Averaging with B-series

Ander Murua
Joint work with P. Chartier and J.M. Sanz-Serna

Euskal Herriko Unibertsitatea / Universidad del Pais Vasco

Genève, June 2009

Example (Fermi-Pasta-Ulam problem)

Hamiltonian system with Hamiltonian function

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2} p_1^T p_1 + \frac{1}{2}(p_2^T p_2 + \omega^2 q_2^T q_2) + U(q_1, q_2),$$

where $p_1, p_2, q_1, q_2 \in \mathbb{R}^m$ and

$$\begin{aligned} U(q_1, q_2) &= \frac{1}{4}\left((q_{1,1} - q_{2,1})^4 + (q_{1,m} + q_{2,m})^4\right) \\ &\quad + \frac{1}{4}\sum_{j=1}^{m-1}(q_{1,j+1} - q_{2,j+1} - q_{1,j} - q_{2,j})^4. \end{aligned}$$

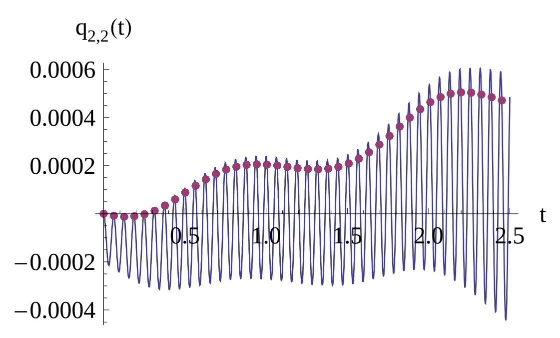
Consider the t-flow $\varphi_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ of that Hamiltonian system.

In particular, take $\epsilon = 2\pi/\omega$, and consider φ_ϵ, that advances the solution in one period if the fast oscillations.

Example (Fermi-Pasta-Ulam problem (cont.))

Component $q_{2,2}(t)$ for $m = 3$, $\omega = 100$, and initial values

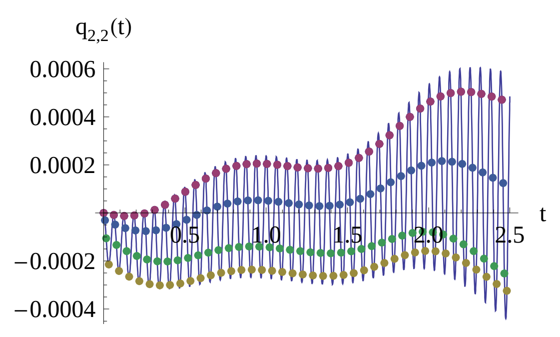
$$p_1(0) = p_2(0) = q_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; q_2(0) = \begin{pmatrix} \omega^{-1} \\ 0 \\ 0 \end{pmatrix}.$$



Example (Fermi-Pasta-Ulam problem (cont.))

Component $q_{2,2}(t)$ for $m = 3$, $\omega = 100$, and initial values

$$p_1(0) = p_2(0) = q_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; q_2(0) = \begin{pmatrix} \omega^{-1} \\ 0 \\ 0 \end{pmatrix}.$$



First rescale time by considering the dimensionless time $\tau = \omega t$, and then apply the (2π)-periodic τ-dependent change of variables

Change of variables

$$\hat{p}_1 = p_1, \quad \hat{p}_2 = \cos(\tau) p_2 + \omega \sin(\tau) q_2,$$
$$\hat{q}_1 = q_1, \quad \hat{q}_2 = \cos(\tau) q_2 - \omega^{-1} \sin(\tau) p_2,$$

(note that at stroboscopic times $\tau_n = 2\pi n$, that change of variables reduces to the identity map.)

