ON THE INVERSE SYMMETRIC QUADRATIC EIGENVALUE PROBLEM

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Abstract. The detailed spectral structure of symmetric, algebraic, quadratic eigenvalue problems has been developed recently. In this paper we take advantage of these canonical forms to provide a detailed analysis of inverse problems of the form: construct the coefficient matrices from the spectral data including the classical eigenvalue/eigenvector data and sign characteristics for the real eigenvalues. An orthogonality condition dependent on these signs plays a vital role in this construction. Special attention is paid to the cases when the leading and trailing coefficients of the quadratic matrix polynomial are prescribed to be positive definite.

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1. Introduction.

Given a quadratic matrix polynomial $L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0$, the *direct* quadratic eigenvalue problem (QEP) is to find scalars λ and nonzero vectors x satisfying $L(\lambda)x = 0$. The scalars λ and the vectors x are called, respectively, eigenvalues and eigenvectors of the quadratic matrix polynomial $L(\lambda)$.

The QEP appears repeatedly in different scientific areas including the analysis of mechanical and acoustic systems, electrical oscillations, fluid mechanics and signal processing. Many applications, mathematical properties, and a variety of numerical techniques for this problem were surveyed by Tisseur and Meerbergen in [22].

In many applications properties of the underlying physical system determine real matrix coefficients L_2, L_1, L_0 (frequently known as the mass, damping and stiffness matrices of the system), while the behaviour of the system can frequently be interpreted in terms of the eigenvalues and eigenvectors. Thus, the process of analysing and deriving the spectral information (eigenvalues and eigenvectors) from the matrix coefficients is the *direct* QEP. The *inverse* QEP is then to validate, determine or estimate the parameters (matrices) of the system consistent with its observed or expected behaviour. In this general setting the "pole assignment problem" (see, for example, [3, 4, 7, 19] and the references there) can also be seen as an inverse QEP. An important reference for inverse eigenvalue problem is the book by Chu and Golub, [6], where a section is dedicated to the inverse QEP.

If the matrix coefficients are not subject to symmetry constraints, a general technique for constructing families of quadratic matrix polynomials with prescribed eigenstructure was proposed in [12]. However, as noted above, many physical systems determine quadratic systems with symmetry constraints on their coefficients. The *inverse symmetric quadratic eigenvalue problem* (ISQEP) calls for the construction of a family of real symmetric quadratic matrix polynomials (possibly with some definiteness restrictions on the coefficients) from an admissible set of spectral data. Although this notion will be extended later in this section, it will be seen that an admissible

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set of spectral data for real symmetric quadratic matrix polynomial must include a complete family of $2n$ real and/or complex eigenvalues, with the property that the non-real eigenvalues (with their associated eigenvectors) appear in conjugate pairs.

This ISQEP has received much attention because of its many applications (see [21]). Thus, in [2, 20] the case of symmetric tridiagonal coefficients is considered and in [5, 11] positivity restrictions are imposed on (some of) the matrix coefficients. It is a common feature of these references that the spectral data is *partially* prescribed. In [12, 14] the problem is considered in greater generality but only partial results were obtained. In this paper the techniques developed there will be completed and a procedure for constructing a family of real symmetric quadratic matrix polynomials with admissible Jordan structure (Definition 2) will be provided in the semisimple case; i.e. when the algebraic and geometric multiplicities of the eigenvalues coincide.

By Jordan structure, or spectral data, we mean a complete family of real and complex eigenvalues with their partial multiplicities and sign characteristics. The sign characteristic is a collection of signs, $+$ and $-$, attached to the elementary divisors corresponding to the real eigenvalues of any self-adjoint matrix polynomial. The notion of sign characteristic plays a fundamental role in the study of the self-adjoint matrix polynomial structures ([10] and [17]) and in the solution of the ISQEP. It will be reviewed in Section 2 (see also [12, Appen. B]). In that section we will also revisit an orthogonality property of the eigenvectors of symmetric matrix polynomials that is important for our developments. This property was studied in [14] for quadratic matrix polynomials with purely non-real spectrum and then in [12] for quasi hyperbolic systems (only real spectrum). It was generalized in [15] to semisimple matrix polynomials of any degree with arbitrary, but admissible, spectrum.

The assumption that the leading coefficient, L_2, is prescribed to be non-singular will be maintained throughout this paper. We first study the ISQEP with no definiteness constraints on the coefficients, and then the case when L_2 is required to be positive definite. The starting point of this study is the fact (see (2.10)) that the coefficients of any self-adjoint matrix polynomial can be obtained from any of its self-adjoint Jordan triples (Section 2.1). Thus, the first goal is the construction of self-adjoint Jordan triples from the prescribed spectral data. This is also the approach taken in the Ph.D. thesis of Maha Al-Ammari [1] - but in a broader context. Section 3 completes that study. It will be shown that, in the semisimple case, any spectral data is admissible provided that two properties are fulfilled:

1. The non-real eigenvalues must be in conjugate pairs and

2. Half of the real spectrum must be of positive type (+1 as sign characteristic) and the other half of negative type (-1 as sign characteristic). However, multiple real eigenvalues of "mixed' ' type (some of them of positive type and some of negative type) are admitted.

Theorem 8 of Section 3 is a central result based on recently developed orthogonality properties of the eigenvectors (Theorem 3). Theorems 13, 14, 15 and 17 are similar results, but admitting positivity constraints on the leading and/or trailing coefficients.

A quadratic matrix polynomial is said to be *diagonalizable* if there is a diagonal quadratic matrix polynomial with the same eigenvalues and partial multiplicities (or the same elementary divisors, or the same invariant factors, or the same Smith normal form). It turns out that semisimple real quadratic matrix polynomials (whether they are symmetric or not) may not be diagonalizable (see [16]). However, if we confine ourselves to semisimple real symmetric quadratic matrix polynomials with positive definite leading coefficient then they are all diagonalizable (Theorem 13).

2. Preliminary notions and definitions. In this section we summarize the spectral properties of semisimple quadratic matrix polynomials required in the subsequent development, namely: self-adjoint Jordan triples (including the sign characteristic), orthogonality of the eigenvectors, and conditions for diagonalizability.

2.1. Selfadjoint Jordan triples. We consider $n \times n$ semisimple real quadratic matrix polynomials,

$$L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0, \tag{2.1}$$

with real and symmetric coefficients $L_i^T = L_i$, $i = 0, 1, 2$, and non-singular leading coefficient, L_2. The notions of self-adjoint Jordan triple and sign characteristic of real symmetric matrix polynomials play a fundamental role in what follows.The reader is referred to [17] (or [15] for a summary). An specific instance of self-adjoint Jordan triple will be used along the paper. It is now introduced.

Assume that $L(\lambda)$ has $2s$ ($s \leq n$) non-real eigenvalues (counting multiplicities) and $2q = 2n-2s$ real eigenvalues. Bearing in mind that the sign characteristic of $L(\lambda)$ as a complex hermitian matrix polynomial or as a real symmetric polynomial is the same ([15, Appen. A]) then, by Proposition 4.2 of [8], half of these real eigenvalues have a positive sign characteristic and the other half a negative sign characteristic.

