# Scaling of Noise and Constructive Aspects of Fluctuations

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**Abstract.** We show how scaling arguments can be applied to analyze the dynamics of stochastic systems that are periodically modulated by an input signal. Information about the behavior of the relevant quantities, such as the signal-to-noise ratio, upon variations of the noise level can be obtained by analyzing the symmetries and invariances of the system. By means of this methodology, it is possible to predict diverse physical manifestations of the cooperative behavior between noise and input signal, as for instance, stochastic resonance and stochastic multiresonance.

## 1 Introduction

Scaling arguments have proved to be very useful to analyze complex situations with little effort, without entering into intricate calculations [1]. This is achieved by simple manipulations which are able to relate to each other the relevant quantities of the system. Thus, scaling arguments have been successfully applied in many classical branches of Physics, such as critical phenomena [2,3], hydrodynamics [4,5], polymer physics [6] and non-linear physics [7,8]; more recently, they have also been used in other fields, such as growth phenomena [9], fractures [10], and economy [11], to mention just a few.

Here, we show how scaling arguments can be applied to analyze the dynamics of a wide class of systems whose dynamics is both modulated periodically and affected by noise. In particular, we will focus on the possibility that the behavior of the system may be enhanced by the addition of noise. This phenomenon, known as stochastic resonance (SR) [12–20], shows a constructive role of noise. This is one of the most counterintuitive facets of noise, since it is able to decrease the randomness displayed by the system. It has been found that SR can appear in a great number of different situations; ranging from systems as simple as a single dipole [21] to systems exhibiting certain degree of complexity, such as neural tissues [22] or pattern-forming systems [18]. Along this paper we present a methodology based upon general scaling arguments which enables one to predict the appearance of such an ordered behavior due to the presence of noise.

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## 2 Preliminary Concepts

The dynamic evolution of the systems we study belongs to the following type:

$$\frac{dx}{dt} = F(x,t) + \xi(t) \ , \tag{1}$$

where $F(x,t)$ is a function of $x$ and $t$, periodic on $t$ with period $2\pi/\omega_0$; and $\xi(t)$ is Gaussian white noise with zero mean and correlation function $\langle \xi(t)\xi(t+\tau)\rangle = 2D\delta(\tau)$, defining the noise level $D$.

Frequently, the variable that enters the dynamic equations is not the quantity we are interested in. For instance, if we are dealing with dipoles in a magnetic field, the dynamics is given by the angle between the dipole and the magnetic field. The representative quantity, however, is not the angle but the component of the magnetic moment along the field. In general, the system is described by a function $v(t)$ of the dynamic variable $x$; i. e., $v(t) \equiv v[x(t)]$. In the case of the dipole, $x$ would be equal to the angle $\theta$ whereas $v$ would be proportional to $\cos(\theta)$.

The response of the system to the periodic component of the force $F(x,t)$ can be analyzed by the averaged power spectrum,

$$P(\omega) = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} dt \int_{-\infty}^{\infty} \langle v(t)v(t+\tau)\rangle \, e^{-i\omega\tau} d\tau \ . \tag{2}$$

To this end we will assume that it consists of a delta function centered at the frequency $\omega_0$ plus a function $Q(\omega)$ which is smooth in the neighborhood of $\omega_0$ and is given by

$$P(\omega) = Q(\omega) + S(\omega_0)\delta(\omega - \omega_0) \ . \tag{3}$$

We have expressed the power spectrum in this form since we are interested in the behavior of the system in the range of frequencies close to the frequency of the input signal. Thereby, the power spectrum explicitly shows the intensity of the deterministic component of the system or output signal, $S(\omega_0)$, and the stochastic component or output noise, $Q(\omega)$. The SNR, defined as the ratio between the signal and noise,

$$\mathrm{SNR} = S(\omega_0)/Q(\omega_0) \ , \tag{4}$$

then indicates the order present in the system.

## 3 Linear Systems

Let us start by discussing in detail the simplest case, which can be treated exactly by using dimensional analysis: the periodically modulated Ornstein-Uhlenbeck process. Here, the input signal modulates the strength of the force in the following way:

$$\frac{dx}{dt} = -\kappa[1 + \alpha \sin(\omega_0 t)]x + \xi(t) \ , \tag{5}$$

where $\kappa$, $\alpha$, and $\omega_0$ are constants. In spite of its simplicity, the previous model encompasses many physical situations of interest since it describes the motion around a minimum in a force field whose intensity varies periodically in time.