The transformed system is a time-dependent Hamiltonian system with Hamiltonian function

$$\omega^{-1} \left(\frac{1}{2} \hat{p}_1^T \hat{p}_1 + U(\hat{q}_1, \cos(\tau)\hat{q}_2 + \omega^{-1} \sin(\tau)\hat{p}_2) \right).$$

Clearly, such a system can be written in the form

$$\frac{d}{d\tau} y \quad = \quad \omega^{-1} f_1(y,\tau) + \omega^{-2} f_2(y,\tau) + \omega^{-3} f_3(y,\tau) + \cdots$$

Backward error analysis of highly oscillatory systems

We consider highly oscillatory systems with frequency ω that, after time rescaling $\tau = \omega t$ (and possibly a (2π)-periodic τ-dependent change of variables that is the identity map at stroboscopic times $\tau_n = 2\pi n$), can be written as

$$\frac{d}{d\tau} y \quad = \quad f(y, \tau; \omega) := \omega^{-1} f_1(y, \tau) + \omega^{-2} f_2(y, \tau) + \omega^{-3} f_3(y, \tau) + \cdots$$

where each $f_j(y, \tau)$ is smooth in y and 2π-periodic in τ.

Consider $\psi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\psi(y(0)) = y(2\pi)$,

- The map ψ is as a smooth near-to-identity map,
- $f_j(\cdot, \tau)$ is (2π)-periodic $\Rightarrow y(\tau_n) = \psi^n(y(0))$ for $\tau_n = 2\pi n$,
- ψ is symplectic if f is Hamiltonian.

Backward error analysis for the near-to-identity map ψ

Given

$$\frac{d}{d\tau}y = \omega^{-1}f_1(y,\tau) + \omega^{-2}f_2(y,\tau) + \omega^{-3}f_3(y,\tau) + \cdots \quad (1)$$

there exists

$$\frac{d}{d\tau}\bar{y} = \omega^{-1}\bar{f}_1(\bar{y}) + \omega^{-2}\bar{f}_2(\bar{y}) + \omega^{-3}\bar{f}_3(\bar{y}) + \cdots \quad (2)$$

such that formally, $y(2\pi n) = \bar{y}(2\pi n)$ for the solutions $y(\tau)$ and $\bar{y}(\tau)$ of (1) and (2) with $\bar{y}(0) = y(0)$.

If (1) is Hamiltonian, then (2) is Hamiltonian with

$$\bar{H}(y) = \omega^{-1}\bar{H}_1(y) + \omega^{-2}\bar{H}_2(y) + \omega^{-3}\bar{H}_3(y) + \cdots$$

Backward error analysis for the near-to-identity map ψ

Given

$$\frac{d}{d\tau} y = \omega^{-1} f_1(y,\tau) + \omega^{-2} f_2(y,\tau) + \omega^{-3} f_3(y,\tau) + \cdots \quad (1)$$

there exists

$$\frac{d}{d\tau}\bar{y} = \omega^{-1}\bar{f}_1(\bar{y}) + \omega^{-2}\bar{f}_2(\bar{y}) + \omega^{-3}\bar{f}_3(\bar{y}) + \cdots \quad (2)$$

such that formally, $y(2\pi n) = \bar{y}(2\pi n)$ for the solutions $y(\tau)$ and $\bar{y}(\tau)$ of (1) and (2) with $\bar{y}(0) = y(0)$.

If (1) is Hamiltonian, then (2) is Hamiltonian with

$$\bar{H}(y) = \omega^{-1}\bar{H}_1(y) + \omega^{-2}\bar{H}_2(y) + \omega^{-3}\bar{H}_3(y) + \cdots$$

In averaging theory [Sanders, Verhulst, Murdock 2007], obtaining

$$\frac{d}{d\tau}\bar{y} = \omega^{-1}\bar{f}_1(\bar{y}) + \omega^{-2}\bar{f}_2(\bar{y}) + \omega^{-3}\bar{f}_3(\bar{y}) + \cdots . \tag{3}$$

from the original HOS system is called stroboscopic averaging.

High order averaging

There exists a formal (2π)-periodic change of variables $y = K(\bar{y}, \tau)$ that transforms the original system

$$\frac{d}{d\tau}y \quad = \quad \omega^{-1}f_1(y,\tau) + \omega^{-2}f_2(y,\tau) + \omega^{-3}f_3(y,\tau) + \cdots$$

into the autonomous system (3).

