).

A fundamental tool to relate two dg-modules is the next one:

A reduction \(c : \{N, M, f, g, \phi\}\), from a dg–module \((N, d_N)\) to another one \((M, d_M)\) is a special type of homology equivalence given by the morphisms \(f, g\) and \(\phi\); where \(f : N_* \rightarrow M_*\), \(g : M_* \rightarrow N_*\) are two morphisms of degree zero and \(\phi : N_* \rightarrow N_{*+1}\) is a homotopy operator. Apart from the conditions \((c1) \ f g = 1_M\), \((c2) \ \phi d_N + d_N \phi + g f = 1_N\), the following ones must be satisfied \((c3) \ f \phi = 0\), \((c4) \ \phi g = 0\), \((c5) \ \phi \phi = 0\).
The most important consequence is that the homology groups of $N$ are naturally isomorphic to the homology groups of $M$. But in general, if $N$ is a dg-algebra, this structure is not transferred to $M$, which inherits an $A_\infty$-algebra structure (the morphisms $f$ and $g$ are not isomorphisms and there is no coherent way in general to transfer the algebra structure to $M$).

More explicitly, an $A_\infty$-algebra is a dg-module with a family of operations $\mu_i : A^{\otimes i} \to A$, of degree $i - 2$, such that for all $i \geq 1$ the following relations are satisfied

$$\sum_{n=1}^{i} \sum_{k=0}^{i-n} (-1)^{n+k+nk} \mu_{i-n+1}(1^k \otimes \mu_n \otimes 1^{i-n-k}) = 0.$$ 

In the same way, we can speak about an $A_\infty$-coalgebra as a dg-module that has to verify the dual properties (in this case the operations are denoted by $\Delta_i$).

Thanks to the perturbation theory, to speak about an $A_\infty$-(co)algebra $M$ is equivalent to give a reduction from a dg-(co)algebra to the dg-module $M$. Let us recall how it is possible.

The key in the homological perturbation theory is the **Basic Perturbation Lemma** (briefly, BPL [2]), which is an algorithm whose input is a reduction of dg–modules

$$c : \{N, M, f, g, \phi\}$$

and a perturbation datum $\delta$ of $d_N$ whose output is a new reduction $c_\delta$. The only requirement is the pointwise nilpotency of the composition $\phi \delta$, that guarantees that the sums involved on the series bellow are finite for each $x \in N$.

\[
\begin{array}{c}
\text{Input:} & c : (N, d_N) & \xrightarrow{f} & (M, d_M) & + \text{ perturbation } \delta \text{ of } d_N \\
& \phi & \downarrow & \delta & \downarrow \\
\text{Output:} & c_\delta : (N, d_N + \delta) & \xrightarrow{f_\delta} & (M, d_M + d_\delta)
\end{array}
\]
where $f_\delta, g_\delta, \phi_\delta, d_\delta$ are given by the formulas

$$
\begin{align*}
d_\delta &= f \delta \Sigma^\delta g; \quad f_\delta = f (1 - \delta \Sigma^\delta \phi); \quad g_\delta = \Sigma^\delta g; \quad \phi_\delta = \Sigma^\delta \phi;
\end{align*}
$$

and $\Sigma^\delta = \sum_{i \geq 0} (-1)^i (\phi^\delta)^i$.

In particular, if we have a reduction from $A$ to $M$, where $A$ is a dg-algebra and $M$ is a dg-module, through homological techniques, it is possible to compute the operations of the $A_\infty$-algebra structure induced on $M$ in terms of the product of $A$. This computation can be done in four steps (described lightly):

- To construct of a new reduction $T(s(c)) : \{T(sA), T(sM), Tf, Tg, T\phi\}$
- To use the BPL with perturbation datum a (simplicial) differential $d_s$ depending on the product of $A$ (see [5]).
- To extract the operations induced on $M$ from the new differential in $T(sM)$.

