An Introduction to the Kosko Subsethood FAM

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Organization of this talk

1. Introduction
2. Basic Notions of Mathematical Morphology
3. The KS-FAM: Motivation and Definition
4. Experimental Results Using Gray-Scale Images
5. Concluding Remarks

Fuzzy Associative Memories (FAMs)

FAMs are fuzzy neural networks that serve as associative memories.

Examples of FAM models:

1. Kosko's max-min and max-product FAMs;
2. Generalized FAMs of Chung and Lee;
3. Max-min FAM of Junbo et al.;
4. Liu's max-min FAM with threshold;
5. Fuzzy logical bidirectional associative memory of Bělohlávek;
6. Implicative fuzzy associative memories.

A new FAM model called Kosko Subsethood FAM (KS-FAM) is based on ideas of Mathematical Morphology (MM).

Mathematical Morphology

Mathematical Morphology (MM) is a theory for the processing and analysis of images using structuring elements (SEs).

Applications of MM include

1. noise removal;
2. skeletonizing;
3. edge detection;
4. automatic target recognition;
5. image segmentation;
6. image restauration.

Elementary Operations of MM

Erosion, dilation, anti-erosion, anti-dilation.

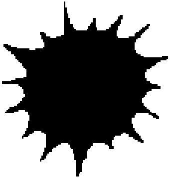
MM from two different points of view:

- MM in the intuitive or geometrical sense: based on inclusion e intersection measures;
- MM in the algebraic sense: defined in a complete lattice setting.

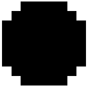
MM in the intuitive or geometrical sense

- Erosion: yields the (crisp or fuzzy) degree of inclusion of the translated SE at every pixel;
- Dilation: yields the (crisp or fuzzy) degree of intersection of the image with the (reflected and) translated SE at every pixel.

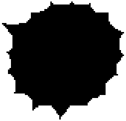
Binary Example



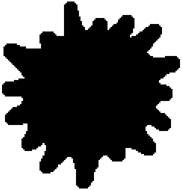
a) Original image



b) Structuring element



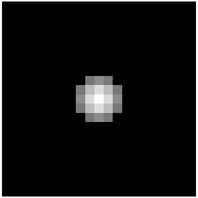
c) Erode image



d) Dilate image

Fuzzy/Grayscale Example

a) Original image (256x256)



b) Structuring element (21x21)

c) Erode image (256x256)

d) Dilate image (256x256)

Fuzzy Morphological Associative Memories

A FAM model is called a fuzzy morphological associative memory (FMAM) if its neurons perform elementary operations of MM.

Many well-known FAM models - including the ones mentioned above - belong to the class of FMAMs (in the algebraic sense).

The KS-FAM introduced in this talk can be viewed as an FMAM model in the intuitive sense.

The Complete Lattice Framework of MM

The algebraic framework of MM is given by complete lattices.

A complete lattice is a partially ordered set $\mathbb{L}$ such that every $Y \subseteq \mathbb{L}$ has an infimum, denoted by $\bigwedge Y$ and a supremum, denoted by $\bigvee Y$ in $\mathbb{L}$.

Examples of complete lattices include $\mathbb{R}_{\pm\infty} = \mathbb{R} \cup \{+\infty, -\infty\}$, $\mathbb{R}^n_{\pm\infty} = (\mathbb{R}_{\pm\infty})^n$, $[0,1]$ and $[0,1]^{\mathbf{X}}$, the class of fuzzy sets over the universe $\mathbf{X}$.

From now on, the symbols $\mathbb{L}$ and $\mathbb{M}$ denote complete lattices.

Basic Operators of MM

Erosion

An operator $\varepsilon : \mathbb{L} \to \mathbb{M}$ represents an (algebraic) erosion if

$$\varepsilon\left(\bigwedge Y\right) = \bigwedge_{y \in Y} \varepsilon(y), \quad \forall Y \subseteq \mathbb{L}.$$

Dilation

An operator $\delta : \mathbb{L} \to \mathbb{M}$ represents a (algebraic) dilation if

$$\delta\left(\bigvee Y\right) = \bigvee_{y \in Y} \delta(y), \quad \forall Y \subseteq \mathbb{L}.$$

Negations

A negation on $\mathbb{L}$ is an involutive bijection $\nu_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}$ that reverses the partial ordering.