Let $r_1, \ldots, r_q$ be the real eigenvalues of positive type, $r_{q+1}, \ldots, r_{2q}$ be those of negative type and construct diagonal matrices of size q:

$$R_+ = \mathrm{Diag}(r_1, \ldots, r_q), \qquad R_- = \mathrm{Diag}(r_{q+1}, \ldots, r_{2q}).$$

Note that the same eigenvalue may appear in both R_+ and R_-. Write the $2s$ conjugate pairs of eigenvalues as follows:

$$\beta_j = \mu_j + i\nu_j, \qquad \beta_{j+1} = \bar{\beta}_j = \mu_j - i\nu_j \qquad (\nu_j > 0), \qquad j = 1, 3, \ldots, 2s-1$$

and define

$$M = \mathrm{Diag}(\mu_1, \mu_3, \ldots, \mu_{2s-1}), \quad N = \mathrm{Diag}(\nu_1, \nu_3, \ldots, \nu_{2s-1}) > 0. \tag{2.2}$$

Let X_+, X_-, $\widetilde{U}$ and $\widetilde{V}$ be real matrices with the property that the j-th column of X_+ (resp. X_-, $\widetilde{U} + i\widetilde{V}$ and $\widetilde{U} - i\widetilde{V}$) is an eigenvector of $L(\lambda)$ associated with r_j (resp. r_{q+j}, β_j and $\bar{\beta}_j$). Now we summarize complete spectral data with a real matrix triple:

$$J = \mathrm{Diag}\left(R_+, R_-, \begin{bmatrix} M & -N \\ N & M \end{bmatrix}\right) \in \mathbb{R}^{2n\times 2n}, \tag{2.3}$$

$$P = \mathrm{Diag}\left(I_q, -I_q, -I_s, I_s\right) \in \mathbb{R}^{2n\times 2n}, \tag{2.4}$$

$$X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} \in \mathbb{R}^{n\times 2n}, \tag{2.5}$$

where $X_+, X_- \in \mathbb{R}^{n\times q}$ and

$$V = \sqrt{2}\begin{bmatrix} v_1 & \cdots & v_s \end{bmatrix} \in \mathbb{R}^{n\times s}, \qquad U = \sqrt{2}\begin{bmatrix} u_1 & \cdots & u_s \end{bmatrix} \in \mathbb{R}^{n\times s}, \tag{2.6}$$

$u_j + iv_j$ and $u_j - iv_j$ being the j-th column of $\widetilde{U} + i\widetilde{V}$ and $\widetilde{U} - i\widetilde{V}$. Notice that U and V are (up to the factor $\sqrt{2}$) the real and imaginary parts of eigenvectors associated with non-real eigenvalues. Then ([15, Thm. 1]):

THEOREM 1. *With X, J, P defined as in* (2.5), (2.3), (2.4)*, (X, J, PX^T) is a real self-adjoint Jordan triple for $L(\lambda)$ of* (2.1).

2.2. Selfdajoint Jordan structures. A self-adjoint Jordan triple (X, J, P) of the quadratic matrix polynomial $L(\lambda)$ of (2.1) has the properties

$$XPX^T = 0 \tag{2.7}$$

and, for the leading coefficient of $L(\lambda)$ we have

$$XJPX^T = L_2^{-1}. \tag{2.8}$$

Indeed, it is possible to express all the coefficients of $L(\lambda)$ in terms of a self-adjoint Jordan triple. Specifically, define the *moment functions* P_k acting on matrices $X \in \mathbb{R}^{n\times 2n}$ as follows:

$$P_k(X) := XJ^kPX^T \tag{2.9}$$

for integers k (and note that k is *any* integer if zero is not in the spectrum and, otherwise, $k \geq 0$).

Then (2.7) gives $P_0(X) = 0$ and the coefficients are defined by the moments in the form (see [15]):

$$\begin{array}{rcl} L_2^{-1} & = & P_1(X), \\ L_1 & = & -L_2P_2(X)L_2 = -P_1(X)^{-1}P_2(X)P_1(X)^{-1}, \\ L_0 & = & -L_2P_3(X)L_2 + L_1P_1(X)L_1 \\ & = & -P_1(X)^{-1}\big(P_3(X) + P_2(X)P_1(X)^{-1}P_2(X))\big)P_1(X)^{-1}. \end{array} \tag{2.10}$$

Alternatively, if $0 \notin \sigma(L)$ then

$$L_0 = -P_{-1}(X)^{-1}. \tag{2.11}$$

We can also use Theorem 14.7.1 of [13] to write L_1 and L_0 in the form:

$$\begin{bmatrix} L_0 & L_1 \end{bmatrix} = -L_2XJ^2 \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1}. \tag{2.12}$$

Notice that (see Remark 2 in [15]), given J and P as in (2.3) and (2.4), for *any* full rank matrix X for which $XPX^T = 0$ and $XJPX^T$ is invertible, (X, J, PX^T) forms a self-adjoint Jordan triple of some real symmetric quadratic matrix polynomial $L(\lambda)$. In fact, the leading coefficient of such a matrix is $L_2 = (XJPX^T)^{-1}$ and L_1 and L_0 are obtained as in (2.12), for example.

Thus, given a semisimple quadratic polynomial $L(\lambda)$ as in (2.1), there is always a self-adjoint Jordan triple (X, J, PX^T) for $L(\lambda)$ given by (2.3)-(2.5) where (J, P) summarizes the spectral data of $L(\lambda)$. Conversely, given matrices (J, P) as in (2.3) and (2.4) that prescribe the eigenvalues, partial multiplicities and sign characteristics associated with the real eigenvalues, then for every real X satisfying conditions (2.7) and (2.8) there is a real symmetric quadratic matrix polynomial $L(\lambda)$ with this prescribed spectral data. In addition, the matrix X determines a matrix of eigenvectors for $L(\lambda)$.

It is our goal to design a procedure to produce matrices X which determine a viable set of right eigenvectors and also satisfy conditions (2.7) and (2.8) for given matrices J and P. It will be shown that if J and P are as in (2.3) and (2.4), and no definiteness condition is imposed on the coefficients, such a matrix X always exists. However, if it is required that L_2 is positive definite, then it may happen that, for some specific matrices J and P, there is no such matrix X.

For example, there is no 2×2 quadratic matrix polynomial with *positive definite* leading coefficient and Jordan structure

$$J = \text{Diag}(1, 2, 3, 4), \quad P = \text{Diag}(1, 1, -1 - 1).$$

The reason is that, by Example 1.5 of [9], for all matrix polynomials of even degree with positive definite leading coefficient, the sign characteristic of the largest real eigenvalue must be positive and that of the smallest real eigenvalue must be negative. Hence, for these matrices J and P, there is no matrix X satisfying (2.7) and (2.8) **and** $L_2 > 0$.

DEFINITION 2.