Where, given a dg-module $(M, d)$, the tensor module of $M$, $T(M)$, is

$$
T(M) = \bigoplus_{n \geq 0} M^\otimes n.
$$

The differential structure in $T(M)$ is provided by the tensor differential, $d_t$. $T(M)$ is endowed with both structures of dg–algebra and dg–coalgebra respectively, by a product $\mu((a_1 \otimes \cdots \otimes a_n) \otimes (a_{n+1} \otimes \cdots \otimes a_{n+p})) = a_1 \otimes \cdots \otimes a_{n+p}$; and a coproduct $\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$.

In particular, if $A$ is a dga-algebra, one can construct the reduced bar construction of $A$, $\overline{B}(A)$, whose underlying module is the tensor module of the suspension of $A$, $T(sA)$.

The total differential $d_\pi$ is given by $d_\pi = d_t + d_s$, the component $d_t$ being the tensor differential on the tensor module and $d_s$ the simplicial differential, that depends on the product on $A$. 
This dg–module is endowed with a structure of dg–coalgebra by the natural coproduct \( \Delta : \overline{B}(A) \to \overline{B}(A) \otimes \overline{B}(A) \) defined in the tensor module.

Analogously, if we have a reduction from \( C \) to \( M \), where \( C \) is a dga–coalgebra and \( M \) is a dg-module, through homological techniques, it is possible to compute the operations of the \( A_\infty \)-coalgebra structure induced in \( M \).

A simple example of this fact is the algebraic structure of the homology groups of the bar construction of a truncated polynomial algebra with coefficients in \( \mathbb{Z} \) (see [1]). Let us recall a little bit this example, in fact, let us see the scheme of the algebraic structure of the homology groups.

• There is an explicit reduction \( c : \{B(Q_p(u, 2n)), E(v, 2n + 1) \otimes \Gamma(\gamma, 2np + 2), f, g, \phi\} \).

• Even though the bar construction has a coalgebra structure, this reduction does not preserve this structure, because the morphisms are not compatible with the coproducts; so we obtain an \( A_\infty \)-coalgebra structure induced on \( E \otimes \Gamma \). This structure is extremely simple, because it only has two operations non-null, \( \Delta_2 \) and \( \Delta_p \).

• Thanks to the null differential in \( E \otimes \Gamma \), the homology groups of \( B(Q_p(u, 2n)) \) are isomorphic to \( E \otimes \Gamma \).

• So from the algebraic point of view, \( H_*(B(Q_p(u, 2n))) \) has an \( A_\infty \)-coalgebra structure defined by \( \Delta_2 \) and \( \Delta_p \).

In particular, when we are speaking about the categories of algebras or coalgebras, we have to emphasize here that the category of Hopf algebras joins the two notions. But, until now, nothing was known about the analogous appropriate notion of \( A_\infty \)-Hopf algebra. So, let us give a new way to understand them via perturbation.
3 The importance of the Hopf relation

Let \((H, d, \mu, \Delta)\) be a dga-Hopf algebra. Because of the Hopf relation, it is possible to define a new algebraic object:

**Definition 3.1.** Let us define a tensor module associated with \(H\), \(BC(H)\), Bar-Cobar of \(H\), as

\[
\{BC(H)\}_{(p,q,n)} = (H^{\otimes p})^{\otimes q} = H_{n}^{p,q},
\]

where an element is described as a matrix

\[
(a_{ij}) = \begin{pmatrix}
  a_{11} & \cdots & a_{1q} \\
  \vdots & \ddots & \vdots \\
  a_{p1} & \cdots & a_{pq}
\end{pmatrix}
\]

and the degree is \(|(a_{ij})| = n + p - q\) where

\[
n = \sum_{i=0,j=0}^{p,q} |a_{ij}|.
\]

\(BC(H)\) is a differential graded module with three differential structures induced,