The change of variables $y = K(\bar{y}, \tau)$ is not unique. Typical choices

- $K(\bar{y}, 0) = \bar{y}$: Stroboscopic averaging,
- $\int_0^{2\pi} K(\bar{y}, \tau)\, d\tau = \bar{y}$.

B-series

We will be dealing with B-series-like expansions of the form

$$B(\alpha, y) = \alpha_\emptyset\, y + \sum_{u \in \mathcal{T}} \frac{\omega^{-|u|}}{\sigma(u)}\, \alpha_u\, F_u(y),$$

where $\mathcal{T}$ is a set of indices, and for each $u \in \mathcal{T}$,

- $\alpha_u \in \mathbb{R}$ are the coefficients of the B-series,
- $F_u : \mathbb{R}^D \to \mathbb{R}^D$ is the *elementary differential,*
- $|u| \in \mathbb{Z}^+$ *is the degree (or order) of* u*,*
- $\sigma(u)$ *is some normalization factor.*

In standard B-series (Hairer and Wanner 1974), $\mathcal{T}$ is the set of rooted trees, and $\sigma(u)$ is the symmetry number of the tree u.

We denote $\mathcal{T}_j = \{u \in \mathcal{T} \ : \ |u| = j\}$, so that

$$B(\alpha, y) = \alpha_\emptyset\, y + \sum_{j \geq 1} \omega^{-j} \sum_{u \in \mathcal{T}_j} \frac{\alpha_u}{\sigma(u)} F_u(y).$$

B-series

We will be dealing with B-series-like expansions of the form

$$B(\alpha, y) = \alpha_\emptyset\, y + \sum_{u \in \mathcal{T}} \frac{\omega^{-|u|}}{\sigma(u)}\, \alpha_u\, F_u(y),$$

where $\mathcal{T}$ is a set of indices, and for each $u \in \mathcal{T}$,

- $\alpha_u \in \mathbb{R}$ are the coefficients of the B-series,
- $F_u : \mathbb{R}^D \to \mathbb{R}^D$ is the *elementary differential,*
- $|u| \in \mathbb{Z}^+$ *is the degree (or order) of* u*,*
- $\sigma(u)$ *is some normalization factor.*

In standard B-series (Hairer and Wanner 1974), $\mathcal{T}$ is the set of rooted trees, and $\sigma(u)$ is the symmetry number of the tree u.

We denote $\mathcal{T}_j = \{u \in \mathcal{T} \; : \; |u| = j\}$, so that

$$B(\alpha, y) = \alpha_\emptyset\, y + \sum_{j \geq 1} \omega^{-j} \sum_{u \in \mathcal{T}_j} \frac{\alpha_u}{\sigma(u)}\, F_u(y).$$

B-series

We will be dealing with B-series-like expansions of the form

$$B(\alpha, y) = \alpha_\emptyset\, y + \sum_{u \in \mathcal{T}} \frac{\omega^{-|u|}}{\sigma(u)}\, \alpha_u\, F_u(y),$$

where $\mathcal{T}$ is a set of indices, and for each $u \in \mathcal{T}$,

- $\alpha_u \in \mathbb{R}$ are the coefficients of the B-series,
- $F_u : \mathbb{R}^D \to \mathbb{R}^D$ is the *elementary differential,*
- $|u| \in \mathbb{Z}^+$ *is the degree (or order) of u,*
- $\sigma(u)$ *is some normalization factor.*

In standard B-series (Hairer and Wanner 1974), $\mathcal{T}$ is the set of rooted trees, and $\sigma(u)$ is the symmetry number of the tree u.