Examples of Negation

1. For $\mathbb{L} = [0, 1]$:
$$\nu_{\mathbb{L}}(x) = \bar{x} = 1 - x.$$
2. For $\mathbb{L} = \mathbb{R}_{\pm\infty} = \mathbb{R} \cup \{-\infty, +\infty\}$:
$$\nu_{\mathbb{L}}(x) = x^* = \begin{cases} -x, & \text{if} \quad x \in \mathbb{R}\,, \\ +\infty, & \text{if} \quad x = -\infty\,, \\ -\infty, & \text{if} \quad x = \infty\,. \end{cases}$$
3. For $\mathbb{L} = \mathbb{R}^{m \times n}_{\pm\infty} = (\mathbb{R}_{\pm\infty})^{m \times n}$:
$$(\nu_{\mathbb{L}}(X))_{ij} = (X^*)_{ij} = (x_{ji})^* \; \forall i = 1, \ldots, n, \; j = 1, \ldots, m.$$

Max Product and Min Product

Let $A \in \mathbb{R}^{m\times n}$ e $B \in \mathbb{R}^{n\times p}_{\pm\infty}$. We have:

- $C = A \boxed{\vee} B$ - max product of A and B: $c_{ij} = \bigvee_{k=1}^{n} (a_{ik} + b_{kj})$.
- $D = A \boxed{\wedge} B$ - min product of A and B: $d_{ij} = \bigwedge_{k=1}^{n} (a_{ik} + b_{kj})$.

We have an (algebraic) erosion

$$\begin{aligned} \varepsilon_A : \mathbb{R}^n_{\pm\infty} &\longrightarrow \mathbb{R}^m_{\pm\infty} \\ x &\longmapsto A \boxed{\wedge} x \end{aligned}$$

We have an (algebraic) dilation

$$\begin{aligned} \delta_A : \mathbb{R}^n_{\pm\infty} &\longrightarrow \mathbb{R}^m_{\pm\infty} \\ x &\longmapsto A \boxed{\vee} x \end{aligned}$$

Morfological Associative Memories (MAMs)

Original Models

Let $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in \mathbb{R}^{n \times k}$ and $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in \mathbb{R}^{m \times k}$. Define the synaptic weight matrices W_{XY} and M_{XY} as follows:

1. $W_{XY} = Y \boxed{\wedge} X^* = \bigwedge_{\xi=1}^{k} \mathbf{y}^\xi \boxed{\wedge} (\mathbf{x}^\xi)^*$
2. $M_{XY} = Y \boxed{\vee} X^* = \bigvee_{\xi=1}^{k} \mathbf{y}^\xi \boxed{\vee} (\mathbf{x}^\xi)^*$

Upon presentation of $\mathbf{x} \in \mathbb{R}^n_{\pm\infty}$ the the MAM W_{XY} and the dual MAM M_{XY} yield the following outputs:

1. $\mathbf{y} = W_{XY} \boxed{\vee} \mathbf{x}$;
2. $\mathbf{z} = M_{XY} \boxed{\wedge} \mathbf{x}$.

Properties of Autoassociative MAMs

Advantages:

1. Infinite absolute storage capacity;
2. One-step convergence if empoyed with feedback.
3. Tolerance of W_{XX} w.r.t. erosive noise;
4. Tolerance of M_{XX} w.r.t. dilative noise;

Disadvantages:

1. Both W_{XX} and M_{XX} are not able to deal with arbitrary noise;
2. Large number of spurious memories.

Kosko's *Subsethood* Measure

Let $\mathbf{X}$ be a finite set and let $\mathbf{a}, \mathbf{b} : \mathbf{X} \rightarrow [0, 1]$ be fuzzy sets. Suppose that $\sum_{x \in \mathbf{X}} \mathbf{a}(x) > 0$:

$$S(\mathbf{a}, \mathbf{b}) = 1 - \frac{\sum_{x \in \mathbf{X}} 0 \vee (\mathbf{a}(x) - \mathbf{b}(x))}{\sum_{x \in \mathbf{X}} \mathbf{a}(x)} = \frac{\sum_{x \in \mathbf{X}} \mathbf{a}(x) \wedge \mathbf{b}(x)}{\sum_{x \in \mathbf{X}} \mathbf{a}(x)}$$

Kosko's subsethood measures the degree of inclusion of **a** in **b**.

