

# Scaling concepts in periodically modulated noisy systems

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## Abstract

We show that scaling arguments are very useful to analyze the dynamics of periodically modulated noisy systems. Information about the behavior of the relevant quantities, such as the signal-to-noise ratio, upon variations of the noise level, can be obtained by analyzing the symmetries and invariances of the system. In this way, it is possible to predict diverse physical manifestations of the cooperative behavior between noise and input signal, as for instance stochastic resonance, spatiotemporal stochastic resonance, and stochastic multiresonance. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The application of scaling concepts in Statistical Physics has revealed their great usefulness to obtain relevant information about statical and dynamical quantities of the systems with little effort, without entering into complex calculations [1]. These techniques, which constitute simple manipulations allowing us to relate different quantities and exponents, have been employed in many classical branches of Statistical Physics, among others: critical phenomena [2,3], hydrodynamics [4,5], polymer physics [6] and non-linear physics [7,8]. More recently, scaling concepts have also been used in fields as, to mention just a few, growth phenomena [9], fractures [10], and economy [11,40]. Frequently, the existence of a scaling regime can be made evident by means of dimensional analysis or by considering the invariances of the equations involving the

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relevant quantities of the system. In this regard, the symmetries present in the underlying equations arise naturally in their physical realization.

Our purpose in this paper is to show how scaling arguments can also be applied to analyze the dynamics of a wide class of systems exhibiting, as a main characteristic, a dynamical evolution which is periodically modulated when the system is affected by noise. In particular, we will focus on the possibility that the response of the system may be enhanced by the addition of noise. This phenomenon, known as stochastic resonance (SR) [12–31], shows a constructive role of noise, sometimes considered counterintuitive, since the addition of noise is able to decrease the randomness of the system. The importance and interest of SR has been revealed by the great number of situations in which it has been predicted and/or observed; ranging from systems as simple as a single dipole [32] to systems exhibiting certain degree of complexity as neural tissues [33] or pattern-forming systems [27].

To illustrate the essentials of the phenomenon we will show an instance of how noise, under some circumstances, is able to increase the order of a system. In Fig. 1 we have displayed the time evolution corresponding to the relaxational dynamics of a single dipole [32] characterized by the angle  $\theta$  between the direction of the dipole moment and the direction of an external applied field.<sup>1</sup> Although apparently Fig. 1a looks deterministic, i.e. periodic, noise is responsible for this order, since the behavior becomes random when noise is sufficiently decreased, as shown in Fig. 1b. This figure, then, makes the fact that noise is not always a source of disorder evident.

Along this paper we present a methodology based upon general scaling arguments which enables one to predict the appearance of such an ordered behavior due to the presence of noise. To this end we have organized the paper in the following way. In Section 2 we present results for the simple case of an Ornstein–Uhlenbeck process. Section 3 deals with the application of scaling concepts to systems described by scale-invariant potentials. In particular, we show the appearance of stochastic resonance for monostable potentials (Section 3.1) and analyze how to deal with spatially extended systems (Section 3.2). In Section 4 we consider potentials which are not scale invariant. We discuss how the terms that break the scale invariance can be treated perturbatively. In this context, we also study the phenomenon of spatiotemporal stochastic resonance, occurring in Rayleigh–Bénard convection, and in general in pattern-forming systems, through the scaling of the Swift–Hohenberg equation. In Section 5 we show how the analysis of the discrete symmetries of a particular class of dynamical systems allows us to predict the appearance of multiple peaks in the signal-to-noise ratio, giving rise

<sup>1</sup> The relaxational dynamics of a single dipole in an external field may be described by the Langevin equation governing the dynamics of the unit vector  $\hat{R}$  along the direction of the dipole moment

$$\frac{d\hat{R}}{dt} = [\mathbf{h}(t) \times \hat{R} + \xi(t)] \times \hat{R}.$$

Here the total field  $\mathbf{h}(t) = k(1 + \alpha \sin(\omega_0 t))\hat{z}$ , with  $\hat{z}$  a unit vector, consists of a constant plus an oscillating field and is characterized by the parameters  $k$  and  $\alpha$ . The noise term  $\xi(t)$  is Gaussian and white, with zero mean and second moment  $\langle \xi_i(t) \xi_j(t + \tau) \rangle = D \delta_{ij} \delta(\tau)$ .

