High order averaging of a family of near-integrable systems

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A class of higly oscillatory ODE systems

We consider systems of the form

$$\frac{d}{dt}x = \frac{1}{\epsilon} A x + r(x),$$

where A is diagonalizable with imaginary eigenvalues, $r : \mathbb{R}^n \to \mathbb{R}^n$ is polynomial, and $\epsilon > 0$ is a small parameter.

This implies that there exist a non-resonant vector of frequencies $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$, $d \leq n$, and matrices $A_1, \ldots, A_d$, such that

$$A = \sum_{j=1}^{d} \omega_j A_j,$$

$[A_j, A_l] = 0$, and each e^{tA_j} is (2π)-periodic in t. Hence, the solutions of the unperturbed system ($\epsilon = 0$) are quasi-periodic in t,

$$\exp(\tfrac{t}{\epsilon} A) = \exp(t \tfrac{\omega_1}{\epsilon} A_1) \cdots \exp(t \tfrac{\omega_d}{\epsilon} A_d).$$

In particular, we are interested in the

Hamiltonian case

$$\frac{d}{dt}x = J^{-1}\nabla H(x), \quad H(x) = \sum_{j=1}^{d} \frac{\omega_j}{\epsilon} I_j(x) + R(x),$$

where the quadratic polynomials $I_j(x) = \frac{1}{2}x^T(JA)x$ are

- in involution,
- first integrals of the unperturbed Hamiltonian system, and
- the t-flow of each $I_j(x)$ is (2π)-periodic in t.

Each $I_j(x)$ is approximatelly preserved along solutions of the perturbed Hamiltonian system. It is known that this is related to the existence of formal first integrals $\tilde{I}_j(x) = I_j(x) + \mathcal{O}(\epsilon)$ of the perturbed system.

Let $\sum_{\mathbf{k}\in\mathbb{Z}^d} R_{\mathbf{k}}(x)e^{i(\mathbf{k}\cdot\omega)\tau}$ be the Fourier series of $R(e^{\tau\sum\omega_j A_j}x)$.

Explicit formulae for formal first integrals

$$
\begin{aligned}
\tilde{I}_j(x) &= I_j(x) + \epsilon \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{\mathbf{k}\cdot e_j}{\mathbf{k}\cdot\omega} R_{\mathbf{k}}(x) \\
&\quad + \sum_{r\geq 2}\epsilon^r \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_r\in Z^d} \frac{\beta^{[j]}_{\mathbf{k}_1\cdots\mathbf{k}_r}}{r}\{\{\cdots\{\{R_{\mathbf{k}_1},R_{\mathbf{k}_2}\},R_{\mathbf{k}_3}\}\cdots\},R_{\mathbf{k}_r}\}(x),
\end{aligned}
$$

Where, given $r\in\mathbb{Z}^+$, $\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}$, and $\mathbf{l}_1,\ldots,\mathbf{l}_s\in\mathbb{Z}^d$,

$$
\begin{aligned}
\beta^{[j]}_{\mathbf{k}} &= \tfrac{\mathbf{k}\cdot e_j}{\mathbf{k}\cdot\omega}, \quad \beta^{[j]}_{\mathbf{0}^r} = 0, \\
\beta^{[j]}_{\mathbf{0}^r\mathbf{k}} &= \tfrac{i}{\mathbf{k}\cdot\omega}\,\beta^{[j]}_{\mathbf{0}^{r-1}\mathbf{k}}, \\
\beta^{[j]}_{\mathbf{k}\mathbf{l}_1\cdots\mathbf{l}_s} &= \tfrac{i}{\mathbf{k}\cdot\omega}(\beta^{[j]}_{\mathbf{l}_1\cdots\mathbf{l}_s} - \beta^{[j]}_{(\mathbf{k}+\mathbf{l}_1)\mathbf{l}_2\cdots\mathbf{l}_s}), \\
\beta^{[j]}_{\mathbf{0}^r\mathbf{k}\mathbf{l}_1\cdots\mathbf{l}_s} &= \tfrac{i}{\mathbf{k}\cdot\omega}(\beta^{[j]}_{\mathbf{0}^{r-1}\mathbf{k}\mathbf{l}_1\cdots\mathbf{l}_s} - \beta^{[j]}_{\mathbf{0}^r(\mathbf{k}+\mathbf{l}_1)\mathbf{l}_2\cdots\mathbf{l}_s}),
\end{aligned}
$$

