Formal averaging of periodic and quasi-periodic vector fields

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Example (Fermi-Pasta-Ulam problem)

Hamiltonian system with Hamiltonian function

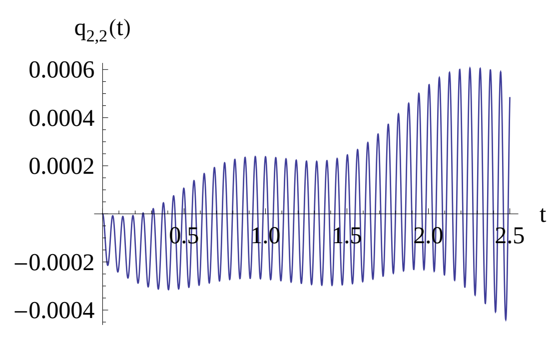
$$\begin{aligned} H(p,\bar{p},q,\bar{q}) &= \frac{1}{2}(p^T p + \bar{p}^T \bar{p}) + \frac{1}{2\epsilon^2} q^T q + U(q,\bar{q}), \\ U(q,\bar{q}) &= \frac{1}{4}\left((\bar{q}_1 - q_1)^4 + (\bar{q}_m + q_m)^4\right) \\ &\quad + \frac{1}{4}\sum_{j=1}^{m-1} (\bar{q}_{j+1} - q_{j+1} - \bar{q}_j - q_j)^4. \end{aligned}$$

We consider $m = 3$, $\epsilon = 1/100$, and initial values

$$\bar{p}(0) = p(0) = \bar{q}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ q(0) = \begin{pmatrix} \epsilon \\ 0 \\ 0 \end{pmatrix}.$$

Example (Fermi-Pasta-Ulam problem (cont.))

Solution for the component $q_2(t)$,



and for $n = 0, 1, 2, 3, \ldots,$

$$q_2(2\pi\epsilon n) \qquad .$$

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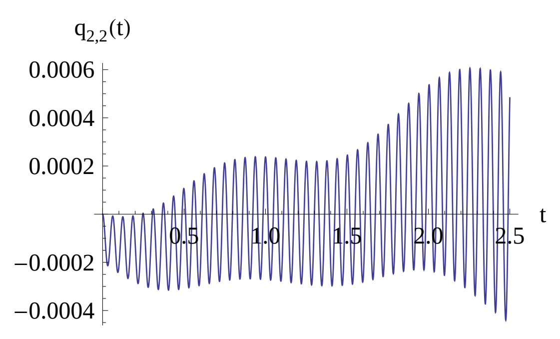

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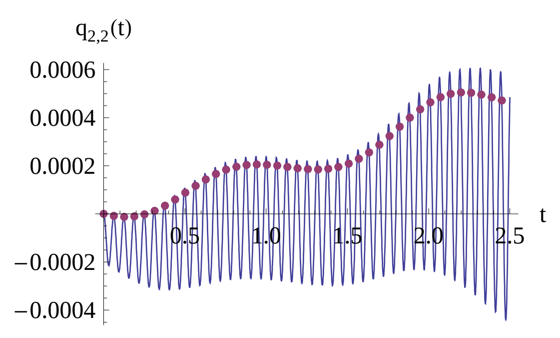

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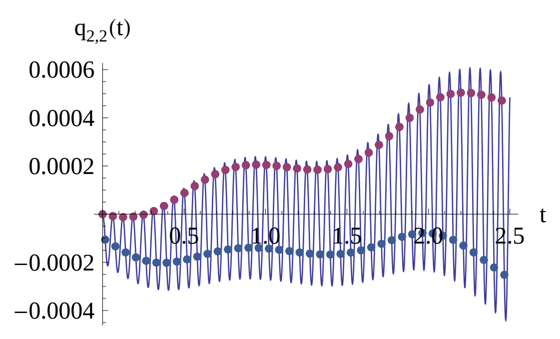

and for $n = 0, 1, 2, 3, \ldots,$

$$q_2(2\pi\epsilon n), \quad \text{and} \quad q_2(\frac{\pi\epsilon}{2} + 2\pi\epsilon n).$$