### 3.1 Dimensional Analysis

Let us now assume the explicit form for the output of the system $v(x) = |x|^\beta$, where $\beta$ is a constant. Considerations based upon dimensional analysis enable us to rewrite the averaged power spectrum as

$$P(\omega, D, \kappa, \alpha, \omega_0, \beta) = \frac{1}{\kappa}\left(\frac{D}{\kappa}\right)^\beta q(\omega/\omega_0, \kappa/\omega_0, \alpha, \beta) + \left(\frac{D}{\kappa}\right)^\beta s(\kappa/\omega_0, \alpha, \beta)\delta\left(1 - \frac{\omega}{\omega_0}\right) , \tag{6}$$

where $q(\omega/\omega_0, \kappa/\omega_0, \alpha)$ and $s(\kappa/\omega_0, \alpha)$ are dimensionless functions. Note that the previous equation is an exact expression for the power spectrum.

From Eq. (6) we can identify the expression for the output signal:

$$S(\omega_0) = \left(\frac{D}{\kappa}\right)^\beta s(\kappa/\omega_0, \alpha, \beta) . \tag{7}$$

In this way, we have easily obtained the exact dependence of the output signal with the noise level. Notice that the output signal depends on the quantity we measure and, consequently, on the exponent $\beta$ [23]. In this respect, inspection of Eq. (7) reveals the presence of three qualitative different situations. For $\beta > 0$ the signal diverges when the noise level $D$ goes to infinity, whereas for $\beta < 0$ the signal diverges when $D$ goes to zero. In the limit case $\beta = 0$, the signal does not depend on the noise level.

The expression for the SNR follows from Eq. (6),

$$\mathrm{SNR} = \kappa \frac{s(\kappa/\omega_0, \alpha, \beta)}{q(\omega/\omega_0, \kappa/\omega_0, \alpha, \beta)} . \tag{8}$$

In contrast to the case for the signal, this result does not depend on the noise level thus indicating that the system is insensitive to the noise. No matter the noise intensity, the SNR has always the same value despite the fact that signal is a monotonic increasing or decreasing function of the noise level.

### 3.2 Scaling

To illustrate how the behavior of the system is modified upon changes on the noise level, in Fig. 1 we have depicted the time evolution of the output of the system for $v(x) = x$ and two different values of the noise level. In both

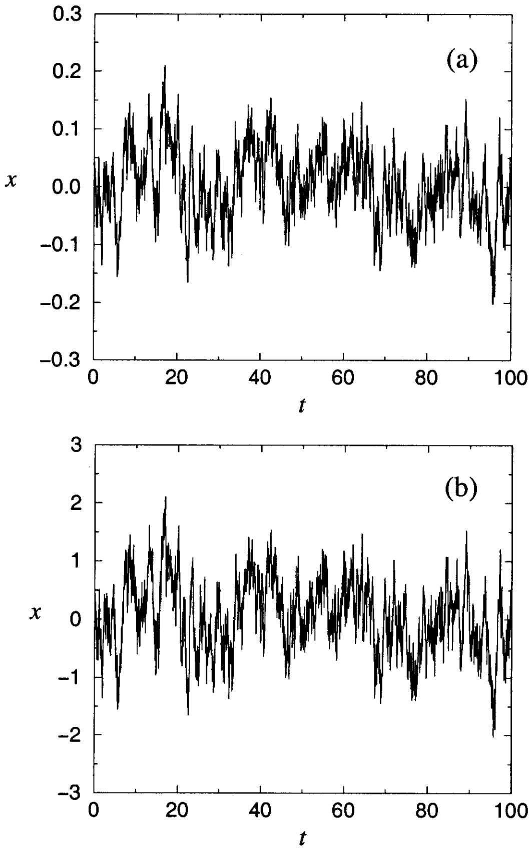


**Fig. 1.** Time evolution of $x$ ($\kappa = 1$, $\alpha = 0.5$, and $\omega_0/2\pi = 0.1$) for the noise levels (a) $D = 0.01$ and (b) $D = 1$.