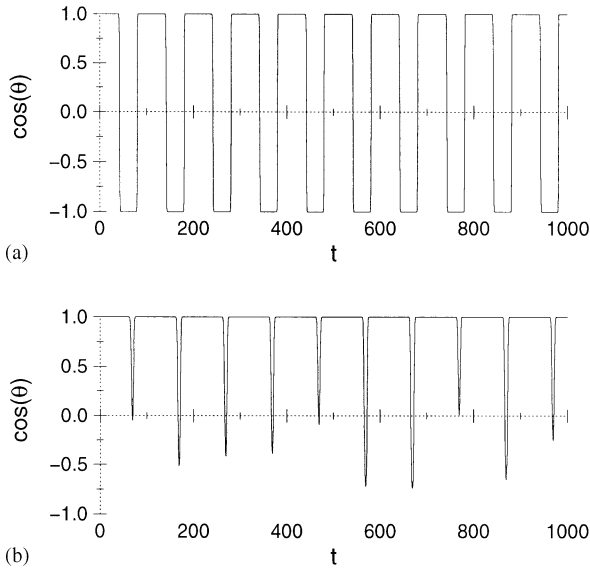


Fig. 1. Time evolution of  $\cos(\theta)$  (see footnote 1) for  $\alpha=3$ ,  $k=1$ ,  $\omega_0/2\pi=0.1$  and the noise levels: (a)  $D=10^{-10}$  and (b)  $D=10^{-44}$ .

to stochastic multiresonance. Finally, in Section 6 we summarize our main results and outline further applications of scaling concepts to this field.

## 2. The periodically modulated Ornstein–Uhlenbeck process

Let us first discuss in detail one of the most simplest cases which can be treated exactly by only using dimensional analysis. Its dynamics is described by an Ornstein–Uhlenbeck process, where the input signal modulates the strength of the force in the following way:

$$\frac{dx}{dt} = -\kappa[1 + \alpha \sin(\omega_0 t)]x + \zeta(t). \tag{1}$$

Here  $\kappa$ ,  $\alpha$ , and  $\omega_0$  are constants and  $\zeta(t)$  is Gaussian white noise with zero mean and correlation function  $\langle \zeta(t)\zeta(t + \tau) \rangle = 2D\delta(\tau)$ , defining the noise level  $D$ . In spite of its simplicity, the previous model encompasses many physical situations of interest since the motion around a minimum in a force field whose intensity varies periodically in time can be described by Eq. (1).

For the sake of generality we will consider that the system is described by means of a function  $v(t)$  of the dynamical variable  $x$ , i.e.,  $v(t) \equiv v[x(t)]$ . The effect of the force

may be analyzed by the averaged power spectrum

$$P(\omega) = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} dt \int_{-\infty}^{\infty} \langle v(t)v(t+\tau) \rangle e^{-i\omega\tau} d\tau. \quad (2)$$

To this end we will assume that it consists of a delta function centered at the frequency  $\omega_0$  plus a function  $Q(\omega)$  which is smooth in the neighborhood of  $\omega_0$  and is given by

$$P(\omega) = Q(\omega) + S(\omega_0)\delta(\omega - \omega_0). \quad (3)$$

We have expressed the power spectrum in this form since we are interested in the behavior of the system in the range of frequencies close to the frequency of the input signal.

The power spectrum, as expressed previously, explicitly shows the intensity of the deterministic component of the system or output signal,  $S(\omega_0)$ , and the stochastic component or output noise,  $Q(\omega)$ . The SNR, defined as the ratio between the signal and noise,

$$\text{SNR} = S(\omega_0)/Q(\omega_0), \quad (4)$$

then indicates the order present in the system.

Let us now assume the explicit form for the output of the system,  $v(x) = |x|^\beta$ , where  $\beta$  is a constant. Considerations based upon dimensional analysis enable us to rewrite the averaged power spectrum as

$$P(\omega, D, \kappa, \alpha, \omega_0, \beta) = \frac{1}{\kappa} \left( \frac{D}{\kappa} \right)^\beta q(\omega/\omega_0, \kappa/\omega_0, \alpha, \beta) + \left( \frac{D}{\kappa} \right)^\beta s(\kappa/\omega_0, \alpha, \beta) \delta \left( 1 - \frac{\omega}{\omega_0} \right), \quad (5)$$

where  $q(\omega/\omega_0, \kappa/\omega_0, \alpha)$  and  $s(\kappa/\omega_0, \alpha)$  are dimensionless functions. Note that the previous equation is an exact expression for the power spectrum since it does not involve any approximation.