*(a) A **real self-adjoint Jordan structure** (for a QEP) is a pair of matrices $(J, P) \in \mathbb{R}^{2n \times 2n} \times \mathbb{R}^{2n \times 2n}$ with the form* (2.3), (2.4) *for four real diagonal matrices $R_+, R_- \in \mathbb{R}^{q \times q}$ and $M, N \in \mathbb{R}^{s \times s}$ $(q + s = n)$ with $N > 0$.*

*(b) A real self-adjoint Jordan structure (J, P) is said to be **admissible** if there is an $X \in \mathbb{R}^{n \times 2n}$ for which equations* (2.7) *holds and $XJPX^T$ is non-singular (see* (2.8)*) .*

Example 1. Consider the real self-adjoint structure with $n = 1$,

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and $\lambda_2 \neq \lambda_1$. With $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ and $x_2 = \pm x_1 \neq 0$, we have

$$XPX^T = x_1^2 - x_2^2 = 0, \qquad XJPX^T = \lambda_1 x_1^2 - \lambda_2 x_2^2 \neq 0.$$

So (J, P) is *admissible* because we can take $x_2 = \pm x_1 \neq 0$ and $\lambda_1 \neq \lambda_2$. If, in addition, we require $\lambda_1 > \lambda_2$ then $XJPX^T > 0$. □

2.3. Orthogonality. The eigenvectors of a semisimple matrix polynomial satisfy an orthogonality property derived from (2.7) and the form of the matrix P in (2.4) (see [15]).

Assume that J and P are given by (2.3) and (2.4), respectively. Let $X \in \mathbb{R}^{n \times 2n}$ be an arbitrary matrix and partition it as in (2.5); i. e.

$$X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix},$$

where $X_+, X_- \in \mathbb{R}^{n \times q}$ and $U, V \in \mathbb{R}^{n \times s}$. It follows from Theorem 2 of [15] that (X, J, PX^T) is a self-adjoint Jordan triple if and only if $XJPX^T$ is invertible and there exists an orthogonal matrix $\Theta \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta \tag{2.13}$$

and, in particular,

$$\text{rank} \begin{bmatrix} X_- & V \end{bmatrix} = \text{rank} \begin{bmatrix} X_+ & U \end{bmatrix} = n. \tag{2.14}$$

These are conditions that any matrix X must satisfy for (X, J, PX^T) to be a self-adjoint Jordan triple.

Observe that, using (2.13), X can be written as

$$X = \begin{bmatrix} X_+ & U \end{bmatrix} \begin{bmatrix} I_{q+s} & \Theta \end{bmatrix} Q \tag{2.15}$$

where

$$Q = \begin{bmatrix} I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_s \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_s & 0 \end{bmatrix}, \tag{2.16}$$

and then

$$XJPX^T = \begin{bmatrix} X_+ & U \end{bmatrix} H(\Theta) \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}, \tag{2.17}$$

where

$$H(\Theta) := \begin{bmatrix} I_n & \Theta \end{bmatrix} \begin{bmatrix} R_+ & 0 & 0 & 0 \\ 0 & M & 0 & -N \\ 0 & 0 & -R_- & 0 \\ 0 & -N & 0 & -M \end{bmatrix} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix}. \tag{2.18}$$

Since $\begin{bmatrix} X_+ & U \end{bmatrix}$ is non-singular, $XJPX^T$ is non-singular or positive definite according as $H(\Theta)$ is non-singular or positive definite, respectively. In conclusion:

THEOREM 3. *Given a real self-adjoint Jordan structure (J, P), there is a semisimple, real, symmetric quadratic matrix polynomial with (J, P) as Jordan structure and non-singular (or positive definite) leading coefficient if and only if there exists an orthogonal matrix Θ such that $H(\Theta)$ of (2.18) is non-singular (resp. positive definite).*

In this case, if $\begin{bmatrix} X_+ & U \end{bmatrix} \in \mathbb{R}^{n\times n}$ is any non-singular matrix and X is defined by (2.15), then (X, J, PX^T) is a self-adjoint Jordan triple which uniquely defines a semisimple real symmetric quadratic matrix polynomial with (J, P) as Jordan structure and non-singular (resp. positive definite) leading coefficient.

The theorem implies that, once Θ is found for which $H(\Theta)$ is non-singular or positive definite, half of the eigenvectors corresponding to real eigenvalues and the real parts of the eigenvectors of the non-real eigenvalues can be freely chosen (provided that they are linearly independent). Then the other half of the real eigenvectors, and the imaginary parts of the non-real ones, are completely determined by the choice of Θ in (2.13).

This orthogonality property was also observed in [1] for general (not necessarily semisimple) complex hermitian or real symmetric quadratic matrix polynomials. However, the expression derived in [1] for the inverse of the leading coefficient is less explicit than that given in (2.8) and (2.17). Our formulation will be convenient in proving that the set of matrices $\begin{bmatrix} X_+ & U \end{bmatrix}$ and Θ such that $XJPX^T$ in (2.17) is invertible is not empty and consequently (as mentioned in [1]), is open and dense (see Section 3).

In other words, in the semisimple case, $\det XJPX^T \neq 0$ is a generic property for matrices J and P of (2.3) and (2.4). Proving this property for *general* real symmetric quadratic matrix polynomials may require a generalization of (2.17) for arbitrary Jordan matrices and sign characteristics.

2.4. Computing the sign characteristic. As introduced in Section 2.1, the sign characteristic attached to the elementary divisors associated with the real eigenvalues of self-adjoint matrix polynomials is not suitable for computations. Two other characterizations were proposed in [10] and [9] for matrices with monic and non-singular leading coefficient, respectively. For computational purposes the so-called *third* characterization is most convenient. We present a brief summary of the results in Section 3.4 of [9][1].

Let $L(\lambda)$ be an $n \times n$ self-adjoint matrix polynomial. We may see $L(\lambda)$ as a real or complex valued matrix function of the real parameter λ. The *eigenfunctions* $\mu_1(\lambda)$, $\ldots$, $\mu_n(\lambda)$ of $L(\lambda)$ are the roots of the characteristic equation of $L(\lambda)$:

$$\det(\mu I_n - L(\lambda)) = 0,$$

and they are real analytic functions of real λ. Clearly, λ_0 is an eigenvalue of $L(\lambda)$ if and only if λ_0 is a zero of $\mu_j(\lambda)$ for some $j = 1, \ldots, n$. Moreover, $\dim \mathrm{Ker} L(\lambda_0)$ is exactly the number of eigenfunctions with a zero at λ_0. The important Theorem 3.7 of [9] (see also [8, Th. 6.10] and [10, Th. 12.5] for the monic case) relates the elementary divisors of the real eigenvalues of $L(\lambda)$ and their sign characteristics to the eigenfunctions.

THEOREM 4. *Let $L(\lambda)$ be an $n \times n$ self-adjoint matrix polynomial with non-singular leading coefficient and let $\mu_1(\lambda)$, $\ldots$, $\mu_n(\lambda)$ be real analytic functions of real λ such that $\det(\mu_j(\lambda) I_n - L(\lambda)) = 0$, for $j = 1, \ldots, n$. Let $\lambda_1 < \cdots < \lambda_r$ be the different real eigenvalues of $L(\lambda)$. For every $i = 1, \ldots, r$, write $\mu_j(\lambda) = (\lambda - \lambda_i)^{m_{ij}} \nu_{ij}(\lambda)$, where $\nu_{ij}(\lambda_i) \neq 0$ is real.*

Then the non-zero numbers among $m_{i1}, \ldots, m_{in}$ are the partial multiplicities of $L(\lambda)$ associated with λ_i, and the sign of $\nu_{ij}(\lambda_i)$ (for $m_{ij} \neq 0$) is the sign characteristic associated with the elementary divisors $(\lambda - \lambda_i)^{m_{ij}}$ of $L(\lambda)$.