Fuzzy Min Product

Let $M \in [0,1]^{m \times n}$ and $\mathbf{x} \in [0,1]^n$. Let $\mathbf{m}_i$ denote the i-th row of M. The fuzzy min product $\mathbf{y} = M \,\tilde{\boxtimes}\, \mathbf{x}$ is given by

$$y_i = S(\bar{\mathbf{m}}_i, \mathbf{x}), \; i = 1, \ldots, m\,.$$

Let $X \in \{0,1\}^{n \times k}$ and $\mathbf{x} \in [0,1]^n$. Consider the binary model $\tilde{\boxtimes} - \mathcal{T}$MAM given by

$$\text{input } \mathbf{x} \rightarrow M_{XX} \,\tilde{\boxtimes}\, \mathbf{x} \rightarrow \text{Defuzzification } \mathcal{T} \rightarrow \text{output } \mathbf{y}$$

Main advantage:

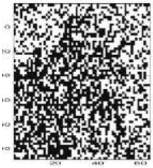
Inexistence of spurious memories.

Main disadvantage:

Requires an additional defuzzification phase ($\mathcal{T}$).

Example of a Fuzzy Min Product $M_{XX} \tilde{\triangle} \mathbf{x}$

Figure: Original Patterns Stored in M_{XX}.



(a) Input $\mathbf{x}$

(b) $M_{XX} \tilde{\triangle} \mathbf{x}$

FAM based on Kosko's Subsethood Measure

Let $\mathbf{h} : [0,1]^p \to \{0,1\}^p$ with $\mathbf{h}(\mathbf{x}) = (h(x_1), \ldots, h(x_p))^t$ be such that

$$h(x_i) = \begin{cases} 1 & \text{if } x_i \geq \bigvee_{j=1}^{p} x_j \\ 0 & \text{else} \end{cases}, \quad \text{for } i = 1, \ldots, p.$$

Definition of KS-FAM:

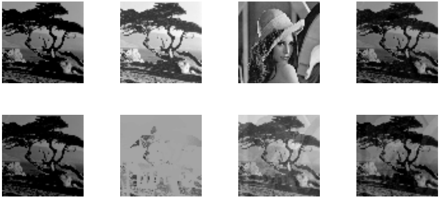
- Let $X \in [0,1]^{n \times k}$ and $Y \in [0,1]^{m \times k}$;
- Choose $Z = [\mathbf{z}^1, \ldots, \mathbf{z}^k] \in \{0,1\}^{p \times k}$ such that $\bigvee_{\xi=1}^{k} \mathbf{z}^\xi = \mathbf{1}$, $\mathbf{z}^\xi \not\leq \mathbf{z}^\gamma$ and $\mathbf{z}^\xi \wedge \mathbf{z}^\gamma = \mathbf{0}$ for $\gamma \neq \xi$;
- For an input pattern $\mathbf{x} \in [0,1]^n$ the output pattern $\mathbf{y} \in [0,1]^m$ is given by:

$$\mathbf{y} = W_{ZY} \boxed{\triangledown} \, \mathbf{w}\,, \quad \text{where } \mathbf{w} = \mathbf{h}(M_{XZ} \, \tilde{\boxed{\vartriangle}} \, \mathbf{x})$$

Models Used in the Experiments

- KS-FAM with $Z = \mathbf{I}_4$ (4×4 identity matrix);
- Hamming Net;
- MAM W_{XX};
- MAM $W_{XX} + \nu$;
- Kosko's Max-Min FAM;
- KAM with Gaussian Kernel Function;
- OLAM;

Variations in Brightness



Variations in Orientation

NRMSEs- Variations in Brightness and Orientation

	Tree	Lena	Church	Cameraman
KS-FAM	**0**	**0**	**0**	**0**
Hamming Net	0.6347	0.8414	**0**	**0**
W_{XX}	0.4771	0.7354	1.6015	0.9509
$W_{XX} + \nu$	0.6032	0.4615	0.6168	0.4765
Kosko's FAM	0.4302	0.8937	1.1586	0.7300
KAM	0.1945	0.1499	0.0566	0.0784
OLAM	0.4986	0.6810	0.2892	0.1937

Gaussian Noise

NRMSEs - Noisy Patterns

	Gaussian Noise ($\sigma^2 = 0.03$)
KS-FAM	**0**
Hamming	**0**
W_{XX}	0.9005
$W_{XX} + \nu$	0.2770
Kosko's FAM	0.8185
KAM	0.0137
OLAM	0.0365

Concluding Remarks

- We presented the Kosko subsethood FAM (KS-FAM) on the basis of ideas from MM.
- The KS-FAM outperformed other AM models in preliminary experiments on gray-scale image recognition.
- Experiments indicate potential utility for applications in pattern recognition.

Thank you!