From Eq. (5) we can identify the expression for the output signal

$$S(\omega_0) = \left( \frac{D}{\kappa} \right)^\beta s(\kappa/\omega_0, \alpha, \beta). \quad (6)$$

Thus, we have easily obtained the exact dependence of the output signal with the noise level. A remarkable aspect that must be emphasized is the fact that the output signal depends on the quantity we measure and consequently on the exponent  $\beta$  [34]. In this respect, inspection of Eq. (6) reveals the presence of three qualitative different situations. For  $\beta > 0$  the signal diverges when the noise level  $D$  goes to infinity, whereas for  $\beta < 0$  the signal diverges when  $D$  goes to zero. Even more interesting is the limit case  $\beta = 0$ , in which the signal does not depend on the noise level.

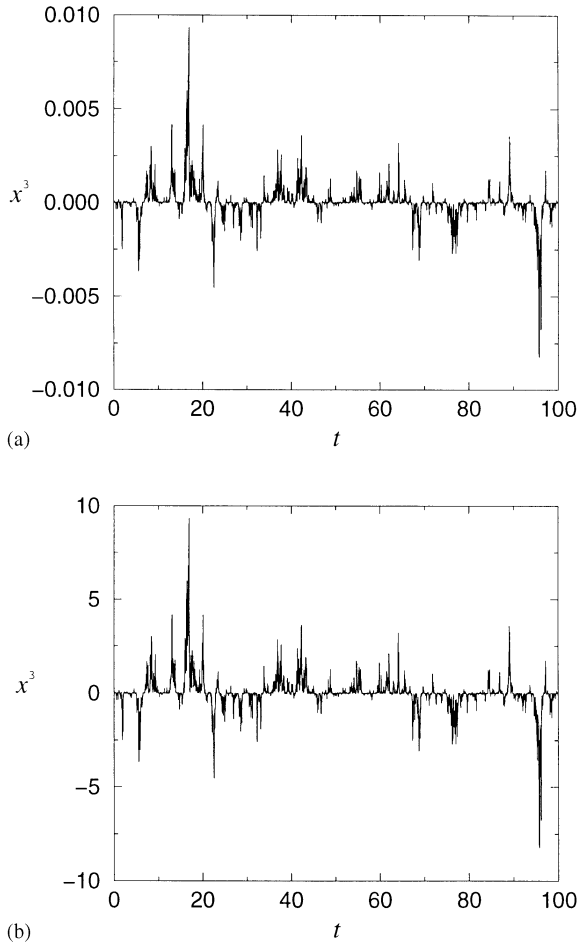


Fig. 2. Time evolution of  $x^3$  ( $\kappa=1$ ,  $\alpha=0.5$ , and  $\omega_0/2\pi=0.1$ ) for the noise levels (a)  $D=0.01$  and (b)  $D=1$ .

The expression for the SNR straightforwardly follows from Eq. (5),

$$\text{SNR} = \kappa \frac{s(\kappa/\omega_0, \alpha, \beta)}{q(\omega/\omega_0, \kappa/\omega_0, \alpha, \beta)}. \tag{7}$$

In contrast to the case for the signal, this result does not depend on the noise level thus indicating that the system is insensitive to the noise. No matter what the noise intensity, the SNR always has the same value despite the fact that the signal is a monotonic increasing or decreasing function of the noise level.

For further illustration of the previous results we have depicted in Fig. 2 the time evolution of the output of the system when  $v(x)=x^3$ , for two values of the noise level. In both cases we have used the same realization of the noise. In the figure we can see how the noise only affects the system by changing its characteristic scales.

Therefore, the dependence of the quantities of interest with the noise level can also be obtained from the inspection of the invariance properties of the equations under scale transformations. Thus when rescaling the noise level,  $x$ , and  $t$  in the following way,

$$\begin{aligned} D &\rightarrow D' \equiv bD, \\ x &\rightarrow x' \equiv b^{\gamma_1}x, \\ t &\rightarrow t' \equiv b^{\gamma_2}t. \end{aligned} \quad (8)$$