Here, we examine the limiting behaviour of the eigenfunctions as $\lambda \to \infty$ and as $\lambda \to 0$. These properties will lead to a simpler characterization of the sign characteristics in the semisimple case with either positive definite leading coefficient, or positive definite trailing coefficient.

2.5. Diagonalizable quadratic matrix polynomials. As mentioned in the introduction, $L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0$ is said to be *diagonalizable* over $\mathbb{C}$ or $\mathbb{R}$ if there is, respectively, a complex or real diagonal quadratic matrix polynomial with the same Jordan structure as $L(\lambda)$. The diagonalizable real and complex quadratic matrix polynomials with non-singular leading coefficient were characterized in [16]. The following results will be needed in the sequel. The first is a re-phrasing of Theorem 6 of that paper.

THEOREM 5. *Let $L(\lambda)$ be an $n \times n$ quadratic matrix polynomial over $\mathbb{C}$ with non-singular leading coefficient, let $\lambda_1, \ldots, \lambda_t \in \mathbb{C}$ be its distinct eigenvalues and for $i = 1, \ldots, t$ let the partial multiplicities of λ_i be $n_{i1} \geq \cdots \geq n_{i,\mu_{g,i}} > 0$, where $\mu_{g,i}$ is the geometric multiplicity of λ_i. Then $L(\lambda)$ is diagonalizable over $\mathbb{C}$ if and only if the following conditions hold:*

$$\sum_{i=1}^{t} \sum_{j=1}^{\mu_{g,i}} n_{ij} = 2n, \tag{2.19}$$

$$1 \leq n_{ij} \leq 2, \quad \text{for} \quad 1 \leq i \leq t, \ \ 1 \leq j \leq \mu_{g,i}, \tag{2.20}$$

and

$$\mu_{g,i} - r_i \leq n - r, \qquad i = 1, 2, \ldots, t, \tag{2.21}$$

[1]See also Section VI.5 of [8]. Appendix B of [12] provides an account of the main facts for semisimple quadratic matrix polynomials.

where r_i is the number of partial multiplicities $n_{i1}, \ldots, n_{i\mu_{g,i}}$ equal to 2, $i = 1, \ldots, t$, and $r = r_1 + \cdots + r_t$.

In contrast, for the real case we have ([16, Th. 7]):

THEOREM 6. *Let $L(\lambda)$ be an $n \times n$* **real** *quadratic matrix polynomial with non-singular leading coefficient and let $\lambda_1, \ldots, \lambda_t \in \mathbb{R}$ and $\alpha_1, \bar{\alpha}_1, \ldots, \alpha_s, \bar{\alpha}_s \in \mathbb{C} \setminus \mathbb{R}$ be its eigenvalues. For $i = 1, \ldots, t$ let $n_{i1} \geq \cdots \geq n_{i,\mu_{g,i}} > 0$ be the partial multiplicities of λ_i. Then $L(\lambda)$ is diagonalizable over $\mathbb{R}$ if and only if the partial multiplicities of α_i and $\bar{\alpha}_i$ are equal to 1, $1 \leq i \leq s$, and conditions* (2.19), (2.20), (2.21) *hold with n replaced by $n - s$.*

Thus, for a semisimple quadratic matrix polynomial it is important to recognise whether it is defined over $\mathbb{C}$ or $\mathbb{R}$. While all complex semisimple matrices are diagonalizable, this may not be the case for real matrix polynomials. For example, the following quadratic matrix polynomial (to reappear in Example 9)

$$L(\lambda) = \begin{bmatrix} 0 & -1/4 & 1/8 \\ -1/4 & 1/4 & -1/8 \\ 1/8 & -1/8 & -1/16 \end{bmatrix} \lambda^2 + \begin{bmatrix} 1/4 & -1/8 & 1/16 \\ -1/8 & -3/16 & 3/32 \\ 1/16 & 3/32 & -3/64 \end{bmatrix} \lambda + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/8 \end{bmatrix}$$

has $\lambda_1 = 0$ as a real eigenvalue with partial multiplicities $n_{11} = n_{12} = 1$, and two semsimple non-real conjugate eigenvalues $\alpha_1 = i$ $\bar{\alpha} = -i$ of multiplicity 2.

Over the complex field

$$D(\lambda) = \mathrm{Diag}[\lambda(\lambda + i), \;\; \lambda(\lambda - i), \;\; \lambda^2 + 1]$$

has the same elementary divisors as $L(\lambda)$, but there is no real diagonal quadratic matrix polynomial with the same elementary divisors as $L(\lambda)$. For example,

$$\mathrm{Diag}[\lambda^2, \;\; \lambda^2 + 1, \;\; \lambda^2 + 1]$$

has the same eigenvalues, but the partial multiplicity of $\lambda_1 = 0$ is $n_{11} = 2$. The fact is that condition (2.21) is violated: for $\lambda_1 = 0$ we have $2 - 0 = \mu_{g,i} - r_i > n - s - r = 3 - 2 - 0 = 1$.

3. Non-singular leading coefficient. Not all $2n \times 2n$ matrices in Jordan form can be the *canonical* Jordan matrix of a quadratic matrix polynomial. For example, $J = \lambda_0 I_{2n}$ is in Jordan form but there is no quadratic matrix polynomial with this matrix as its Jordan form. The reason is that the maximal geometric multiplicity of the eigenvalues of any $n \times n$ matrix polynomial is n. Thus, if $L(\lambda)$ is semisimple and p is the maximal multiplicity of any real eigenvalue, then $p \leq n$.

Now assume that $L(\lambda)$ is real and symmetric and let s be the number of conjugate pairs of non-real eigenvalues of $L(\lambda)$. If this matrix is semisimple then $n = s + q$, where q is the number of real eigenvalues of positive type (or of negative type) counting multiplicities. Consequently,

$$s \geq p - q \tag{3.1}$$

is a condition that must be satisfied by any semisimple real symmetric quadratic matrix polynomial. In particular, if all eigenvalues are distinct then $p = 1$ and (3.1) is always satisfied.

It should be noticed that $p - q$ can be either positive or negative. However:

PROPOSITION 7. *With the above notation, assume that $L(\lambda)$ is a semisimple real symmetric quadratic matrix polynomial with non-singular leading coefficient. It is diagonalizable over $\mathbb{R}[\lambda]$ if and only if $p - q \leq 0$.*

Proof. If $L(\lambda)$ is semisimple then, by definition, all partial multiplicities are equal to 1 and conditions (2.19) and (2.20) are satisfied with n replaced by $n-s$. For condition (2.21) we have $r_i = 0$ for $i = 1, \ldots, 2q$, $r = 0$, $\mu_{g,i} \le p$ for $i = 1, \ldots, 2q$ and there is an index i such that $\mu_{g,i} = p$. Thus

$$\mu_{g,i} - r_i \le n - s - r, \ i = 1, \ldots, 2q \quad \Leftrightarrow \quad p \le n - s = q.$$

By Theorem 6, $L(\lambda)$ is diagonalizable if and only if $p \le q$ as claimed. □

The basic condition (3.1) is also sufficient for the existence of semisimple real quadratic matrix polynomials with prescribed Jordan structure.