Consider a Hamiltonian system

$$\frac{d}{dt}y = J^{-1}\nabla H(y;\epsilon), \quad H(y;\epsilon) := \epsilon^{-1}H^0(y) + R(y),$$

Let $\varphi_\tau : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be such that $\varphi_{t/\epsilon}$ is the t-flow of that system. Assume that the τ-flow φ^0_τ of H^0 is (2π)-periodic. ($H^0(y)$ and $R(y)$ may depend on ϵ).
It can be shown that $\varphi_{2\pi}$ is a near-to-identity map,

Backward error analysis

There exists $\mathcal{H}(Y;\epsilon) = \mathcal{H}_0(Y) + \epsilon\mathcal{H}_1(Y) + \epsilon^2\mathcal{H}_2(y) + \cdots,$

$$\frac{d}{dt}Y = J^{-1}\nabla\mathcal{H}(Y;\epsilon),$$

such that, $Y(2\pi\epsilon n) = y(2\pi\epsilon n)$ if $Y(0) = y(0) = y_0$.

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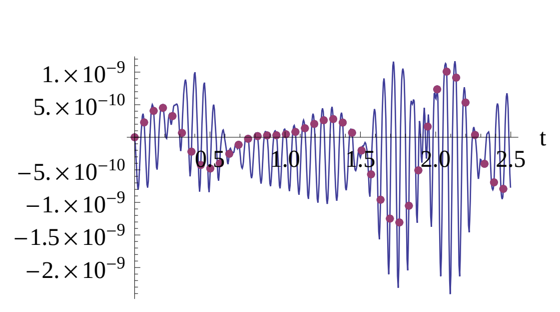
Example (Fermi-Pasta-Ulam problem)

We consider

$$\tilde{\mathcal{H}}(Y,\epsilon) := \mathcal{H}_0(Y) + \epsilon^2\mathcal{H}_2(Y) + \epsilon^4\mathcal{H}_4(Y) = \mathcal{H}(Y;\epsilon) + \mathcal{O}(\epsilon^6),$$

and plot the variation $\tilde{\mathcal{H}}(y(t);\epsilon) - \tilde{\mathcal{H}}(y(0);\epsilon)$



Smooth invariant

Under the previous assumtions for

$$\frac{d}{dt}y = J^{-1}\nabla H(y;\epsilon), \quad H(y;\epsilon) := \epsilon^{-1}H^0(y) + R(y),$$

consider $\mathcal{H}(Y;\epsilon) = \mathcal{H}_0(Y) + \epsilon\mathcal{H}_1(Y) + \epsilon^2\mathcal{H}_2(y) + \cdots$ as before, then $\mathcal{H}(y;\epsilon)$ is a first integral of the original system.

Indeed, for $t_n = 2\pi\epsilon n$, $n = 1, 2, \ldots$

$$H(Y(t_n);\epsilon) = H(y(t_n);\epsilon) = \mathrm{Const}$$

and by a interpolating argument, $H(Y(t);\epsilon) = \mathrm{Const}$, and thus

$$\{H, \mathcal{H}\} \equiv 0.$$

Numerical integration of HOS with ϵ-independent time-steps

Integrate the smooth system

$$\frac{d}{dt}Y = J^{-1}\nabla H(Y;\epsilon), \quad Y(0) = y_0$$

instead of the highly oscillatory one. Different options

- Symbolic-numeric algorithms using explicit knowledge of $\mathcal{H}$
- Purely numerical schemes that try to approximate $Y(t)$ by using H as input (HMSM, ...).

Motivated by that, we aim at

- Obtaining formulae for $\mathcal{H}(Y;\epsilon)$ and its solutions $Y(t)$
- Such formulae should be as explicit as possible and
- Of universal character

Time-dependent $(2\pi\epsilon)$-periodic change of variables

Let φ^0_τ be the τ-flow of H^0, consider

$$y = \varphi^0_{t/\epsilon}(\hat{y}).$$

Notice that for $t_n = 2\pi\epsilon n$, it reduces to the identity map.