Eq. (1) must be independent of  $b$  for the adequate values of the exponents  $\gamma_1$  and  $\gamma_2$ . Consequently, by substituting Eqs. (8) into Eq. (1) we obtain

$$b^{\gamma_1 - \gamma_2} \frac{dx}{dt} = -\kappa[1 + \alpha \sin(\omega_0 b^{\gamma_2} t)] b^{\gamma_1} x + b^{1/2 - \gamma_2/2} \zeta(t), \quad (9)$$

which is left unchanged when  $\gamma_1 = 1/2$  and  $\gamma_2 = 0$ . Since the power spectrum, given in Eq. (2), transforms under the scaling (8) as

$$P'(\omega') = b^{2\beta\gamma_1 + \gamma_2} Q(\omega) + b^{2\beta\gamma_1} S(\omega_0) \delta(\omega' - \omega'_0), \quad (10)$$

then the output signal and SNR scale with the noise level as  $S(\omega_0) \sim D^\beta$  and  $\text{SNR} \sim D^0$ .

### 3. Scale-invariant potentials

In this section we will discuss two different cases which share as a common characteristic the scale invariance of the potential. In Section 3.1 we will deal with nonlinear systems by using the analysis introduced in the previous section, whereas in Section 3.2 we will extend those ideas to study spatial effects in a linear system.

#### 3.1. Nonlinear systems

The previous results we have obtained are exact since no approximations have been made. In order to analyze more interesting situations a similar scheme can be followed together with some assumptions about the behavior of the quantities of interest. Below we will discuss some of such situations.

The class of systems we will now consider are described by only one relevant degree of freedom whose dynamics is governed by the following Langevin equation

$$\frac{dx}{dt} = -\kappa[1 + \alpha \sin(\omega_0 t)] x^{1+2n} + \zeta(t), \quad (11)$$

where  $\kappa$  and  $\alpha$  ( $< 1$ ) are constants,  $n$  is an integer number and  $\zeta(t)$  is Gaussian white noise with zero mean and correlation function  $\langle \zeta(t) \zeta(t + \tau) \rangle = 2D\delta(\tau)$ .

From its definition [Eq. (4)] we see that the SNR has dimensions of inverse of time. The existence of a characteristic time  $\tau$  in our system will enable us to propose the

form of the SNR through the simple scaling law

$$\text{SNR} = f(\alpha, \omega_0 \tau) \tau^{-1}, \tag{12}$$

where  $\tau^{-1} = D^{n/(1+n)} \kappa^{1/(1+n)}$  and  $f(\alpha, \omega_0 \tau)$  is a dimensionless function, provided that  $v(x)$  does not introduce any other characteristic time, as occurs when  $v(x) = |x|^\beta$ . We will suppose that for a given value of  $\tau$  the limit of the SNR when  $\omega_0$  goes to zero exists. As such, the following expression for small driving frequencies holds:

$$\text{SNR} = f(\alpha, 0) \tau^{-1}. \tag{13}$$

Let us now discuss the main characteristics of this model upon varying the exponent  $n$ . If  $n = 0$ , this system is equivalent to the one corresponding to Eq. (1), then one finds the result  $\text{SNR} = f(\alpha, \omega_0 \kappa^{-1}) \kappa$ , which does not depend on the noise level as shown in Section 2. More interesting is the behavior obtained for the case  $n > 0$ . The scaling of the SNR indicates that it increases when increasing the noise level, achieving the behavior  $\text{SNR} \sim D$  as  $n$  goes to infinity. We note the fact that the only assumption introduced here concerns to the existence of the SNR in the limit case of frequency of the external signal going to zero. A particular and common situation illustrating this case corresponds to a quartic potential, obtained when  $n = 1$ , for which the SNR increases as  $\sqrt{D}$ .

### 3.2. Spatially extended systems

We will now proceed to extend the analysis of Section 2 to the case in which spatial effects are considered. To be explicit, we will treat the Swift–Hohenberg (SH) equation [35–37], which models Rayleigh–Bénard convection near the convective instability. Far from being specific to this problem, the main ideas can also be applied to other spatially extended systems in the vicinity of a symmetry-breaking bifurcation.

The stochastic SH equation in dimensionless spatial units is given by

$$\frac{\partial \psi}{\partial t} = h(t) \psi - q(1 + \nabla^2)^2 \psi - g \psi^3 + \xi(\mathbf{r}, t). \tag{14}$$

Here the control parameter  $h(t) = -\kappa + \alpha \sin(\omega_0 t)$ , with  $\kappa$ ,  $\alpha$ , and  $\omega_0$  positive constants, accounts for the presence of an external periodic forcing, due, for instance, to variations of the temperature difference between the plates of the convective cell [35,37,38,41]. The parameters  $g$  and  $q$  depend on the characteristics of the system and  $\xi(\mathbf{r}, t)$  is Gaussian white noise with zero mean and second moment  $\langle \xi(\mathbf{r}, t) \xi(\mathbf{r}', t') \rangle = 2D \delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$ .