Main ingredients to obtain our explicit formal results

- Combinatorial-algebraic tools (B-series) developped for the numerical analysis of non-stiff ODEs. We rewite the solutions of the original highly oscillatory system as a (generalized) B-series, with highly oscillatory coefficients, which correspond to the solution of an ODE on the coefficient group (the Butcher group).
- Expressing the solution $x(t)$ of the original system as $x(t) = z(t/\epsilon, \tau\omega/\epsilon)$, where $z(\tau, \theta)$ is an appropriately chosen solution of the transport equation

$$\begin{aligned} \partial_\tau z + \omega \cdot \partial_\theta z &= A\,z + \epsilon\, r(z), \\ z(0, \mathbf{0}) &= x(0). \end{aligned}$$

Related ongoing work

- Use of B-series approach to analyse numerical resonances of existing integration schemes.
- Rigorous estimates for formal first integrals and high order averaging on the ODE case, by estimating the corresponding B-series coefficients.
- New numerical integrators based on the associated transport equation.
- Application to PDEs: Variants of Schrödinger, vawe equation, ...

We go back to the general ODE case of the form

$$\frac{d}{d\tau}x = A\,x + \epsilon\, r(x), \quad x(0) = x_0 \in \mathbb{R}^n,$$

with real analytic $r(x)$, and A diagonalizable with imaginary eigenvalues. Guided with our work using B-series, we seek to express formaly its solution as

$$x(\tau) = e^{\tau A}x_0 + \epsilon\, Z_1(\tau, \tau\omega, x_0) + \epsilon^2\, Z_2(\tau, \tau\omega, x_0) + \cdots$$

where for each $j = 1, 2, \ldots,$

$$\begin{array}{rcl} Z_j : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ (\tau, \theta, x) & \mapsto & Z_j(\tau, \theta, x) \end{array}$$

is polynomial in τ and real analytic in (θ, x).

Theorem

Consider the transport equation associated to the original ODE,

$$\partial_\tau z + \omega \cdot \partial_\theta z = A\, z + \epsilon\, r(z), \quad z(0, \mathbf{0}) = x_0.$$

If $r(x)$ *is real analytic and* $\omega \in \mathbb{R}^d$ *satisfies a Diophantine condition*

$$\forall \mathbf{k} \in \mathbb{Z}^d / \{\mathbf{0}\}, \quad |\mathbf{k} \cdot \omega| \geq c |\mathbf{k}|^{-\nu},$$

then, there is a unique formal solution

$$z(\tau, \theta) = e^{\sum \theta_j A_j} x_0 + \epsilon\, Z_1(\tau, \theta, x_0) + \epsilon^2\, Z_2(\tau, \theta, x_0) + \cdots$$

such that for each $j = 1, 2, \ldots,$

$$\begin{array}{rcl} Z_j : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ (\tau, \theta, x) & \mapsto & Z_j(\tau, \theta, x) \end{array}$$

is polynomial in τ *and real-analytic in* (θ, x)*.*

Consider the family of formal maps $\Phi_{\tau,\theta} : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Phi_{\tau,\theta}(x) \quad := \quad e^{\sum \theta_j A_j} x + \epsilon\, Z_1(\tau,\theta,x) + \epsilon^2\, Z_2(\tau,\theta,x) + \cdots$$

so that $z(\tau,\theta) = \Phi_{\tau,\theta}(x_0)$ is the polynomial in time solution of the transport equation, and $x(\tau) = \Phi_{\tau,\omega\tau}(x_0)$ is the solution of the original ODE.