The system with $H(y;\epsilon) = \epsilon^{-1} H^0(y) + R(y)$ is transformed into a non-autonomous Hamiltonian system with $\hat{H}(\hat{y}, t/\epsilon)$, where

$$\hat{H}(\hat{y}, \tau) := R(\varphi^0_\tau(\hat{y})) = \sum_{k\in\mathbb{Z}} e^{ik\tau} \hat{H}_k(\hat{y}).$$

($\varphi^0_\tau(\hat{y})$ is (2π)-periodic in τ). Clearly,

$$\frac{d}{dt}\hat{y} \;\; = \;\; \sum_{k\in\mathbb{Z}} e^{ikt/\epsilon} f_k(\hat{y}).$$

From now on, we consider systems of that general form. The $f_k(\hat{y})$ may depend on ϵ, but not reflected in the notation.

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Standard high order averaging [Sanders, Verhulst, Murdock 2007]

Under suitable assumptions on the HOS

$$\frac{d}{dt}y = \sum_{k\in\mathbb{Z}} e^{ikt/\epsilon} f_k(y).$$

there exists a formal $(2\pi\epsilon)$-periodic change of variables $y = K(Y, t/\epsilon)$ that transforms the original HOS into the (*averaged*) autonomous equations

$$\frac{d}{dt}Y = F(Y;\epsilon) := F_0(Y) + \epsilon F_1(Y) + \epsilon^2 F_2(Y) + \cdots$$

The change of variables $y = K(Y, \tau)$ is not unique:

- Stroboscopic averaging: $K(Y, 0) = Y$, which implies $Y(2\pi\epsilon n) = y(2\pi\epsilon n)$ for all $n \in \mathbb{Z}$.
- $\int_0^{2\pi} K(Y, \tau)\, d\tau = Y$,

Stroboscopic averaging of HOS autonomous Hamiltonian systems

Back to the HOS Hamiltonian system $H(y) = \epsilon^{-1}H^0(y) + R(y)$, the $(2\pi\epsilon)$-periodic change of variables

$$y = \varphi^0_{t/\epsilon}(K(Y, t/\epsilon))$$

transforms the original HOS autonomous Hamiltonian H into a smooth autonomous Hamiltonian $\mathcal{H}$.

Comments:

- The smooth Hamiltonian $\mathcal{H}$ is exactly the same as that derived from backward error analysis applied to $\varphi_{2\pi\epsilon}$.
- The (stroboscopically) averaged system of general HOS

$$\frac{d}{dt}y = \sum_{k\in\mathbb{Z}} e^{ikt/\epsilon} f_k(y).$$

can also be derived from backward error analysis applied to the near-identity map ψ such that $\psi(y(0)) = y(2\pi\epsilon)$ for any solution $y(t)$.

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We next focus on obtaining, using standard techniques in numerical analysis (B-series, ...), universal formulae for the averaging of non-autonomous periodic HOS.

B-series expansion of solution of the HOS

For the solutions of $\dot{y} = \sum_k e^{ikt/\epsilon} f_k(y)$,

$$y(t) = y(0) + \sum_{u \in \mathcal{T}} \frac{\alpha_u(t)}{\sigma_u} \mathcal{F}_u(y(0)),$$

$\mathcal{T}$ is the set of rooted trees labelled by $k \in \mathbb{Z}$, and for each $u \in \mathcal{T}$,

- the coefficients $\alpha_u(\tau)$ are linear combinations of $t^j e^{ikt/\epsilon}$,
- The elementary differentials $\mathcal{F}_u : \mathbb{R}^d \to \mathbb{R}^d$ are smooth,

($\sigma_u \in \mathbb{Z}$ is a normalization factor.)

Elementary coefficients

$$\alpha_u(t) = \int_0^t e^{ikt'/\epsilon} \alpha_{u_1}(t') \cdots \alpha_{u_m}(t')\, dt', \quad u = [u_1 \cdots u_m]_k.$$