It is interesting to point out the fact that the spatial effects enter the equation through both the term  $q(1 + \nabla^2)^2 \psi$  and the boundary conditions. Here, however, we will assume that the system is infinite, thus neglecting the boundary effects. Hence, when dealing with scaling arguments, the spatial dependence is taken into account only through the parameter  $q$ .

With the purpose of analyzing the time evolution of the system we will consider the convective heat flux, which in this model is given by

$$J(t) = c \int \psi(\mathbf{r}, t)^2 d\mathbf{r}, \quad (15)$$

where  $c$  is a constant depending on the physical characteristics of the system. This quantity is in fact the order parameter of the transition from the homogeneous state to the state where spatial structures develop. The existence of spatial ordering can be revealed by the time-averaged structure factor

$$F(k) = \langle \hat{\psi}_k \hat{\psi}_{-k} \rangle_t, \quad (16)$$

where  $\hat{\psi}_k$  is the spatial Fourier transform of the field  $\psi$  and  $\langle \rangle_t$  indicates time and noise average. The occurrence of a sharp peak in this magnitude makes the presence of an ordered spatial structure manifest.

If the noise level is sufficiently small the field  $\psi$  is small, except for a possible initial transient, and the nonlinear term of Eq. (14) can be neglected. Therefore, the potential describing the dynamics is scale invariant. In this situation, by proceeding as in the previous section it is easy to see how the characteristic quantities scale with noise. We note that, when the linearized equation is considered, any dimensionless parameter cannot depend on the noise level, because only  $D$  involves the dimensions of the field  $\psi$ . Thus  $\psi$  scales with the noise as  $\psi \sim \sqrt{D}$ . Since the SNR has dimensions of the inverse of time, then for  $J(t)$  it is given by

$$\text{SNR} = \omega_0 f_1(\gamma). \quad (17)$$

Here  $f_1$  is a dimensionless function which depends on the set of dimensionless parameters  $\kappa/\omega_0$ ,  $\alpha/\omega_0$  and  $q/\omega_0$ , denoted by  $\gamma$ . We then conclude that the SNR does not depend on the noise level. However, both the output noise and signal scale with the noise as  $D^2$ . This fact indicates that the output signal increases when noise increases. In this regard the structure factor also follows a scaling law since  $F(k) \sim D$ .

#### 4. Breaking of scale invariance

The class of systems we have discussed in the previous section are characterized by the fact that their dynamics exhibit scale-invariant potentials. In this section we will show that scaling arguments can also be applied when this requirement is not fulfilled.

##### 4.1. Low noise level

We will first analyze the case of low noise level. To be explicit, we will consider the Langevin equation

$$\frac{dx}{dt} = -\kappa[1 + \alpha \sin(\omega_0 t)](x + ax^{1+2n}) + \xi(t). \quad (18)$$



When the noise level is sufficiently small the nonlinear term can be neglected. Then, the SNR does not depend on  $D$ . In order to analyze how the SNR behaves upon varying  $D$ , we must take into account the nonlinear term. To this purpose we will assume that the effects of the nonlinear contribution  $\alpha x^{1+2n}$  on a given quantity, the SNR in this case, can be replaced by the ones of an effective linear term  $abD^n \kappa^{-n} x$ , where  $b \equiv b(\alpha, \omega_0 \kappa^{-1})$  is a dimensionless positive function. The explicit form of  $b$  may depend on the quantity we are considering but it is always a positive function. Consequently the previous equation transforms into

$$\frac{dx}{dt} = -\kappa(1 + abD^n \kappa^{-n})[1 + \alpha \sin(\omega_0 t)]x + \zeta(t), \tag{19}$$

which can be rewritten in the form

$$\frac{dx}{dt} = -\tilde{\kappa}[1 + \alpha \sin(\omega_0 t)]x + \zeta(t), \tag{20}$$

where  $\tilde{\kappa} = (1 + abD^n \kappa^{-n})\kappa$  is an effective parameter. The SNR is then

$$\text{SNR} = f(\alpha, \omega_0 \tilde{\kappa}^{-1})\tilde{\kappa}, \tag{21}$$

which for small frequencies ( $\omega_0 \kappa^{-1} \ll 1$ ) leads to

$$\text{SNR} = f(\alpha, 0)\kappa(1 + abD^n \kappa^{-n}). \tag{22}$$

In this regard, we have been able to predict the behavior of the SNR as a function of  $D$ , for low noise level, by means of simple scaling arguments. It is interesting to point out the fact that for low noise level, when  $a$  is positive, the SNR is an increasing function of  $D$ , whereas when  $a$  is negative the SNR decreases with  $D$ . Thus, if the SNR decreases for high noise level, as usually happens, the system may exhibit SR when its dynamics around the minimum of the potential can be approximated by Eq. (18), with  $a$  positive.