THEOREM 8. *Let $J, P \in \mathbb{R}^{2n\times 2n}$ be matrices of the form* (2.3) *and* (2.4). *Then there exists a semisimple real symmetric quadratic matrix polynomial $L(\lambda)$ for which (J, P) is an admissible real self-adjoint Jordan structure (Definition 2) if and only if condition* (3.1) *holds.*

Proof. We have already seen that (3.1) is necessary for the existence of a semisimple real symmetric quadratic matrix polynomial.

Conversely, assume that (3.1) holds and $J \in \mathbb{R}^{2n\times 2n}$ is given as in (2.3). We divide the proof into two parts according as

$$p - q \le 0 \quad \text{or} \quad p - q > 0.$$

(i) $\underline{p - q \le 0}$. If there is a real symmetric quadratic matrix polynomial with a Jordan structure (J, P) for which $p \le q$ then it is diagonalizable (Proposition 7). Let us confirm that a diagonal quadratic matrix polynomial can be constructed with the desired spectral data. First, recall that $R_+ = \mathrm{Diag}(r_1, \ldots, r_q)$ collects the eigenvalues of J of positive type, $R_- = \mathrm{Diag}(r_{q+1}, \ldots, r_{2q})$ those of negative type and $\mu_j \pm i\nu_j$, $j = 1, 3, \ldots, 2s-1$, are the non-real complex conjugate eigenvalues of J.

If $p \le q$ then it is not difficult to see that the diagonal entries of R_- can be arranged in such a way that $r_i \ne r_{q+i}$, $1 \le i \le q$. For $i = 1, \ldots, q$ put

$$\begin{array}{ll} a_i(\lambda) = (\lambda - r_i)(\lambda - r_{q+i}) & \text{if} \quad r_i > r_{q+i}, \\ a_i(\lambda) = -(\lambda - r_i)(\lambda - r_{q+i}) & \text{if} \quad r_i < r_{q+i}, \end{array}$$

and for $i = 1, \ldots, s$ put

$$a_{q+i} = \lambda^2 - 2\mu_i\lambda + \mu_i^2 + \nu_i^2.$$

It is clear that $L(\lambda) = \mathrm{Diag}(a_1(\lambda), \ldots, a_n(\lambda))$ is real symmetric, quadratic and J is a Jordan matrix for $L(\lambda)$. Now, using Theorem 4, we conclude that the sign characteristics of the real eigenvalues of $L(\lambda)$ are as prescribed.

However, a general technique can be designed that provides infinitely many quadratic matrix polynomials with (J, P) as their real self-adjoint Jordan structure.

Since the diagonal elements of R_- can be arranged in such a way that $r_i \ne r_{q+i}$, $1 \le i \le q$, there is a permutation matrix $\widehat{\Theta} \in \mathbb{R}^{q\times q}$ such that $R_+ - \widehat{\Theta} R_- \widehat{\Theta}^T$ is non-singular.

Take any $n \times n$ non-singular matrix and write it in the form $\begin{bmatrix} X_+ & U \end{bmatrix}$, where $X_+ \in \mathbb{R}^{n\times q}$ and $U \in \mathbb{R}^{n\times s}$. Define the permutation matrix

$$\Theta = \begin{bmatrix} \widehat{\Theta} & 0 \\ 0 & I_s \end{bmatrix}, \tag{3.2}$$

and a matrix X as in (2.15). Then, with P of (2.4), $XPX^T = 0$ and (cf. (2.17))

$$XJPX^T = \left[\begin{array}{cc} X_+ & U \end{array}\right] H(\Theta) \left[\begin{array}{c} X_+^T \\ U^T \end{array}\right],$$

where $H(\Theta)$ is defined in (2.18) and can be written in the form

$$H(\Theta) = \left[\begin{array}{cc} R_+ & 0 \\ 0 & M \end{array}\right] - \Theta \left[\begin{array}{cc} 0 & 0 \\ 0 & N \end{array}\right] - \left[\begin{array}{cc} 0 & 0 \\ 0 & N \end{array}\right] \Theta^T - \Theta \left[\begin{array}{cc} R_- & 0 \\ 0 & M \end{array}\right] \Theta^T, \quad (3.3)$$

where M and N are the diagonal matrices of (2.2). With Θ of (3.2),

$$H(\Theta) = \left[\begin{array}{cc} R_+ - \widehat{\Theta} R_- \widehat{\Theta}^T & 0 \\ 0 & -2N \end{array}\right].$$

But $\widehat{\Theta}$ has been chosen so that $R_+ - \widehat{\Theta} R_- \widehat{\Theta}^T$ is non-singular. Thus, $H(\Theta)$ is non-singular and, since $\left[\begin{array}{cc} X_+ & U \end{array}\right]$ is also non-singular, so is $XJPX^T$ of (2.17).

We conclude that (X, J, PX^T) is a real self-adjoint Jordan triple generating a semisimple quadratic matrix polynomial whose coefficients are given by (2.10) and, in particular, L_2 is non-singular.

(ii) $\underline{p - q > 0}$. This case is more involved because there is no *diagonal* quadratic matrix polynomial with the prescribed real self-adjoint Jordan structure (J, P) (Proposition 7).

Let λ_0 be an eigenvalue of J of highest algebraic multiplicity p and let p_+ and p_- be the number of times that λ_0 appears in the diagonal of R_+ and R_-, respectively. Let $r_1^+, \dots r_{q-p_+}^+$ and $r_1^-, \dots, r_{q-p_-}^-$ be the remaining elements of R_+ and R_-. Since $p_+ + p_- = p > q$ the diagonal elements of R_+ and R_- can be paired as follows:

$$(\lambda_0, r_1^-), \dots, (\lambda_0, r_{q-p_-}^-), (r_1^+, \lambda_0), \dots, (r_{q-p_+}^+, \lambda_0), (\lambda_0, \lambda_0), \dots, (\lambda_0, \lambda_0),$$

where (λ_0, λ_0) appears $q - (q - p_-) - (q - p_+) = p - q$ times. Thus, we can assume without lost of generality that

$$R_+ = \mathrm{Diag}(\lambda_0 I_{q-p_-}, r_1^+, \dots, r_{q-p_+}^+, \lambda_0 I_{p-q}), \quad R_- = \mathrm{Diag}(r_1^-, \dots, r_{q-p_-}^-, \lambda_0 I_{q-p_+}, \lambda_0 I_{p-q}).$$