cases we have used the same realization of the noise. In this figure one can see how the noise only affects the system by changing its characteristic scales. Therefore, the dependence of the quantities of interest with the noise level can also be obtained from the inspection of the invariance properties of the equations under scale transformations. Thus, when rescaling the noise level, $x$, and $t$ in the following way:

$$\begin{aligned} D &\to D' \equiv bD \ , \\ x &\to x' \equiv b^{\gamma_1} x \ , \\ t &\to t' \equiv b^{\gamma_2} t \ , \end{aligned} \tag{9}$$

Eq. (5) must be independent of $b$ for the adequate values of the exponents $\gamma_1$ and $\gamma_2$. Consequently, by substituting Eqs. (9) into Eq. (5) we obtain

$$b^{\gamma_1-\gamma_2}\frac{dx}{dt} = -\kappa[1+\alpha\sin(\omega_0 b^{\gamma_2}t)]b^{\gamma_1}x + b^{1/2-\gamma_2/2}\xi(t) \ , \qquad (10)$$

which is left unchanged when $\gamma_1 = 1/2$ and $\gamma_2 = 0$. Since the power spectrum, given in Eq. (2), transforms under the scaling (9) as

$$P'(\omega') = b^{2\beta\gamma_1+\gamma_2}Q(\omega) + b^{2\beta\gamma_1}S(\omega_0)\delta(\omega'-\omega_0') \ , \qquad (11)$$

then, the output signal and SNR scale with the noise level as $S(\omega_0) \sim D^{\beta}$ and SNR $\sim D^0$.

# 4 Nonlinear Systems

The results we have obtained so far are exact since no approximations has been made. In order to analyze more interesting situations, a similar scheme can be followed. However, some assumptions about the behavior of the quantities of interest should be made. In the following, we will discuss some of such situations.

## 4.1 Scale Invariant Potentials

We consider the class of systems described by the following Langevin equation:

$$\frac{dx}{dt} = -\kappa[1+\alpha\sin(\omega_0 t)]x^{1+2n} + \xi(t) \ , \qquad (12)$$

where $\kappa$ and $\alpha$ ($< 1$) are constants and $n$ is an integer number.

Let us assume that $v(x)$ does not introduce any characteristic time, as occurs when $v(x) = |x|^{\beta}$. Since the SNR has dimensions of the inverse of time, it follows from the simple scaling law

$$\mathrm{SNR} = f(\alpha, \omega_0\tau)\tau^{-1} \ , \qquad (13)$$

where characteristic time $\tau$ is given by $\tau^{-1} = D^{n/(1+n)}\kappa^{1/(1+n)}$ and $f(\alpha,\omega_0\tau)$ is a dimensionless function. We assume that for a given value of $\tau$ the limit of the SNR when $\omega_0$ goes to zero exists. As such, the following expression for small driving frequencies holds

$$\mathrm{SNR} = f(\alpha, 0)\tau^{-1}. \qquad (14)$$

The main characteristics of this model upon varying the exponent $n$ are as follows: if $n=0$, this system is equivalent to the one corresponding to Eq. (5), then one finds the result SNR $= f(\alpha,\omega_0\kappa^{-1})\kappa$, which does not depend on the noise level as shown in Sec. 3; for the case $n > 0$, the scaling of the SNR indicates that it increases when the noise level increases, achieving the behavior SNR $\sim D$ as $n$ goes to infinity. The only assumption introduced here concerns to the existence of the SNR in the limit case of frequency of the external signal going to zero. A particular and common situation illustrating this case corresponds to a quartic potential, obtained when $n = 1$, for which the SNR increases as $\sqrt{D}$.

### 4.2 Breaking of Scale Invariance

The class of systems discussed previously is characterized by the dynamics coming from scale invariant potentials. In this section we will show that scaling arguments can also be applied when that requirement is not fulfilled.