When considering perturbations to the linear system, the analysis we have performed can also be carried out in spatially extended systems. In order to illustrate this fact, we will now study the effect of the nonlinear term in the SH equation [Eq. (14)]. Since for low noise level the SNR does not depend on  $D$ , as shown previously, the lowest order correction in  $D$  to the constant value of the SNR comes from the nonlinear term. To elucidate its form, it is convenient to rewrite Eq. (14) in the following way:

$$\frac{\partial \psi}{\partial t} = [-(\kappa + g\psi^2) + \alpha \sin(\omega_0 t) - q(1 + \nabla^2)^2] \psi + \zeta(\mathbf{r}, t). \tag{23}$$

Due to the fact that for low noise level  $\psi^2 \sim D$ , in first approximation  $\kappa + g\psi^2$  can be interpreted as an effective parameter  $\tilde{\kappa} \equiv \kappa + g\omega_0^{-1} f_2(\gamma)D$ , with  $f_2$  being a positive dimensionless function. Consequently, Eq. (23) has the same form as the linearized counterpart for which the scaling law for the SNR [Eq. (17)] has been derived. We note that this case differs from the previous one in the form in which the nonlinearity

enters the equation, therefore we must follow an alternative scheme. By replacing  $\kappa$  by  $\tilde{\kappa}$  in Eq. (17) and expanding around  $\kappa$  we then obtain

$$\text{SNR} = \omega_0 f_1(\gamma) + \frac{\partial f_1}{\partial \kappa} g \omega_0^{-1} f_2(\gamma) D, \quad (24)$$

which includes the lowest order correction in  $D$  to the SNR due to the nonlinear term. An important consequence of this result is the fact that the knowledge of the dependence of the SNR on the parameter  $\kappa$ , for the linearized equation, enables us to predict the presence of SR when the nonlinear term is considered. If  $f_1$  is an increasing function of  $\kappa$  then  $\partial f_1 / \partial \kappa$  is positive and consequently the SNR is an increasing function of  $D$  for low noise level. Since for high  $D$  the SNR decreases, one then concludes that it exhibits a maximum thus indicating the presence of SR. In fact, it has been shown that the SH equation may exhibit SR since for a certain range of the values of the parameter  $\partial f_1 / \partial \kappa$  is positive [27]. We would like to emphasize the fact that the previous scaling argument is robust upon varying the nonlinear term. For example, if one replaces the term  $g\psi^3$  by  $g|\psi|\psi$  one obtains a SNR that increases as  $\sqrt{D}$  instead of as  $D$ .

#### 4.2. High noise level

As in the former case, the high noise level limit can also be treated by means of scaling arguments. To this purpose we will assume that the dynamics of the system in this case may be approximated by

$$\frac{dx}{dt} = -\kappa [1 + \alpha \sin(\omega_0 t)] x^n - l x^m + \zeta(t), \quad (25)$$

where  $l$ ,  $n$ , and  $m$  are positive constants. If  $n = m$ , Eq. (25) is equivalent to Eq. (11), as follows by changing only the values of the parameters. If  $n > m$ , Eq. (25) also leads to Eq. (11), since when  $n > m$ , for high noise level, the term  $l x^m$  can be neglected. Therefore we consider the case in which  $n < m$ .