We denote $\mathcal{T}_j = \{u \in \mathcal{T} \ : \ |u| = j\}$, so that

$$B(\alpha, y) = \alpha_\emptyset\, y + \sum_{j \geq 1} \omega^{-j} \sum_{u \in \mathcal{T}_j} \frac{\alpha_u}{\sigma(u)}\, F_u(y).$$

Assumption

For each pair $(u,v) \in \mathcal{T}_j \times \mathcal{T}_k$*, there exist* $w_1, \ldots, w_m \in \mathcal{T}_{j+k}$ *and* $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ *such that*

$$\left(\frac{\partial}{\partial y} F_u(y)\right) F_v(y) = \sum_{j=1}^{m} \lambda_j \, F_{w_j}(y).$$

Let us denote $\mathcal{F} = \{\emptyset\} \cup \{u_1 \cdots u_m \ : \ u_1, \ldots, u_m \in \mathcal{T}\}$.

Let $\mathbb{R}^{\mathcal{F}}$ be the set of real functions $\mathcal{F} \to \mathbb{R}$. Consider also

- $\mathfrak{g} = \{\beta \in \mathbb{R}^{\mathcal{F}} \ : \ \beta_\emptyset = 0, \ \beta_{uv} = 0 \text{ if } u, v \in \mathcal{F}/\{\emptyset\}\}$.
- $\mathcal{G} = \{\alpha \in \mathbb{R}^{\mathcal{F}} \ : \ \alpha_\emptyset = 1, \ \alpha_{uv} = \alpha_u \alpha_v \text{ if } u, v \in \mathcal{F}\}$.

Theorem

Under the assumption above, there exists a binary operation

$$\begin{array}{rcl} \mathbb{R}^{\mathcal{F}} \times \mathbb{R}^{\mathcal{F}} & \to & \mathbb{R}^{\mathcal{F}} \\ (\alpha, \beta) & \mapsto & \alpha\beta \end{array}$$

where for each $u \in \mathcal{F}$, $\exists v_j, w_j \in \mathcal{F}/\{\emptyset\}$ *such that*

$$(\alpha\beta)_u = \beta_\emptyset \alpha_u + \alpha_\emptyset \beta_u + \sum_{j=1}^{m} \alpha_{v_j} \beta_{w_j},$$

with $|v_j| + |w_j| = |u|$ *satisfying that*

- $B(\beta, B(\alpha, y)) = B(\alpha\beta, y)$ *if* $\alpha \in \mathcal{G}$
- $\frac{\partial}{\partial y} B(\alpha, y) \cdot B(\beta, y) = B(\beta\alpha, y)$ *if* $\beta \in \mathfrak{g}$.

- $\mathcal{G}$ has with that binary operation a group structure, and
- $\mathfrak{g}$ has a pre-Lie algebra structure, and also a Lie algebra structure with $[\alpha, \beta] := \alpha\beta - \beta\alpha$.

Theorem

Under the assumption above, there exists a binary operation

$$\begin{array}{rcl} \mathbb{R}^{\mathcal{F}} \times \mathbb{R}^{\mathcal{F}} & \to & \mathbb{R}^{\mathcal{F}} \\ (\alpha, \beta) & \mapsto & \alpha\beta \end{array}$$

where for each $u \in \mathcal{F}$, $\exists v_j, w_j \in \mathcal{F}/\{\emptyset\}$ *such that*

$$(\alpha\beta)_u = \beta_\emptyset \alpha_u + \alpha_\emptyset \beta_u + \sum_{j=1}^{m} \alpha_{v_j} \beta_{w_j},$$

with $|v_j| + |w_j| = |u|$ *satisfying that*

- $B(\beta, B(\alpha, y)) = B(\alpha\beta, y)$ *if* $\alpha \in \mathcal{G}$
- $\frac{\partial}{\partial y} B(\alpha, y) \cdot B(\beta, y) = B(\beta\alpha, y)$ *if* $\beta \in \mathfrak{g}$.

- $\mathcal{G}$ has with that binary operation a group structure, and
- $\mathfrak{g}$ has a pre-Lie algebra structure, and also a Lie algebra structure with $[\alpha, \beta] := \alpha\beta - \beta\alpha$.