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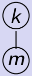
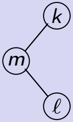
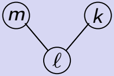
Examples for rooted trees with less than 4 vertices

u	$\mathcal{F}_u(y)$	$\alpha_u(t)$
	$f_k(y)$	$\int_0^t e^{\frac{ikt_1}{\epsilon}}\, dt_1$
	$f'_m(y) f_k(y)$	$\int_0^t \int_0^{t_2} e^{\frac{i(kt_1+mt_2)}{\epsilon}}\, dt_1\, dt_2$
	$f'_\ell(y) f'_m(y) f_k(y)$	$\int_0^t \int_0^{t_3} \int_0^{t_2} e^{\frac{i(kt_1+mt_2+\ell t_3)}{\epsilon}}\, dt_1\, dt_2\, dt_3$
	$f''_\ell(y)(f_m(y), f_k(y))$	$\int_0^t \int_0^{t_2} e^{\frac{i(kt_1+mt_1+\ell t_2)}{\epsilon}}\, dt_1\, dt_2$

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$$\alpha_u(t,\tau) = \sum_{k\in\mathbb{Z}} \alpha_u^k(t) e^{ik\tau},$$

where each $\alpha_u^k(t)$ is a polynomial in t.

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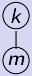
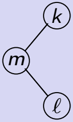
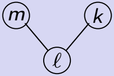
	$f'_m(y) f_k(y)$	$\int_0^t \int_0^{t_2} e^{\frac{i(kt_1+mt_2)}{\epsilon}}\, dt_1\, dt_2$
	$f'_\ell(y) f'_m(y) f_k(y)$	$\int_0^t \int_0^{t_3} \int_0^{t_2} e^{\frac{i(kt_1+mt_2+\ell t_3)}{\epsilon}}\, dt_1\, dt_2\, dt_3$
	$f''_\ell(y)(f_m(y), f_k(y))$	$\int_0^t \int_0^{t_2} e^{\frac{i(kt_1+mt_1+\ell t_2)}{\epsilon}}\, dt_1\, dt_2$

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where each $\alpha_u^k(t)$ is a polynomial in t.

Relation with modulated Fourier series

For any solution of $\dot{y} = \sum_k e^{ikt/\epsilon} f_k(y)$,

$$
\begin{aligned}
y(t) &= y(0) + \sum_{u\in\mathcal{T}} \frac{\alpha_u(t, t/\epsilon)}{\sigma_u} \mathcal{F}_u(y(0)), \\
&= y(0) + \sum_{u\in\mathcal{T}} \sum_{k\in\mathbb{Z}} \frac{\alpha_u^k(t)}{\sigma_u} e^{ikt/\epsilon} \mathcal{F}_u(y(0)), \\
&= y(0) + \sum_{k\in\mathbb{Z}} \left(\sum_{u\in\mathcal{T}} \frac{\alpha_u^k(t)}{\sigma_u} \mathcal{F}_u(y(0)) \right) e^{ikt/\epsilon} \\
&= y^0(t) + \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{ikt/\epsilon} z^k(t).
\end{aligned}
$$

Modulated Fourier series versus B-series approach

In the case of exact solution of the Hamiltonian system with $H(y; \epsilon) = \epsilon^{-1} H^0(y) + R(y)$:

- Modulated Fourier approach [Hairer& Lubich 2000]: Expand exact solution $y(t)$ in the form

$$y(t) = y^0(t) + \sum_{k \in \mathbb{Z} \setminus \{0\}} e^{ikt/\epsilon} z^k(t)$$

  by Taylor-expanding $R(y)$ around $y^0(t)$.

- Our approach: Expand $\hat{y}(t) = \varphi_{-t/\epsilon}(y(t))$ as a B-series in terms of the Fourier modes $\hat{H}_k$ of

$$\hat{H}(\hat{y}, \tau) := R(\varphi^0_\tau(\hat{y})) = \sum_{k \in \mathbb{Z}} e^{ikt/\epsilon} \hat{H}_k(\hat{y}).$$