The uniqueness result on the polynomial in time solution of the transport equation can be used to prove:

Theorem

$\forall (\tau,\theta), (\tau',\theta') \in \mathbb{R} \times \mathbb{T}^d$

$$\Phi_{\tau',\theta'} \circ \Phi_{\tau,\theta} = \Phi_{\tau+\tau',\theta+\theta'}.$$

We thus have that

$$\Phi_{\tau,\theta} = \Phi_{0,\theta_1 e_1} \circ \cdots \circ \Phi_{0,\theta_d e_d} \circ \Phi_{\tau,\mathbf{0}},$$

where e_j is the jth unit vector in $\mathbb{R}^d$.

- Each $\Phi_\tau^{[j]} := \Phi_{0,\tau e_j}$ is (2π)-periodic in τ, and (since $\Phi_\tau^{[j]} \circ \Phi_{\tau'}^{[j]} = \Phi_{\tau+\tau'}^{[j]}$,) it is the τ-flow of an autonomous ODE.
- $\Phi_{\tau,\mathbf{0}}$ is the τ-flow of the averaged ODE,

$$\frac{d}{d\tau} X = \epsilon \tilde{r}(X), \quad \text{where} \quad \epsilon \tilde{r}(X) := \frac{d}{d\tau} \Phi_{\tau,\mathbf{0}}(X)\Big|_{\tau=0}.$$

The time-dependent change of variables

$$x = \Phi_{\tau\omega_1}^{[1]} \circ \cdots \circ \Phi_{\tau\omega_d}^{[d]}(X)$$

transforms the original ODE into the averaged one $(X(0) = x(0))$.

Theorem

If in addition to previous assumptions, there exist skew-symmetric $S \in \mathbb{R}^{n\times n}$ and real analytic $R : \mathbb{R}^n \to \mathbb{R}$ such that SA is symmetric and $r(x) = S\nabla R(x)$, so that

$$\frac{d}{d\tau}x = S\nabla H(x), \quad H(x) = \sum_{j=1}^{d} \omega_j I_j(x) + \epsilon R(x),$$

with $I_j(x) = \frac{1}{2}x^T(SA)x$, then

- $(\partial_x \Phi_{\tau,\theta}(x))^T \, S \, (\partial_x \Phi_{\tau,\theta}(x)) \equiv S.$
- *each $\Phi_\tau^{[j]}$ is the τ-flow of a Hamiltonian $\tilde{I}_j(x) = I_j(x) + \mathcal{O}(\epsilon)$,*
- *$H(x), \tilde{I}_1(x), \ldots, \tilde{I}_d(x)$ are in involution.*

Example (A Fermi-Pasta-Ulam type problem (from HLW))

A Hamiltonian system with $n = 10$ ($x = (p, q) \in \mathbb{R}^{10}$),

$$\begin{aligned} H(p,q) &= \sum_{j=2}^{5} \left(\frac{1}{2} p_j^2 + \frac{\lambda_j^2}{2} q_j^2 \right) + \epsilon \left(\frac{1}{2} p_1^2 + \frac{1}{2} q_1^2 + \epsilon\, U(q) \right), \\ U(q) &= \frac{1}{8} q_1^2 q_2^2 + \epsilon \left(\frac{\sqrt{70}}{20} + q_2 + q_3 + \frac{5}{2} q_4 + q_5 \right)^4, \end{aligned}$$

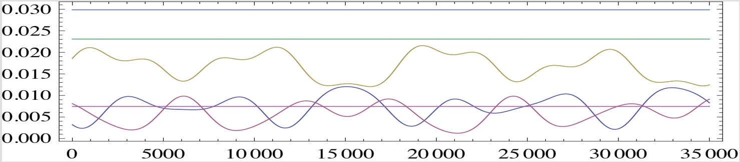
where $\lambda_2 = \lambda_3 = 1$, $\lambda_4 = 2$, $\lambda_5 = \sqrt{2}$. The quadratic part is the sum of the energies

$$J_1 = \frac{\epsilon}{2}(p_1^2 + q_1^2), \qquad J_j = \frac{1}{2} p_j^2 + \frac{\lambda_j^2}{2} q_j^2, \quad j = 2, \ldots, 5.$$

of five uncoupled harmonic oscillators with frequencies $\lambda = \epsilon$, $\lambda_2 = \lambda_3 = 1$, $\lambda_4 = 2$, $\lambda_5 = \sqrt{2}$.