- the tensor differential \(d_t: H_{n}^{p,q} \to H_{n-1}^{p,q}\); 

\[
d_t = - \sum_{i=1,j=1}^{p,q} (-1)^{P(i,j)}d_{i,j},
\]

where

\[
d_{ij} = -(-1)^{P(i,j)}
\begin{pmatrix}
  a_{11} & \cdots & a_{1q} \\
  \vdots & \ddots & \vdots \\
  a_{i1} & \cdots & d(a_{ij}) & a_{iq} \\
  \vdots & \ddots & \vdots \\
  a_{p1} & \cdots & a_{pq}
\end{pmatrix}
\]

\[
P(i, j) = \sum_{k<i \cup (k=i, l<j)} |a_{kl}| + (i - 1) - (j - 1)
\]
- the simplicial differential $d_s: H^{p,q}_n \rightarrow H^{p-1,q}_n$; $d_s = \sum_{k=0}^{p} (-1)^k \delta_k$, depending on the product of $H$:

$$\delta_k = (-1)^{sg(k,k+1)+sg_B(k)} \begin{pmatrix}
a_{11} & \cdots & a_{1q} \\
\vdots & \ddots & \vdots \\
a_{p1} & \cdots & a_{pq}
\end{pmatrix}
$$

where

$$sg(k, k + 1) = |a_{k+1,1}|(|a_{k,2}| + \cdots + |a_{k,q}|) + |a_{k+1,2}|(|a_{k,3}| + \cdots + |a_{k,q}|) + \cdots + |a_{k+1,q-1}| |a_{k,q}|$$

$$sg_B(k) = \sum_{j=1,l=1}^{k,q} |a_{jl}|$$

- the cosimplicial differential $d_c: H^{p,q}_n \rightarrow H^{p,q+1}_n$; $d_c = \sum_{i=0}^{q+1} (-1)^i \delta^c_i$, depending on the coproduct of $H$:

$$\delta^c = (-1)^{sg_C(k)} \begin{pmatrix}
a_{11} & \cdots & a_{1,k-1} & \Delta(a_{1,k}) & a_{1,k+1} & \cdots & a_{1q} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{p1} & \cdots & a_{p,k-1} & \Delta(a_{p,k}) & a_{p,k+1} & \cdots & a_{pq}
\end{pmatrix}
$$

where

$$sg_C(k) = \sum_{j=1,l=1}^{p,k-1} |a_{jl}|.$$

**Theorem 3.2.** With the above definitions, the morphism

$$d_{BC} = dt + dc + (-1)^q d_s : H^{p,q}_n \rightarrow H^{p,q}_{n-1} \oplus H^{p-1,q}_n \oplus H^{p,q+1}_n$$

is a differential in the total complex of $BC(H)$. 

8
Definition 3.3. Given $H$ a connected dga-Hopf algebra, it is possible to define a new algebraic object $\widehat{BC}H$ as a differential graded module

$$\widehat{BC}(H) = \Lambda \oplus \overline{H}^{1,1} \oplus (\overline{H}^{2,1} \oplus \overline{H}^{1,2}) \oplus \cdots \oplus \sum_{i+j=k} \overline{H}^{i,j} \oplus \cdots$$

with the differentials $d_t$, $d_s$ and $d_c$ induced in a natural way from $BC(H)$ to $\widehat{BC}(H)$.

In the rest of the paper, given a dg-module $M$ we will denote by $BC(M)$ the tensor module $\{M^{p,q}_{n}\}_{p,q,n}$ with the tensor differential induced on it. Analogously $\widehat{BC}(M)$ is the differential graded module

$$\widehat{BC}(M) = \Lambda \oplus \overline{M}^{1,1} \oplus (\overline{M}^{2,1} \oplus \overline{M}^{1,2}) \oplus \cdots \oplus \sum_{i+j=k} \overline{M}^{i,j} \oplus \cdots$$

with the tensor differential induced.

4 What about $A_{\infty}$-Hopf algebras?

We can now formulate our main results related with $A_{\infty}$-Hopf algebras. To start with, we make the following definition.