Averaging with B-series

There exist $\bar{\beta}_u, \bar{\alpha}_u(t), \kappa_u(\tau)$, $u \in \mathcal{T}$, ($\bar{\alpha}_u(t)$ polynomial, $\kappa_u(\tau)$ (2π)-periodic) such that for any solution $y(t)$ of the HOS

$$y(t) = K(Y(t), t/\epsilon), \quad \frac{d}{dt} Y(t) = F(y(t)),$$

where

$$\begin{aligned} F(Y) &= \sum_{u \in \mathcal{T}} \frac{\bar{\beta}_u}{\sigma_u} \mathcal{F}_u(Y), \\ Y(t) &= Y(0) + \sum_{u \in \mathcal{T}} \frac{\bar{\alpha}_u(t)}{\sigma_u} \mathcal{F}_u(Y(0)), \\ K(Y, \tau) &= Y + \sum_{u \in \mathcal{T}} \frac{\kappa_u(\tau)}{\sigma_u} \mathcal{F}_u(Y), \end{aligned}$$

Not unique (but uniquely determined for prescribed $\kappa_u(0)$, $u \in \mathcal{T}$).

That is, $\alpha(t,\tau) = \bar{\alpha}(t)\kappa(\tau)$ with $\frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t)\bar{\beta}$.

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Stroboscopic averaging: $K(Y,0) = Y$

So that $Y(2\pi n\epsilon) = y(2\pi n\epsilon)$. Thus, $\kappa(0) = \mathbb{1}$ which implies $\bar{\alpha}(0) = \mathbb{1}$, and

$$\bar{\alpha}_u(t) = \alpha_u(t,0), \quad \kappa_u(\tau) = \alpha_u(0,\tau), \quad \bar{\beta}_u = \frac{d}{dt}\bar{\alpha}_u(t)\Big|_{t=0}.$$

'Classical' averaging: $\frac{1}{2\pi}\int_0^{2\pi} K(Y,\tau)d\tau = Y$

$Y(2\pi n\epsilon) \neq y(2\pi n\epsilon)$. We have $\frac{1}{2\pi}\int_0^{2\pi}\kappa(\tau)d\tau = \mathbb{1}$, and thus

$$\bar{\alpha}_u(t) = 1/(2\pi)\int_0^{2\pi}\alpha_u(t,\tau)d\tau, \quad \kappa_u(\tau) = (\bar{\alpha}(0)^{-1}\alpha(0,\tau))_u,$$

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Given $\omega \in \mathbb{R}^d$ non-resonant and $\theta(t) = \theta_0 + \omega t$,

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Expansion of solutions of quasi-periodic vector fields

$$y(t) = y_0 + \sum_{w \in \mathcal{W}} \alpha_w(t, \theta(t))\, f_w(y_0),$$

where $\mathcal{W}$ is the set of 'words' $w = k_1 \cdots k_r$ on the alphabet $\mathbb{Z}^d$,

$$\begin{aligned} \alpha_{k_1 \cdots k_r}(t) &= \int_0^t \int_0^{t_r} \cdots \int_0^{t_2} e^{i\omega\cdot(k_1 t_1 + \cdots + k_r t_r)}\, dt_1 \cdots dt_r, \\ f_{k_1 \cdots k_r}(y) &= \frac{\partial}{\partial y} f_{k_2 \cdots k_r}(y) f_{k_1}(y), \end{aligned}$$

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Indeed, consider for each $k \in \mathbb{Z}^d$, the Lie operator associated to f_k,

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$$\begin{aligned} \frac{d}{dt}\Phi_t[g](y_0) &= \frac{d}{dt} g(y(t)) = \frac{\partial}{\partial y} g(y(t)) f(y(t), \theta(t)) \\ &= \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\theta(t)} E_k[g](y(t)) = \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\theta(t)} \Phi_t E_k[g](y_0). \end{aligned}$$

Linear non-autonomous quasi-periodic differential equation

$$\frac{d}{dt}\Phi_t = \Phi_t \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\theta(t)} E_k, \quad \Phi_0 = I.$$

$$\Phi_t = I + \sum_{k_1 \cdots k_r \in \mathcal{W}} \alpha_{k_1 \cdots k_r}(t)\, E_{k_1} \cdots E_{k_r} \quad \text{(by Picard iteration)}$$

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Each $\alpha_w(t)$ can be written as a linear combination of terms of the form $t^j e^{i(k\cdot\theta(t))}$. Actually, $\alpha_w(t) = \alpha_w(t, \theta(t))$, where

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Recursive formulae for $\alpha_w(t,\theta)$