Recalling (3.3), we aim to find an orthogonal matrix $\widetilde{\Theta}$ such that

$$H(\widetilde{\Theta}) = \left[\begin{array}{cc} R_+ & 0 \\ 0 & M \end{array}\right] - \widetilde{\Theta} \left[\begin{array}{cc} 0 & 0 \\ 0 & N \end{array}\right] - \left[\begin{array}{cc} 0 & 0 \\ 0 & N \end{array}\right] \widetilde{\Theta}^T - \widetilde{\Theta} \left[\begin{array}{cc} R_- & 0 \\ 0 & M \end{array}\right] \widetilde{\Theta}^T$$

is non-singular. Given that $\lambda_0 \neq r_i^-$ and $\lambda_0 \neq r_i^+$ it is enough to find an orthogonal matrix Θ such that

$$\widetilde{H}(\Theta) = \left[\begin{array}{cc} \lambda_0 I_{p-q} & 0 \\ 0 & M \end{array}\right] - \Theta \left[\begin{array}{cc} 0 & 0 \\ 0 & N \end{array}\right] - \left[\begin{array}{cc} 0 & 0 \\ 0 & N \end{array}\right] \Theta^T - \Theta \left[\begin{array}{cc} \lambda_0 I_{p-q} & 0 \\ 0 & M \end{array}\right] \Theta^T$$

is non-singular. For, if such a matrix Θ is found and $\widetilde{\Theta} = \mathrm{Diag}(I_{2q-p_+-p_-}, \Theta)$, then $\det H(\widetilde{\Theta}) \neq 0$.

Now condition (3.1) ($s \geq p - q$) enters the scene. Write

$$N = \mathrm{Diag}(N_1, N_2), \quad M = \mathrm{Diag}(M_1, M_2),$$

with N_1 and M_1 of size $(p-q)\times(p-q)$ and define

$$\Theta = \begin{bmatrix} 0 & I_{p-q} & 0 \\ I_{p-q} & 0 & 0 \\ 0 & 0 & I_{s-(p-q)} \end{bmatrix}. \tag{3.4}$$

Then

$$\widetilde{H}(\Theta) = \begin{bmatrix} \lambda_0 I_{p-q} - M_1 & -N_1 & 0 \\ -N_1 & M_1 - \lambda_0 I_{p-q} & 0 \\ 0 & 0 & -2N_2 \end{bmatrix}$$

Bearing in mind that the diagonal elements of N are all different from zero we conclude that the eigenvalues of $\widetilde{H}(\Theta)$ are those of $-2N_2$ and $\pm|z_j|$ where

$$z_j = (\lambda_0 - \mu_j) \pm i\nu_j, \quad j = 1,\dots,p-q.$$

Hence $\widetilde{H}(\Theta)$ is a non-singular matrix as claimed.

We complete the proof as in the first case: Take any $n\times n$ non-singular real matrix and write it as $\begin{bmatrix} X_+ & U \end{bmatrix}$, where $X_+ \in \mathbb{R}^{n\times q}$ an $U \in \mathbb{R}^{n\times s}$. Define

$$\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix}\Theta, \quad X = X(\Theta) = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix}$$

as in (2.13). Then $XPX^T = 0$ and $XJPX^T$ is non-singular. Therefore (X, J, PX^T) is a real self-adjoint Jordan triple that generates a unique semisimple real symmetric quadratic matrix polynomial whose coefficients are given by (2.10) or, alternatively, $L_2 = (XJPX^T)^{-1}$ and L_1 and L_0 are given by (2.12). □

Let us illustrate these techniques with two examples:

Example 9. : (a) Assume that

$$J = \text{Diag}\left(-2, -1, \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}\right), \qquad P = \text{Diag}(1, -1, -1, 1)$$

(with the structure of (2.3) and (2.4)) and we are to find a quadratic matrix polynomial with (J, P) as the real self-adjoint Jordan structure. In other words, we are to construct a real, symmetric and quadratic $L(\lambda)$ with eigenvalues $r_1 = -2$, $r_2 = -1$, $\beta_1 = -2 + i$ and $\bar{\beta}_1 = -2 - i$ and such that the sign characteristic of $r_1 = -2$ is $+1$ and that of $r_2 = -1$ is -1.

Since there are no repeated real eigenvalues we can follow the proof of Theorem 8: $R_+ = [-2]$, $R_- = [-1]$ and $\widehat{\Theta} = [1]$. Thus $\Theta = I_2$ and

$$H(\Theta) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Now let $\begin{bmatrix} X_+ & U \end{bmatrix}$ be *any* invertible matrix; say

$$\begin{bmatrix} X_+ & U \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

We use (2.10) to compute coefficients L_2 and L_1. Thus,

$$P_1(X) = \begin{bmatrix} X_+ & U \end{bmatrix} H(\Theta) \begin{bmatrix} X_+^T \\ U^T \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ -4 & -8 \end{bmatrix}$$

and

$$L_2 = P_1(X)^{-1} = \begin{bmatrix} -1/4 & 1/8 \\ 1/8 & -3/16 \end{bmatrix}.$$

Now we compute L_1. First,

$$P_2(X) = XJ^2PX^T = \begin{bmatrix} 20 & 16 \\ 16 & 32 \end{bmatrix}$$

and so (2.10) gives

$$L_1 = -L_2P_2(X)L_2 = \begin{bmatrix} -3/4 & 3/8 \\ 3/8 & -11/16 \end{bmatrix}.$$

Finally, since 0 is not a prescribed eigenvalue, (2.11) gives $L_0 = (-P_{-1}(X))^{-1}$ where

$$P_{-1}(X) = XJ^{-1}PX^T = \begin{bmatrix} 12/5 & 4/5 \\ 4/5 & 8/5 \end{bmatrix}$$

Thus

$$L_0 = -P_{-1}(X)^{-1} = \begin{bmatrix} -1/2 & 1/4 \\ 1/4 & -3/4 \end{bmatrix}$$

and therefore

$$L(\lambda) = \begin{bmatrix} -1/4 & 1/8 \\ 1/8 & -3/16 \end{bmatrix}\lambda^2 + \begin{bmatrix} -3/4 & 3/8 \\ 3/8 & -11/16 \end{bmatrix}\lambda + \begin{bmatrix} -1/2 & 1/4 \\ 1/4 & -3/4 \end{bmatrix}.$$

It is easily seen that, as prescribed, the eigenvalues of $L(\lambda)$ are -1, -2, $-2+i$ and $-2-i$. The sign characteristics of the real eigenvalues can be computed using Theorem 4. It is found that the sign characteristic of -1 is -1 and that of -2 is $+1$, as desired.

Notice that $\Theta = I_2$ and so

$$X = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

is a matrix of eigenvectors of $L(\lambda)$. Thus, we have determined a real symmetric quadratic matrix polynomial with prescribed real self-adjoint Jordan structure and, by choosing $\begin{bmatrix} X_+ & U \end{bmatrix}$ and Θ, a matrix of eigenvectors for that matrix. □

(b) Assume that

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad P = \mathrm{Diag}(1,-1,-1,-1,1,1) \tag{3.5}$$

so that the prescribed elementary divisors are λ, λ, λ^2+1 and λ^2+1 and the sign characteristic associated with the eigenvalue 0 is both $+1$ and -1. According to Theorem 6 (see the example following this theorem) there is no diagonal quadratic matrix polynomial with these elementary divisors. Taking into account that $s = 2 > p - q = 1$ there is a real symmetric quadratic matrix polynomial with (J, P) as self-adjoint Jordan structure, but that matrix polynomial cannot be diagonal. Let us apply the technique developed in the proof of Theorem 8 to find one such matrix polynomial.