Definition 4.1. An $A_{\infty}$-Hopf algebra $M$ is a dg-module with a family of operations

$$h^{i,j} : M^\otimes i \to M^\otimes j$$

with $i, j \in \mathbb{N}$, of degree $i + j - 3$, such that

- The family $\{h^{i,1}\} : M^\otimes i \to M$ defines an $A_{\infty}$-algebra on $M$.
- The family $\{h^{1,j}\} : M \to M^\otimes j$ defines an $A_{\infty}$-coalgebra on $M$.
- The extension of $\{h^{i,j}\}_{i,j \in \mathbb{N}}$ to $(\widehat{BC}(M), d_t)$ defines a differential on it.
If we consider a reduction from $H$ to $M$, where $H$ is a dga-Hopf algebra and $M$ is a dg-module, then the information about the $A_\infty$-algebra structure induced on $M$ and the information about the $A_\infty$-coalgebra structure can be extracted, as well as the operations defining an $A_\infty$-Hopf algebra structure.

**Proposition 4.2.** Given $H$ a Hopf algebra and $c : H \to M$ a reduction, this induces a new one of dg-module

$$bc(c) : \{(BC(H), d_t), (BC(M), d_t), bc(f), bc(g), bc(\phi)\},$$

where the morphisms $bc_f, bc_g, bc_\phi$ are defined by the formulas

$$bc(f)|_{BC(H)_n} = f \otimes \cdots \otimes f;$$

$$bc(g)|_{BC(M)_n} = g \otimes \cdots \otimes g;$$

$$bc(\phi)|_{BC(H)_n} = \sum_{k=0}^{n-1} 1 \otimes \cdots \otimes 1 \otimes \phi \otimes gf \otimes \cdots \otimes gf.$$

**Theorem 4.3.** Given $c : H \to M$ a reduction, where $H$ is a simply connected dga-Hopf algebra and $M$ is a dg-module, if we consider the reduction defined in proposition 4.2, $bc(c) : \{(BC(H), d_t), (BC(M), d_t), bc(f), bc(g), bc(\phi)\}$ together with the perturbation datum $d_c + (-1)^q d_s$, thanks to the basic perturbation lemma, it is possible to define a new reduction

$$bc(c)_{d_c + (-1)^q d_s} : \{(\widehat{BC}(H), d_t + d_c + (-1)^q d_s), (\widehat{BC}(M), d_t + d_\infty), bc(f)_{\infty}, bc(g)_{\infty}, bc(\phi)_{\infty}\}$$

such that $M$ inherits an $A_\infty$-Hopf algebra structure, i.e.:

- the projection over the elements of $\widehat{BC}(M)^{1,*}$, gives the $A_\infty$-algebra of $M$;
- the projection over the elements of $\widehat{BC}(M)^{*,1}$, gives the $A_\infty$-coalgebra of $M$;
the homotopy operators of higher order of $M$ are operations $h^{i,j}: M^\otimes i \to M^\otimes j$ of degree $i + j - 3$, with $i > 1, j > 1$, defined by the formulas of the BPL.

Indeed, the explicit formulas of $h^{i,j}$ are

$$h^{i,j} = f\delta(-1)^{i-1}\sigma(\phi\delta^{i-2}(\phi\delta')^{j-1})g + f\delta'(-1)^{i-1}\sigma(\phi\delta^{i-1}(\phi\delta')^{j-2})g$$

We obtain automatically the next result

**Corollary 4.4.** Given $H$ a simply connected dga-Hopf algebra with product $\mu$, coproduct $\Delta$, $M$ a dga-module and $c:\{H, M, f, g, \phi\}$ a reduction between them. Then, $M$ inherits an $A_{\infty}$-Hopf algebra structure.

5 Summary

As we have just seen, if we are interested in the algebraic structure of the homology of a Hopf algebra, in general, we know that this structure will not be a Hopf algebra, but yes an $A_{\infty}$-Hopf algebra, that we can determine with simple methods explicitly, thanks to the perturbation theory.

References