We will first analyze the case of low noise level. To be explicit, we will consider the Langevin equation

$$\frac{dx}{dt} = -\kappa[1 + \alpha \sin(\omega_0 t)] \left(x + ax^{1+2n}\right) + \xi(t) \ . \tag{15}$$

When the noise level is sufficiently small the nonlinear term can be neglected. Then, the SNR does not depend on $D$. In order to analyze how the SNR behaves upon varying $D$, we must take into account the nonlinear term. To this purpose, we will assume that the effects of the nonlinear contribution $ax^{1+2n}$ on a given quantity, the SNR in this case, can be replaced by the ones of an effective linear term $abD^n\kappa^{-n}x$, where $b \equiv b(\alpha, \omega_0\kappa^{-1})$ is a dimensionless positive function. The explicit form of $b$ may depend on the quantity we are considering but it is always a positive function. Consequently, the previous equation transforms into

$$\frac{dx}{dt} = -\kappa \left(1 + abD^n\kappa^{-n}\right) [1 + \alpha \sin(\omega_0 t)] \, x + \xi(t) \ , \tag{16}$$

which can be rewritten in the form

$$\frac{dx}{dt} = -\tilde{\kappa}[1 + \alpha \sin(\omega_0 t)]x + \xi(t) \ , \tag{17}$$

where $\tilde{\kappa} = (1 + abD^n\kappa^{-n})k$ is an effective parameter. The SNR is then

$$\mathrm{SNR} = f(\alpha, \omega_0\tilde{\kappa}^{-1})\tilde{\kappa} \ , \tag{18}$$

which for small frequencies ($\omega_0\kappa^{-1} \ll 1$) leads to

$$\mathrm{SNR} = f(\alpha, 0)\kappa \left(1 + abD^n\kappa^{-n}\right) \ , \tag{19}$$

Thereby, have found the behavior of the SNR as a function of the noise level by simple scaling arguments. It is interesting to point out that for low noise level, when $a$ is positive, the SNR is an increasing function of $D$, whereas when $a$ is negative the SNR decreases with $D$. Thus, if the SNR decreases for high noise level, as usually happens, the system may exhibit SR when its dynamics around the minimum of the potential can be approximated by Eq. (15) with $a$ positive.

The high noise level limit can also be treated by means of scaling arguments. To this purpose, we will assume that the dynamics of the system in this case may be approximated by

$$\frac{dx}{dt} = -\kappa[1 + \alpha \sin(\omega_0 t)]x^n - lx^m + \xi(t) \ , \tag{20}$$

where $l$, $n$, and $m$ are positive constants. If $n = m$, Eq. (20) is equivalent to Eq. (12), as follows by just changing the values of the parameters. If $n > m$, Eq. (20) also leads to Eq. (12), since, for high noise level, the term $lx^m$ can be neglected when $n > m$. Therefore, we consider the case in which $n < m$.

The previous equation can be rewritten in the following way:

$$\frac{dx}{dt} = -l\left[1 + \frac{\kappa}{lx^{m-n}} + \frac{\kappa\alpha}{lx^{m-n}}\sin(\omega_0 t)\right]x^m + \xi(t) \ . \tag{21}$$

Since for high noise level $x$ is large ($x \sim D^{1/(1+m)}$), the periodic force acts as a small perturbation to the dynamics of the system. Proceeding in a similar way as in the case for low noise level, one can introduce the effective parameter $\tilde{\alpha} \equiv bl^{-(n+1)/(1+m)}\kappa D^{-(m-n)/(1+m)}\alpha$, with $b$ now a dimensionless constant. Since the term $\kappa/(lx^{m-n})$ can be neglected, Eq. (21) reads

$$\frac{dx}{dt} = -l\,[1 + \tilde{\alpha}\sin(\omega_0 t)]\,x^m + \xi(t) \ , \tag{22}$$

which has the same form as Eq. (12). We then obtain

$$\text{SNR} = f(\tilde{\alpha}, \omega_0\tau)\tau^{-1} \ , \tag{23}$$

with $\tau = D^{-(m-1)/(1+m)}l^{-2/(1+m)}$. Since $\tilde{\alpha}$ and $\omega_0\tau$ are small for high $D$, then

$$\text{SNR} = \frac{1}{2}f''\tilde{\alpha}^2\tau^{-1} \ , \tag{24}$$

where $f''$ is the second derivative of $f(\tilde{\alpha}, \omega_0\tau)$, with respect to $\tilde{\alpha}$, evaluated at $\tilde{\alpha} = 0$ and $\omega_0\tau = 0$. Explicitly,

$$\text{SNR} = \frac{1}{2}f''\alpha^2\kappa^2 b^2 l^{-2n/(1+m)}D^{-1+2n/(1+m)} \ . \tag{25}$$

Note that when the forcing term does not depend on $x$, i. e. $n = 0$, the SNR always decreases as SNR $\sim D^{-1}$, irrespective of the value of $m$. From this expression one can elucidate some interesting situations. For instance if $m = 2n - 1$, the SNR tends to a constant value for high noise level, whereas if $m < 2n - 1$ it diverges. Hence, in this situation, for $m < 2n - 1$ the response of the system is always enhanced when the noise level is increased. Thus, noise is unable to destroy the coherent response of the system to the periodic input signal.