Theorem

Under the assumption above, there exists a binary operation

$$\begin{array}{rcl} \mathbb{R}^{\mathcal{F}} \times \mathbb{R}^{\mathcal{F}} & \to & \mathbb{R}^{\mathcal{F}} \\ (\alpha, \beta) & \mapsto & \alpha\beta \end{array}$$

where for each $u \in \mathcal{F}$, $\exists v_j, w_j \in \mathcal{F}/\{\emptyset\}$ *such that*

$$(\alpha\beta)_u = \beta_\emptyset \alpha_u + \alpha_\emptyset \beta_u + \sum_{j=1}^{m} \alpha_{v_j} \beta_{w_j},$$

with $|v_j| + |w_j| = |u|$ *satisfying that*

- $B(\beta, B(\alpha, y)) = B(\alpha\beta, y)$ *if* $\alpha \in \mathcal{G}$
- $\frac{\partial}{\partial y} B(\alpha, y) \cdot B(\beta, y) = B(\beta\alpha, y)$ *if* $\beta \in \mathfrak{g}$.

- $\mathcal{G}$ has with that binary operation a group structure, and
- $\mathfrak{g}$ has a pre-Lie algebra structure, and also a Lie algebra structure with $[\alpha, \beta] := \alpha\beta - \beta\alpha$.

Averaging in terms of B-series

Consider a nonautonomous (2π)-periodic ODE written in the form

$$\frac{d}{d\tau}y = B(\beta(\tau), y), \quad \text{where} \quad \beta(\tau) \in \mathfrak{g} \text{ and } \beta(\tau + 2\pi) = \beta(\tau).$$

Its solutions $y(\tau)$ can be expanded as $y(\tau) = B(\alpha(\tau), y(0))$ with $\alpha(\tau) \in \mathcal{G}$, where

$$\frac{d}{d\tau}B(\alpha(\tau), y(0)) = B(\beta(\tau), B(\alpha(\tau), y(0))), \quad B(\alpha(0), y(0)) = y(0).$$

That certainly holds if

$$\forall u \in \mathcal{T}, \quad \frac{d}{d\tau}\alpha(\tau)_u = (\alpha(\tau)\beta(\tau))_u, \quad \alpha(0)_u = 0,$$

or more compactly,

$$\frac{d}{d\tau}\alpha(\tau) = \alpha(\tau)\beta(\tau), \quad \alpha(0) = I,$$

which uniquely determines $\alpha(\tau) \in \mathcal{G}$ in terms of $\beta(\tau) \in \mathfrak{g}$.

Averaging in terms of B-series

Consider a nonautonomous (2π)-periodic ODE written in the form

$$\frac{d}{d\tau} y = B(\beta(\tau), y), \quad \text{where} \quad \beta(\tau) \in \mathfrak{g} \text{ and } \beta(\tau + 2\pi) = \beta(\tau).$$

Its solutions $y(\tau)$ can be expanded as $y(\tau) = B(\alpha(\tau), y(0))$ with $\alpha(\tau) \in \mathcal{G}$, where

$$\frac{d}{d\tau} B(\alpha(\tau), y(0)) = B(\beta(\tau), B(\alpha(\tau), y(0))), \quad B(\alpha(0), y(0)) = y(0).$$

That certainly holds if

$$\forall u \in \mathcal{T}, \quad \frac{d}{d\tau} \alpha(\tau)_u = (\alpha(\tau)\beta(\tau))_u, \quad \alpha(0)_u = 0,$$

or more compactly,

$$\frac{d}{d\tau} \alpha(\tau) = \alpha(\tau)\beta(\tau), \quad \alpha(0) = I,$$

which uniquely determines $\alpha(\tau) \in \mathcal{G}$ in terms of $\beta(\tau) \in \mathfrak{g}$.