If $r \in \mathbb{Z}^+$, $k \in \mathbb{Z}^d - \{0\}$, $l \in \mathbb{Z}^d$, and $w \in \mathcal{W} \cup \{\emptyset\}$,

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\alpha_k(t,\theta) &= \frac{i}{k\cdot\omega}(e^{i(k\cdot\theta_0)} - e^{i(k\cdot\theta)}),\\
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Formal high order averaging: Find a factorization

$$I + \sum_{w \in \mathcal{W}} \alpha_w(t,\theta) E_w \;=\; \Big(I + \sum_{w \in \mathcal{W}} \bar{\alpha}_w(t)\, E_w\Big)\Big(I + \sum_{w \in \mathcal{W}} \kappa_w(\theta)\, E_w\Big)$$

$$\frac{d}{dt}\Big(I + \sum_{w \in \mathcal{W}} \bar{\alpha}_w(t)\, E_w\Big) \;=\; \Big(I + \sum_{w \in \mathcal{W}} \bar{\alpha}_w(t)\, E_w\Big)\Big(\sum_{w \in \mathcal{W}} \bar{\beta}_w\, E_w\Big).$$

Or more compactly,

$$\alpha(t,\theta) = \bar{\alpha}(t)\kappa(\theta), \quad \frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t)\bar{\beta}.$$

Pseudo-stroboscopic averaging: $\kappa(\theta_0) = \mathbb{1}$

$$\bar{\alpha}_w(t) = \alpha_w(t,\theta_0), \quad \kappa_w(\theta) = \alpha_w(0,\theta), \quad \bar{\beta}_w = \frac{d}{dt}\bar{\alpha}_w(t)\Big|_{t=0}.$$

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Pseudo-stroboscopic averaging for quasi-periodic problems

We get $y(t) = K(Y(t), \theta(t))$, $(\theta(t) = \theta_0 + \omega t,)$ where

$$\frac{d}{dt} Y = F(Y), \quad Y(0) = y_0,$$

with

$$\begin{aligned} K(Y, \theta) &= Y + \sum_{w \in \mathcal{W}} \kappa_w(\theta)\, f_w(Y), \\ F(Y) &= \sum_{w \in \mathcal{W}} \bar{\beta}_w\, f_w(Y), \\ Y(t) &= y_0 + \sum_{w \in \mathcal{W}} \bar{\alpha}_w(t)\, f_w(y_0). \end{aligned}$$

Recursion for coefficients of averaged equation

$$\begin{aligned}
\bar\beta_k &= 0, \quad \bar\beta_0 = 1, \quad \bar\beta_{0^{r+1}} = 0, \\
\bar\beta_{0^r k} &= \frac{i}{k\cdot\omega}(\bar\beta_{0^{r-1}k} - \bar\beta_{0^r} e^{i(k\cdot\theta_0)}), \\
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But does the factorization

$$\alpha(t,\theta) = \bar{\alpha}(t)\kappa(\theta), \quad \frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t)\bar{\beta}, \quad \kappa(\theta_0) = \mathbb{1}$$

of $\alpha(t,\theta)$ exist? First observe that

$$(\tfrac{\partial}{\partial t} + \omega \cdot \nabla_\theta)\alpha(t,\theta) = \alpha(t,\theta)\beta(\theta), \quad \alpha(0,0) = \mathbb{1},$$

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Sketch of proof: We first show that $\exists \kappa(\theta), \bar{\beta}$ such that

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B-series expansion of exact solution of quasi-periodic ODEs

$$y(t) = B(\alpha(t), y_0), \quad \text{where} \quad \alpha_u(t) = \alpha_u(t, \theta(t)) = \sum_{w \in \mathcal{W}_u} \alpha_w(t, \theta(t)).$$

Averaging formulae in B-series form

$$Y(t) = B(\bar{\alpha}(t), y_0), \quad \dot{Y} = B(\bar{\beta}, Y), \quad Y(t) = B(\kappa(\theta(t)), Y(t)),$$

- Quasi-stroboscopic averaging:

$$\bar{\alpha}(t) = \alpha(t, \theta_0), \quad \bar{\beta} = \frac{d}{dt}\bar{\alpha}(t)\Big|_{t=0}, \quad \kappa(\theta) = \alpha(0, \theta).$$