The previous equation can be rewritten in the following way:

$$\frac{dx}{dt} = -l \left[ 1 + \frac{\kappa}{l x^{m-n}} + \frac{\kappa \alpha}{l x^{m-n}} \sin(\omega_0 t) \right] x^m + \zeta(t). \quad (26)$$

Since for high noise level  $x$  is large ( $x \sim D^{1/(1+m)}$ ) the periodic force acts as a small perturbation to the dynamics of the system. Thereby, proceeding in a similar way as in the case for low noise level, one can introduce the effective parameter  $\tilde{\alpha} \equiv b l^{-(n+1)/(1+m)} \kappa D^{-(m-n)/(1+m)} \alpha$ , with  $b$  now a dimensionless constant. Since the term  $\kappa / (l x^{m-n})$  can be neglected, Eq. (26) reads

$$\frac{dx}{dt} = -l [1 + \tilde{\alpha} \sin(\omega_0 t)] x^m + \zeta(t), \quad (27)$$

which has the same form as Eq. (11). We then obtain

$$\text{SNR} = f(\tilde{\alpha}, \omega_0 \tau) \tau^{-1}, \quad (28)$$

with  $\tau = D^{-(m-1)/(1+m)} l^{-2/(1+m)}$ . Since  $\tilde{\alpha}$  and  $\omega_0 \tau$  are small for high  $D$ , then

$$\text{SNR} = \frac{1}{2} f'' \tilde{\alpha}^2 \tau^{-1}, \tag{29}$$

where  $f''$  is the second derivative of  $f(\tilde{\alpha}, \omega_0 \tau)$ , with respect to  $\tilde{\alpha}$ , evaluated at  $\tilde{\alpha} = 0$  and  $\omega_0 \tau = 0$ . Explicitly,

$$\text{SNR} = \frac{1}{2} f'' \alpha^2 \kappa^2 b^2 l^{-2n/(1+m)} D^{-1+2n/(1+m)}. \tag{30}$$

Note that when the forcing term does not depend on  $x$ , i.e.  $n = 0$ , the SNR always decreases as  $\text{SNR} \sim D^{-1}$ , irrespective of the value of  $m$ . From this expression one can elucidate some interesting situations. For instance if  $m = 2n - 1$ , the SNR tends to a constant value for high noise level, whereas if  $m < 2n - 1$  it diverges. Hence, in this situation, for  $m < 2n - 1$  the response of the system is always enhanced when the noise level is increased. Thus, noise is unable to destroy the coherent response of the system to the periodic input signal.

### 5. Discrete symmetries

In the cases discussed previously, we have considered the invariance of the system under a continuous scaling of the noise level. Another interesting possibility is that the system may only be invariant for a discrete set of values of the noise level. This fact implies that the quantities which remain invariant under that transformation exhibit a periodic behavior with the logarithm of the noise level.

In order to treat this aspect explicitly we now consider the following Langevin dynamics:

$$\frac{dx}{dt} = -F(x, t)x + \sqrt{2D}\zeta(t), \tag{31}$$

where  $F(x, t)$  is a given function,  $\zeta(t)$  is Gaussian white noise with zero mean and second moment  $\langle \zeta(t)\zeta(t + \tau) \rangle = \delta(\tau)$ , and  $D$  is a constant, defining the noise level. Here the input signal enters the system through  $F(x, t)$ , and we will assume it to be periodic in time with frequency  $\omega_0/2\pi$ . The output of the system is given by  $v(x) = |x|^n$ , with  $n$  a positive constant.

The transformations

$$\begin{aligned} x &\rightarrow x' \equiv e^\gamma x, \\ D &\rightarrow D' \equiv e^{2\gamma} D, \end{aligned} \tag{32}$$

with  $\gamma$  a constant leaves Eq. (31) and the SNR [Eq. (4)] invariant, provided that

$$F(x, t) = F(xe^\gamma, t). \tag{33}$$

As a consequence, for the class of systems for which the previous requirement holds, for a certain value of  $\gamma$ , the SNR has the same value at  $D$  and at  $e^{2\gamma}D$ . This fact

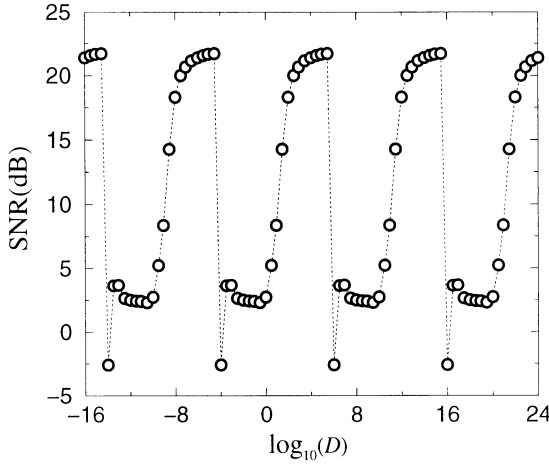


Fig. 3. SNR corresponding to  $F(x, t)$  given through Eq. (35) with parameter values  $\kappa_1 = 100$ ,  $\kappa_2 = 1$ ,  $T = 10$ ,  $\alpha = 0.5$  and  $\omega_0/2\pi = 1$ .