As $p-q = 1$ and $s-(p-q) = 1$ matrix Θ of (3.4) is $\Theta = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and then (3.3) gives

$$H(\Theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \Theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Theta^T - \Theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Theta^T,$$

so that

$$H(\Theta) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

which is non-singular. If, for example

$$\begin{bmatrix} X_+ & U \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

then

$$P_1(X) = \begin{bmatrix} -4 & -4 & 0 \\ -4 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix} \quad \text{and} \quad L_2 = P_1(X)^{-1} = \begin{bmatrix} 0 & -1/4 & 1/8 \\ -1/4 & 1/4 & -1/8 \\ 1/8 & -1/8 & -1/16 \end{bmatrix}.$$

As 0 is a prescribed eigenvalue we use (2.12) to compute coefficients L_1 and L_0. First,

$$\begin{bmatrix} X_+ & U \end{bmatrix} \begin{bmatrix} I_3 & \Theta \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} X_+ & U & X_- & V \end{bmatrix}.$$

Then

$$X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \end{bmatrix},$$

and

$$\begin{bmatrix} L_0 & L_1 \end{bmatrix} = -L_2 X J^2 \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/4 & -1/8 & 1/16 \\ 0 & 0 & 0 & -1/8 & -3/16 & 3/32 \\ 0 & 0 & -1/8 & 1/16 & 3/32 & -3/64 \end{bmatrix}.$$

In conclusion,

$$L(\lambda) = \begin{bmatrix} 0 & -1/4 & 1/8 \\ -1/4 & 1/4 & -1/8 \\ 1/8 & -1/8 & -1/16 \end{bmatrix} \lambda^2 + \begin{bmatrix} 1/4 & -1/8 & 1/16 \\ -1/8 & -3/16 & 3/32 \\ 1/16 & 3/32 & -3/64 \end{bmatrix} \lambda + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/8 \end{bmatrix}$$

is a real symmetric quadratic matrix polynomial with the prescribed self-adjoint Jordan structure (J, P) of (3.5). □

3.1. Constructing infinitely many symmetric quadratic matrix polynomials. We have seen in Theorem 8 that, given J and P of the form (2.3) and (2.4) and provided that the basic condition (3.1) is satisfied, there is always a full rank matrix X such that $\det(XJPX^T) \neq 0$. This matrix X is a matrix of eigenvectors for the unique semisimple real symmetric quadratic matrix polynomial having (X, J, PX^T) as self-adjoint Jordan triple. It is obtained from an arbitrary non-singular matrix $X_1 = \begin{bmatrix} X_+ & U \end{bmatrix}$ and an orthogonal matrix Θ for which $\det H(\Theta) \neq 0$, and $H(\Theta)$ is defined by (2.18). As announced in [1], $\det(XJPX^T) \neq 0$ for "almost all" matrices X provided that condition (3.1) is satisfied. This can be explained as follows: Let $\mathcal{S}$ be the set of orthogonal matrices Θ such that $\det H(\Theta) \neq 0$ and let $\mathrm{Gl}_n(\mathbb{R})$ denote the set of $n \times n$ real invertible matrices. Any matrix $X \in \mathbb{R}^{n\times 2n}$ obtained by formula (2.15) from matrices $X_1 = \begin{bmatrix} X_+ & U \end{bmatrix} \in \mathrm{Gl}_n(\mathbb{R})$ and $\Theta \in \mathcal{S}$ satisfies $\det(XJPX^T) \neq 0$. Now, $\mathrm{Gl}_n(\mathbb{R})$ and $\mathcal{S}$ are open and dense sets in $\mathbb{R}^{n\times n}$ and $\mathcal{O}_n$ (the set of $n \times n$ real orthogonal matrices), respectively. Thus, "almost" all matrices $X_1 \in \mathbb{R}^{n\times n}$ and $\Theta \in \mathcal{O}_n$ produce matrices X such that $\det(XJPX^T) \neq 0$. Hence, a procedure of constructing infinitely many semisimple real symmetric quadratic matrix polynomials with prescribed spectral data (J, P) is as follows:

1. Let $X_1 \in \mathrm{Gl}_n(\mathbb{R})$, $\Theta \in \mathcal{S}$, and split X_1 into submatrices $X_1 = \begin{bmatrix} X_+ & U \end{bmatrix}$ with $X_+ \in \mathbb{R}^{n\times q}$ and $U \in \mathbb{R}^{n\times s}$ (recall that $2q$ and $2s$ are the number of real and non-real eigenvalues, respectively).
2. With X_1 and Θ construct a full rank matrix X as in (2.13).
3. Define $L_2 = (XJPX^T)^{-1}$ and L_1 and L_0 as in (2.12).

Then the matrix polynomial $L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0$ is semisimple, real and symmetric with (J, P) as Jordan structure (eigenvalues and sign characteristic). The matrix X is a matrix of eigenvectors for $L(\lambda)$.

EXAMPLE 10. Consider the following spectral data:

Elementary divisors	$\lambda - 1$	$\lambda - 1$	$\lambda - 1$	$\lambda + 1$	$\lambda^2 + \lambda + 1$
Sign characteristic	$+1$	-1	-1	$+1$	

The corresponding Jordan structure is:

$$J = \mathrm{Diag}\left(1, -1, 1, 1, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) \qquad P = \mathrm{Diag}(1, 1, -1, -1, -1, 1).$$

Define (cf. (2.16))

$$Q = \begin{bmatrix} I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_s \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_s & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

There is *no diagonal* quadratic matrix polynomial with this Jordan structure. However, for almost any $X_1 = \begin{bmatrix} X_+ & U \end{bmatrix} \in \mathbb{R}^{n\times n}$ and $\Theta \in \mathcal{O}_n$ we can use (2.15) to construct a real self-adjoint Jordan triple that defines a unique real symmetric quadratic matrix polynomial with the desired Jordan structure.

This technique can be implemented numerically using (2.12) and MATLAB code to obtain coefficients L_0 and L_1. An example follows where X_1 is chosen as a random matrix (along this paper random matrix will mean a matrix containing pseudorandom values drawn from the standard normal distribution) and Θ is the Q-factor of a QR factorization of a random matrix:

```
>> X1=randn(3); Th=randn(3); [Th,R]=qr(Th); X=X1*[eye(3) Th]*Q;
>> L2=(X*J*P*X')^(-1),L=-L2*X*J^2*[X;X*J]^(-1); L1=L(:,4:6), L0=L(:,1:3)
L2 =                                          L1 =
  -44.4598   21.9310   25.6340                   72.2346  -27.1021  -49.2375
   21.9310  -12.1663  -13.0700                  -27.1021    9.7963   19.6616
   25.6340  -13.0700  -13.2789                  -49.2375   19.6616   29.7717
L0 =
  -27.7748    5.1711   23.6035
    5.1711    2.3700   -6.5916
   23.6035   -6.5916  -16.4928
```

4. Positive definite leading coefficient.

The real self-adjoint Jordan structures prescribed in the two cases of Example 9 are admissible but the leading coefficients of the real symmetric quadratic matrix polynomials with that Jordan structure cannot be positive definite. In fact, in the *first case* the sign characteristics assigned to the real eigenvalues -1 and -2 are, respectively, -1 and $+1$ and, for all matrix polynomials of even degree with positive definite leading coefficient, the largest and smallest real eigenvalues must be of positive and negative type, respectively. In the *second case* of Example 9, the prescribed spectral data satisfies this condition but, as we will see in a moment, *all semisimple real symmetric quadratic matrix polynomials with positive definite leading coefficient are diagonalizable* and there is no diagonal real quadratic matrix polynomial with λ, λ, λ^2+1 and λ^2+1 as elementary divisors. This diagonalizability property follows from careful analysis of the sign characteristics admissible for semisimple self-adjoint matrix polynomials; the topic of the next section.