Example (cont.)

As in [HLW], we take $\epsilon = 1/70$, $p(0) = (-0.2, 0.6\delta, 0.7\delta, -0.9\delta, 0.8\delta)^T$, $q(0) = (1, 0.3\delta, 0.8\delta, -1.1\delta, 0.7\delta)^T$ ($\delta = \sqrt{\epsilon}$), and plot (versus τ) the evolution of J_i, $i = 1, \ldots, 5$, and $J_2 + J_3 + J_4$, for $0 \le \tau \le 500/\epsilon^2$.



We get formal first integrals $\tilde{I}_1(x)$ and $\tilde{I}_2(x)$ corresponding to $\omega = (1, \sqrt{2})$, $I_1(x) = J_2(x) + J_3(x) + J_4(x)$, $I_2(x) = \frac{\sqrt{2}}{2} J_5(x)$, $R(x) = \frac{1}{2}(p_1^2 + q_1^2) + \epsilon U(q)$. Moreover, the averaged Hamiltonian is of the form

$$\tilde{R}(x) = \frac{1}{2}(p_1^2 + q_1^2) + \epsilon K(x),$$

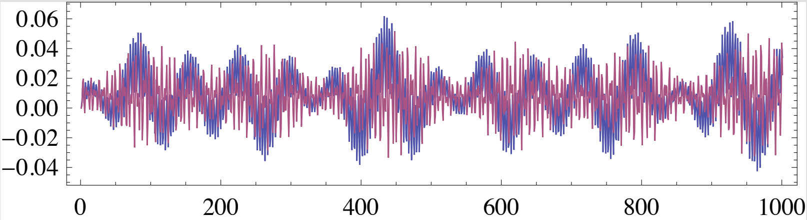
which gives an additional formal first integral $\tilde{I}_3(x) = \frac{1}{2}(p_1^2 + q_1^2) + \mathcal{O}(\epsilon)$.

Example (cont.)

We compute the second order truncation of $\tilde{I}_1(x)$,

$$\tilde{I}_1(x) = I_1(x) + \sum_{k \in \mathbb{Z}^d} \beta_k^{[1]} R_k + \sum_{k,\ell \in \mathbb{Z}^d} \beta_{k\ell}^{[1]} \{R_\ell, R_k\},$$

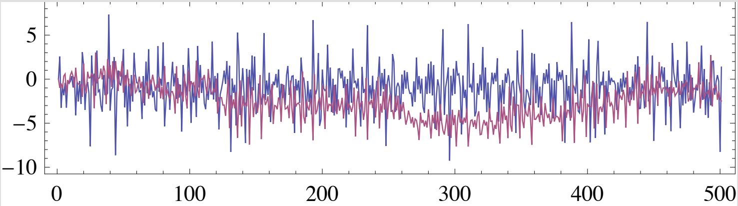
and plot $|\tilde{I}_1(x(\tau)) - \tilde{I}_1(x(0))|/\epsilon^5$ versus τ for $\epsilon = 1/70$, $\epsilon = 1/140$.

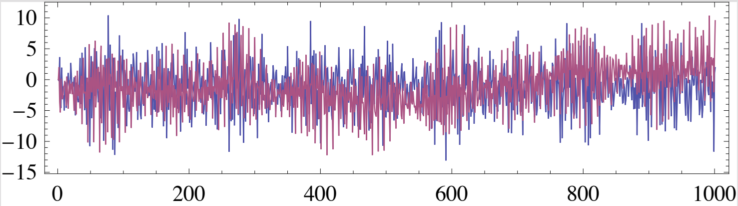


Example (cont.)

$$|\tilde{I}_2(x(\tau)) - \tilde{I}_2(x(0))|/\epsilon^7 \quad \text{and} \quad |\tilde{I}_3(x(\tau)) - \tilde{I}_3(x(0))|/\epsilon^5$$

versus τ for $\epsilon = 1/70$, $\epsilon = 1/140$.