## 5 Discrete Symmetries

So far, we have considered the invariance of the system under a continuous scaling of the noise level. It is also possible that the system may remain invariant only for a discrete set of values of the noise level.

In order to study this aspect explicitly, we now consider the following Langevin dynamics:

$$\frac{dx}{dt} = -G(x,t)x + \xi(t) \ , \tag{26}$$

where $G(x,t)$ is a given function. Here, the input signal enters the system through $G(x,t)$, and we will assume it to be periodic in time with frequency $\omega_0/2\pi$. The output of the system is given by $v(x) = |x|^n$, with $n$ a positive constant.

The transformations

$$\begin{aligned} x &\to x' \equiv e^{\gamma} x \ , \\ D &\to D' \equiv e^{2\gamma} D \ , \end{aligned} \tag{27}$$

with $\gamma$ a constant, leave Eq. (26) and the SNR [Eq. (4)] invariant provided that

$$G(x,t) = G(xe^{\gamma},t) \ . \tag{28}$$

Consequently, for the class of systems in which Eq. (28) holds, the value of the SNR at D is the same as at $e^{2\gamma}D$. This fact occurs when $G(x,t) = g[\ln(x),t]$, where $g$ is a periodic function of its first argument, with periodicity $\gamma$ if $\gamma$ is the lower positive number satisfying Eq. (28). Therefore, the SNR is a periodic function of the logarithm of the noise level. Both signal and noise, however, are not invariant under this transformation, but change as

$$\begin{aligned} S' &= e^{2\gamma n} S \ , \\ Q' &= e^{2\gamma n} Q \ . \end{aligned} \tag{29}$$

In order to be more explicit, we consider the case in which

$$G(x,t) = \Theta_T[\ln(x^2)][1 + \alpha \cos(\omega_0 t)] \ , \tag{30}$$

where $\alpha$ and $\omega_0$ are constants, and $\Theta_T(s)$ is a square wave of period $T$, defined by

$$\Theta_T(s) = \begin{cases} \kappa_1 & \text{if } \sin(2\pi s/T) > 0 \ , \\ \kappa_2 & \text{if } \sin(2\pi s/T) \le 0 \ , \end{cases} \tag{31}$$

with $\kappa_1$ and $\kappa_2$ constants.

Since the SNR has dimensions of the inverse of time, its behavior is closely related to the characteristic temporal scales of the system. Thus, variations of the relaxation time manifest in the SNR. When $T$ is sufficiently large, for some values of the noise level the system may be approximated by

$$\frac{dx}{dt} = -\kappa_i[1 + \alpha \sin(\omega_0 t)]x + \xi(t) \ , \tag{32}$$

where $i = 1, 2$, depending on the noise level. In such a situation the SNR is given by

$$\mathrm{SNR} = f(\alpha, \omega_0 \kappa_i^{-1})\kappa_i \ , \tag{33}$$

with $f$ a dimensionless function. For a sufficiently low frequency, the SNR is proportional to $\kappa_i$ [$\mathrm{SNR} = f(\alpha, 0)\kappa_i$], i.e. proportional to the inverse of the relaxation time. Consequently, there are two set of values of $D$ for which the SNR differs in approximately $\kappa_1/\kappa_2$. In this case, multiple maxima in the SNR appear as a consequence of the form in which the relaxation time of the system changes with the noise level. The appearance of multiple maxima then implies the presence of stochastic multiresonance [19].

## 6 Conclusions

We have shown how scaling arguments can easily be applied to derive the main characteristics of a broad variety of periodically modulated noisy systems. Scaling of the noise level shows that the signal-to-noise ratio may increase when the noise level is increased, making the presence of stochastic resonance manifest. Thereby, under some circumstances, the constructive role played by noise is merely a consequence of the form in which the system scales upon variations of the noise level and may arise directly from dimensional analysis. The methodology we have outlined is not restricted only to the systems we have explicitly considered here, but can be applied also to a much broader variety of situations due to the generality of the assumptions involved in the scaling arguments.

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