- Classical averaging:

$$\bar{\alpha}(t) = \int_{\mathcal{T}^d} \alpha(t, \theta)\, d\theta, \quad \bar{\beta} = \frac{d}{dt}\bar{\alpha}(t)\Big|_{t=0},$$

$$\kappa(\theta) = \bar{\alpha}(0)^{-1}\alpha(0, \theta).$$

Future work

- Prove rigorous results (bound independently the coefficients and the smooth y-dependent maps),
- Study approximate preservarion of formal invariants of HOS,
- Analyze existing numerical methods for highly oscillatory systems
- Desing and analyze new numerical methods. In particular, heterogenous multiscale methods based on

  microintegration+
  numerical evaluation of averaged equations+
  macrointegration

Algebraic/geometric properties of exact solution

For $u_1, u_2, u_3 \in \mathcal{T}$

- $\alpha_{u_1 \circ u_2} + \alpha_{u_2 \circ u_1} = \alpha_{u_1}\alpha_{u_2}$, $\rightarrow$ symplectic for each fixed t, τ.
- In addition

$$\alpha_{u_1 \circ u_2 u_3} + \alpha_{u_2 \circ u_1 u_3} + \alpha_{u_3 \circ u_1 u_2} = \alpha_{u_1}\alpha_{u_2}\alpha_{u_3}.$$

  Hence, for each fixed t, τ, it represents the 1-flow of a vector field in the Lie algebra generated by the f_k.

- Classical averaging does not preserve the properties of $\alpha(t, \tau)$:

$$1/(2\pi) \int_0^{2\pi} (\alpha_{u_1 \circ u_2}(t,\tau) + \alpha_{u_2 \circ u_1}(t,\tau))\, d\tau \neq 1/(2\pi)^2 \left(\int_0^{2\pi} \alpha_{u_1}(t,\tau) d\tau\right) \left(\int_0^{2\pi} \alpha_{u_2}(t,\tau) d\tau\right)$$

- Stroboscopic averaging: Properties inherited by $\bar{\alpha}(t) = \alpha(t, 0)$ (and $\bar{\beta}$). Stroboscopic averaging is well defined for periodic vector fields on manifolds.

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$$\neq 1/(2\pi)^2 \left(\int_0^{2\pi} \alpha_{u_1}(t, \tau) d\tau \right) \left(\int_0^{2\pi} \alpha_{u_2}(t, \tau) d\tau \right)$$

- Stroboscopic averaging: Properties inherited by $\bar{\alpha}(t) = \alpha(t, 0)$ (and $\bar{\beta}$). Stroboscopic averaging is well defined for periodic vector fields on manifolds.

Algebraic/geometric properties of exact solution

For $u_1, u_2, u_3 \in \mathcal{T}$

- $\alpha_{u_1 \circ u_2} + \alpha_{u_2 \circ u_1} = \alpha_{u_1} \alpha_{u_2}$, $\rightarrow$ symplectic for each fixed t, τ.
- In addition

$$\alpha_{u_1 \circ u_2 u_3} + \alpha_{u_2 \circ u_1 u_3} + \alpha_{u_3 \circ u_1 u_2} = \alpha_{u_1} \alpha_{u_2} \alpha_{u_3}.$$

  Hence, for each fixed t, τ, it represents the 1-flow of a vector field in the Lie algebra generated by the f_k.

- Classical averaging does not preserve the properties of $\alpha(t, \tau)$:

$$1/(2\pi) \int_0^{2\pi} (\alpha_{u_1 \circ u_2}(t, \tau) + \alpha_{u_2 \circ u_1}(t, \tau))\, d\tau$$
$$\neq 1/(2\pi)^2 \left(\int_0^{2\pi} \alpha_{u_1}(t, \tau) d\tau \right) \left(\int_0^{2\pi} \alpha_{u_2}(t, \tau) d\tau \right)$$

- Stroboscopic averaging: Properties inherited by $\bar{\alpha}(t) = \alpha(t, 0)$ (and $\bar{\beta}$). Stroboscopic averaging is well defined for periodic vector fields on manifolds.