Averaging in terms of B-series

Consider a nonautonomous (2π)-periodic ODE written in the form

$$\frac{d}{d\tau}y = B(\beta(\tau), y), \quad \text{where} \quad \beta(\tau) \in \mathfrak{g} \text{ and } \beta(\tau + 2\pi) = \beta(\tau).$$

Its solutions $y(\tau)$ can be expanded as $y(\tau) = B(\alpha(\tau), y(0))$ with $\alpha(\tau) \in \mathcal{G}$, where

$$\frac{d}{d\tau}B(\alpha(\tau), y(0)) = B(\beta(\tau), B(\alpha(\tau), y(0))), \quad B(\alpha(0), y(0)) = y(0).$$

That certainly holds if

$$\forall u \in \mathcal{T}, \quad \frac{d}{d\tau}\alpha(\tau)_u = (\alpha(\tau)\beta(\tau))_u, \quad \alpha(0)_u = 0,$$

or more compactly,

$$\frac{d}{d\tau}\alpha(\tau) = \alpha(\tau)\beta(\tau), \quad \alpha(0) = I,$$

which uniquely determines $\alpha(\tau) \in \mathcal{G}$ in terms of $\beta(\tau) \in \mathfrak{g}$.

Averaging in terms of B-series coefficients

Theorem

Given a (2π)-periodic $\beta(\tau) \in \mathfrak{g}$, there exist $\bar{\beta} \in \mathfrak{g}$, $\bar{\alpha}(\tau) \in \mathcal{G}$, and a (2π)-periodic $\kappa(\tau) \in \mathcal{G}$ such that

$$\alpha(\tau) = \bar{\alpha}(\tau)\kappa(\tau), \quad \frac{d}{d\tau}\bar{\alpha}(\tau) = \bar{\alpha}(\tau)\bar{\beta}, \quad \bar{\alpha}(0) = I.$$

Furthermore, it holds that

$$\frac{d}{d\tau}\kappa(\tau) = \kappa(\tau)\bar{\beta} - \beta(\tau)\kappa(\tau).$$

$$\bar{\beta}_u = \int_0^{2\pi} \beta(\tau)_u \, d\tau - \sum_{j=1}^{m} \int_0^{2\pi} (\kappa(\tau)_{v_j} \bar{\beta}_{w_j} - \beta(\tau)_{v_j} \kappa(\tau)_{w_j}) d\tau,$$

$$\kappa(\tau)_u = \tau\bar{\beta}_u - \int_0^{\tau} \beta(\sigma)_u d\sigma + \sum_{j=1}^{m} \int_0^{\tau} (\kappa(\sigma)_{v_j}\bar{\beta}_{w_j} - \beta(\sigma)_{v_j}\kappa(\sigma)_{w_j}) d\sigma.$$

Averaging in terms of B-series coefficients

Theorem

Given a (2π)-periodic $\beta(\tau) \in \mathfrak{g}$, there exist $\bar{\beta} \in \mathfrak{g}$, $\bar{\alpha}(\tau) \in \mathcal{G}$, and a (2π)-periodic $\kappa(\tau) \in \mathcal{G}$ such that

$$\alpha(\tau) = \bar{\alpha}(\tau)\kappa(\tau), \quad \frac{d}{d\tau}\bar{\alpha}(\tau) = \bar{\alpha}(\tau)\bar{\beta}, \quad \bar{\alpha}(0) = I.$$

Furthermore, it holds that

$$\frac{d}{d\tau}\kappa(\tau) = \kappa(\tau)\bar{\beta} - \beta(\tau)\kappa(\tau).$$

$$\begin{aligned}
\bar{\beta}_u &= \int_0^{2\pi} \beta(\tau)_u \, d\tau - \sum_{j=1}^{m} \int_0^{2\pi} (\kappa(\tau)_{v_j} \, \bar{\beta}_{w_j} - \beta(\tau)_{v_j} \, \kappa(\tau)_{w_j}) d\tau, \\
\kappa(\tau)_u &= \tau\bar{\beta}_u - \int_0^{\tau} \beta(\sigma)_u d\sigma + \sum_{j=1}^{m} \int_0^{\tau} (\kappa(\sigma)_{v_j}\bar{\beta}_{w_j} - \beta(\sigma)_{v_j}\kappa(\sigma)_{w_j}) d\sigma.
\end{aligned}$$