occurs when  $F(x, t) = q[\ln(x), t]$ , where  $q$  is a periodic function of its first argument, with periodicity  $\gamma$ , if  $\gamma$  is the lower positive number satisfying Eq. (33). Therefore, the SNR is a periodic function of the logarithm of the noise level. We note that both signal and noise are not invariant under this transformation, contrarily they change as

$$\begin{aligned} S' &= e^{2\gamma n} S, \\ Q' &= e^{2\gamma n} Q. \end{aligned} \quad (34)$$

In order to illustrate the previous results we have analyzed a representative explicit expression of  $F(x, t)$ . The corresponding Langevin equation has been numerically integrated by means of a standard second-order Runge–Kutta method for stochastic differential equations [39]. As the output of the system we have used  $v(x) = x^2$ . We will consider the case in which

$$F(x, t) = \Theta_T[\log_{10}(x^2)][1 + \alpha \cos(\omega_0 t)], \quad (35)$$

where  $\alpha$  and  $\omega_0$  are constants, and  $\Theta_T(s)$  is a square wave of period  $T$  defined by

$$\Theta_T(s) = \begin{cases} \kappa_1 & \text{if } \sin(2\pi s/T) > 0, \\ \kappa_2 & \text{if } \sin(2\pi s/T) \leq 0, \end{cases} \quad (36)$$

with  $\kappa_1$  and  $\kappa_2$  constants. In Fig. 3 we have plotted the SNR corresponding to the previous form of  $F(x, t)$ , for particular values of the parameters. This figure clearly manifests the periodicity of the SNR as a function of the noise level and the presence of multiple maxima at  $D = D_0 e^{mT}$ , with  $m$  being any integer number and  $D_0$  the noise level corresponding to the maximum with  $m = 0$ . The appearance of multiple maxima then implies the presence of stochastic multiresonance [28].

From this example one can infer the mechanism responsible for the appearance of this phenomenon. Due to the fact that the SNR has dimensions of the inverse of time,

its behavior is closely related to the characteristic temporal scales of the system. Thus variations of the relaxation time manifest in the SNR. In this example, when  $T$  is sufficiently large, for some values of the noise level the system may be approximated by

$$\frac{dx}{dt} = -\kappa_i [1 + \alpha \sin(\omega_0 t)]x + \sqrt{2D}\zeta(t), \quad (37)$$

where  $i = 1, 2$ , depending on the noise level. In such a situation the SNR is given by

$$\text{SNR} = f(\alpha, \omega_0 \kappa_i^{-1}) \kappa_i, \quad (38)$$

with  $f$  a dimensionless function. For a sufficiently low frequency, the SNR is proportional to  $\kappa_i$  [ $\text{SNR} = f(\alpha, 0)\kappa_i$ ], i.e. proportional to the inverse of the relaxation time. Consequently, there are two set of values of  $D$  for which the SNR differs in approximately  $10 \log_{10}(\kappa_1/\kappa_2)$  dB, as one can see in Fig. 3. We then conclude that multiple maxima in the SNR appears as a consequence of the form in which the relaxation time of the system changes with the noise level.

## 6. Concluding remarks

In this paper we have shown how scaling arguments can be used to derive the main characteristics of a broad variety of periodically modulated noisy systems, focusing on the phenomenon of stochastic resonance. Application of scaling arguments then leads to the knowledge of the dependence of the signal-to-noise ratio on the noise level. In particular, we have shown that the signal-to-noise ratio may increase when the noise level is increased. This fact makes the presence of stochastic resonance manifest, implying that the addition of noise enhances the response of the system to a periodic signal. In this context, the constructive role played by noise, sometimes considered counterintuitive, under some circumstances is merely a consequence of the form in which the system scales upon variations of the noise level and may arise directly from dimensional analysis.

Finally, it is worth pointing out the fact that in this paper we have presented explicit results concerning to representative situations of interest. However, the methodology we have outlined can be applied to a broad variety of different situations due to the general assumptions involved in the scaling arguments. Our results, then, reinforce the usefulness of scaling concepts in this field, adding new examples to the already wide variety of cases encountered in Statistical Physics.

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