The first two terms in the series defining $\tilde{I}_j(x) - I_j(x)$:

$$\epsilon \sum_{\mathbf{k}>\mathbf{0}} \frac{\mathbf{k}\cdot e_j}{\mathbf{k}\cdot\omega} \left(R_{\mathbf{k}}(x) + R_{-\mathbf{k}}(x)\right)$$
$$+\epsilon^2 \sum_{\mathbf{k}>\mathbf{0}} i\, \frac{\mathbf{k}\cdot e_j}{(\mathbf{k}\cdot\omega)^2} \left(\{R_{\mathbf{0}}, R_{\mathbf{k}} - R_{-\mathbf{k}}\}(x) + \{R_{\mathbf{k}}, R_{-\mathbf{k}}\}(x)\right)$$
$$+\epsilon^2 \sum_{\mathbf{l}>\mathbf{k}>\mathbf{0}} \left(B^{[j]}_{\mathbf{k},\mathbf{l}}(x) + B^{[j]}_{-\mathbf{k},\mathbf{l}}(x) + B^{[j]}_{\mathbf{k},-\mathbf{l}}(x) + B^{[j]}_{-\mathbf{k},-\mathbf{l}}(x)\right) + \mathcal{O}(\epsilon^3),$$

where

$$B^{[j]}_{\mathbf{k},\mathbf{l}}(x) = \frac{i}{(\mathbf{l}+\mathbf{k})\cdot\omega}\left(\frac{\mathbf{k}\cdot e_j}{\mathbf{k}\cdot\omega} - \frac{\mathbf{l}\cdot e_j}{\mathbf{l}\cdot\omega}\right)\{R_{\mathbf{l}}, R_{\mathbf{k}}\}(x).$$

Properties of formal first integrals $\tilde{I}_j(x)$

- $H(x), \tilde{I}_1(x), \ldots, \tilde{I}_d(x)$ are in involution,
- the t-flow $\Phi_t^{[j]}$ of each $\tilde{I}_j(x)$ is (2π)-periodic in t,

High order averaging

Any solution of the original system satisfies

$$x(t) = \Phi^{[1]}_{\frac{t}{\epsilon}\omega_1} \circ \cdots \circ \Phi^{[d]}_{\frac{t}{\epsilon}\omega_d}(X(t)),$$

where $X(t)$ is the solution of the averaged system

$$\begin{aligned} \frac{d}{dt}X &= J^{-1}\nabla\tilde{R}(X), \quad X(0) = x(0), \\ \tilde{R}(x) &= \sum_{j=1}^{d} \frac{\omega_j}{\epsilon}\left(I_j(x) - \tilde{I}_j(x)\right) + R(x). \end{aligned}$$

More explicitly, the averaged Hamitonian is

$$\begin{aligned}\tilde{R}(x) &= \sum_{j=1}^{d} \frac{\omega_j}{\epsilon}\left(I_j(x) - \tilde{I}_j(x)\right) + R(x) \\ &= R_{\mathbf{0}}(x) \\ &\quad + \sum_{r\geq 2} \epsilon^{r-1} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_r \in Z^d} \frac{\tilde{\beta}_{\mathbf{k}_1\cdots\mathbf{k}_r}}{r} \{\{\cdots\{\{R_{\mathbf{k}_1}, R_{\mathbf{k}_2}\}, R_{\mathbf{k}_3}\}\cdots\}, R_{\mathbf{k}_r}\}(x),\end{aligned}$$

where

- $\tilde{\beta}_{\mathbf{k}} = 1 - \sum_{j=1}^{d} \omega_j \, \beta_{\mathbf{k}}^{[j]}$, and for $r > 1$, $\tilde{\beta}_{\mathbf{k}_1\cdots\mathbf{k}_r} = -\sum_{j=1}^{d} \beta_{\mathbf{k}_1\cdots\mathbf{k}_r}^{[j]}$,
- first order averaged Hamiltonian $R_{\mathbf{0}}(x)$ is

$$R_{\mathbf{0}}(x) = \lim_{t\to 0} \frac{1}{t} \int_0^t R(e^{\tau A} x)\, d\tau.$$