Alternative algorithm

Given $\beta(\tau) \in \mathfrak{g}$,

- Find $\alpha(\tau) \in \mathcal{G}$ such that

$$\frac{d}{d\tau}\alpha(\tau) = \alpha(\tau)\beta(\tau), \quad \alpha(0) = I.$$

- For each $u \in \mathcal{T}$, obtain $\bar{\alpha}(\tau)_u$ as the polynomial of degree $|u|$ that interpolates $\alpha(\tau)_u$ for $\tau = 2\pi n$, $n \in \mathbb{Z}$,
- Obtain $\bar{\beta}$ as

$$\bar{\beta}_u = \left. \frac{d}{d\tau}\bar{\alpha}(\tau)_u \right|_{\tau=0}.$$

Last two steps of the algorithm are equivalent to applying numerical differentiation formulae with data $\alpha(\tau)_u$ for $\tau = 2\pi n$, $n \in \mathbb{Z}$.

Applications (work in progress)

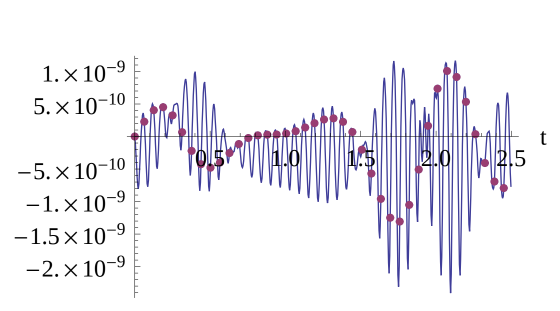
- Practical computation of several terms of the higher order averaged equations (possible application beyond numerical analysis)
- Numerical methods that accurately approximate averaged solutions (oscillatory solution can always be recovered locally):

  1. Integration of averaged equations obtained by combining microintegration and numerical differentiation
  2. B-series methods designed to fit the B-series expansion of the solution of the averaged equations (splitting methods, RK-like methods$\cdots$)

Example (Fermi-Pasta-Ulam problem)

Variation $\bar{H}(y(t)) - \bar{H}(y(0))$ of the averaged Hamiltonian

$$\bar{H}_1(y) + \omega^{-2}\bar{H}_3(y) + \omega^{-4}\bar{H}_5(y).$$



Actually, it can be proven that formally, $\{H, \bar{H}\} = 0$, so that $\bar{H}$ is a formal invariant of the original problem. Hence,

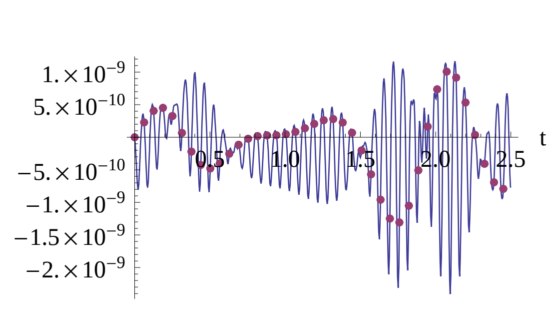
$$\tilde{I}(p_1, p_2, q_1, q_2) := H(p_1, p_2, q_1, q_2) - \bar{H}(p_1, p_2, q_1, q_2)$$

is a formal invariant of the original system.

Example (Fermi-Pasta-Ulam problem)

Variation $\bar{H}(y(t)) - \bar{H}(y(0))$ of the averaged Hamiltonian

$$\bar{H}_1(y) + \omega^{-2}\bar{H}_3(y) + \omega^{-4}\bar{H}_5(y).$$



Actually, it can be proven that formally, $\{H, \bar{H}\} = 0$, so that $\bar{H}$ is a formal invariant of the original problem. Hence,

$$\tilde{I}(p_1, p_2, q_1, q_2) := H(p_1, p_2, q_1, q_2) - \bar{H}(p_1, p_2, q_1, q_2)$$

is a formal invariant of the original system.