4.1. Distribution of characteristic signs; general degree.

We now examine the distribution of the real eigenvalues of $L(\lambda)$ with respect to their sign characteristics. For this purpose we quote two results from [18] concerning polynomials of general degree, ℓ.

THEOREM 11. *Let $L(\lambda)$ be an $n\times n$ semisimple symmetric matrix polynomial with $L_\ell > 0$ and maximal and minimal real eigenvalues $\lambda_{\max}$ and $\lambda_{\min}$, respectively. For any $\alpha \le \lambda_{\max}$, let $p(\alpha)$ denote the number of real eigenvalues (counting multiplicities) of $L(\lambda)$ of positive type in $(\alpha,+\infty)$ and $n(\alpha)$ the number of real eigenvalues (counting multiplicites) of $L(\lambda)$ of negative type in $[\alpha,+\infty)$. Then $n(\alpha) \le p(\alpha)$ for all $\alpha \in [\lambda_{\min}, \lambda_{\max}]$.*

This theorem implies that, if $L_\ell > 0$ then, for each real eigenvalue of negative type, there is at least one larger real eigenvalue of positive type.

THEOREM 12. *Let $L(\lambda)$ be an $n \times n$ semisimple symmetric matrix polynomial with $L_0 > 0$ and maximal and minimal real eigenvalues $\lambda_{\max}$ and $\lambda_{\min}$, respectively. For $\alpha < 0$ let $p_-(\alpha)$ denote the number of real eigenvalues (counting multiplicities) of $L(\lambda)$ of positive type in $(\alpha,0]$ and $n_-(\alpha)$ the number of real eigenvalues (counting multiplicites) of $L(\lambda)$ of negative type in $[\alpha,0)$. For $\alpha > 0$ let $p_+(\alpha)$ denote the number of real eigenvalues (counting multiplicities) of $L(\lambda)$ of positive type in $(0,\alpha]$ and $n_+(\alpha)$ the number of real eigenvalues (counting multiplicites) of $L(\lambda)$ of negative type in $[0,\alpha)$. Then $n_-(\alpha) \le p_-(\alpha)$ for all $\alpha \in [\lambda_{\min}, 0)$ and $n_+(\alpha) \ge p_+(\alpha)$ for all $\alpha \in (0, \lambda_{\max}]$.*

We also note that, when $L_\ell > 0$, the largest real eigenvalue (if any) has positive type. Similarly, if $L_0 > 0$ and λ_z is the positive eigenvalue of $L(\lambda)$ closest to zero (provided that $L(\lambda)$ has a positive real eigenvalue) then it must be of negative type because $n_+(\lambda_z) \ge p_+(\lambda_z)$.

4.2. The ISQEP with positive definite leading coefficient. Now we have a condition that all semisimple real symmetric quadratic matrix polynomials with positive leading coefficient must satisfy:

$$n(\alpha) \leq p(\alpha) \quad \text{for all } \alpha \in [\lambda_{\min}, \lambda_{\max}]. \tag{4.1}$$

On the other hand, we recall that (Proposition 7)

$$p - q \leq 0, \tag{4.2}$$

is a necessary and sufficient condition for a semisimple real symmetric quadratic matrix polynomial be diagonalizable. It turns out that condition (4.1) implies (4.2). Thus, (4.1) is sufficient for the existence of a semisimple real symmetric quadratic matrix polynomials with positive definite leading coefficient. All this will be shown in the next Theorem. Notice first that if $p - q \leq 0$ then condition (3.1) is automatically satisfied.

THEOREM 13. *Let $J, P \in \mathbb{R}^{2n\times 2n}$ be matrices of the form* (2.3) *and* (2.4). *Then there exists a semisimple real symmetric quadratic matrix polynomial $L(\lambda)$ with (J, P) as real self-adjoint Jordan structure and positive definite leading coefficient if and only if condition* (4.1) *holds. In particular, all these quadratic matrix polynomials are diagonalizable.*

Proof. The necessity of condition (4.1) is already established. Let us show that it is also sufficient for the construction of a monic diagonal quadratic matrix polynomial with (J, P) as Jordan structure. Let $\lambda_1,\dots, \lambda_{2n}$ be the prescribed eigenvalues and assume, without lost of generality, that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2q}$ are the real eigenvalues and $\lambda_{2q+2j-1} = \mu_j + i\nu_j$, $\lambda_{2q+2j} = \mu_j - i\nu_j$, $j = 1,\dots, s$ with $\nu_j > 0$. Recall that q real eigenvalues are of positive type and q of negative type.

Condition (4.1) guarantees that the real eigenvalues of $L(\lambda)$ can be split into two groups: $\lambda_{i_1} \geq \cdots \geq \lambda_{i_q}$ of positive type, and $\lambda_{j_1} \geq \cdots \geq \lambda_{j_q}$ of negative type, with $\lambda_{i_k} > \lambda_{j_k}$, $k = 1,\dots, q$. This implies that $q \geq p$ and (4.2) follows.

Now define

$$\begin{aligned} a_k(\lambda) &= (\lambda - \lambda_{i_k})(\lambda - \lambda_{j_k}), && k = 1,\dots, q \\ a_{q+k}(\lambda) &= \lambda^2 - 2\mu_k\lambda + \mu_k^2 + \nu_k^2, && k = 1,\dots, s \end{aligned} \tag{4.3}$$

and then $L(\lambda) = \mathrm{Diag}[a_1(\lambda),\dots, a_n(\lambda)]$. This matrix is a monic semisimple real symmetric quadratic matrix polynomial with the desired eigenvalues. Using Theotem 4, it is easily seen that $L(\lambda)$ also has the prescribed sign characteristic. □

4.3. Positive definite trailing coefficient. Theorem 12 provides two conditions that the sign characteristic of any semisimple real symmetric quadratic matrix polynomial with positive definite trailing coefficient must satisfy, namely,

$$n_-(\alpha) \leq p_-(\alpha), \quad \text{for all} \quad \alpha \in [\lambda_{\min}, 0) \tag{4.4}$$

and

$$n_+(\alpha) \geq p_+(\alpha), \quad \text{for all} \quad \alpha \in (0, \lambda_{\max}]. \tag{4.5}$$

As in the case of positive definite leading coefficient, these conditions are also sufficient for the existence of a diagonal real quadratic matrix polynomial with positive definite trailing coefficient.

Theorem 14. *Let $J, P \in \mathbb{R}^{2n\times 2n}$ be matrices of the form* (2.3) *and* (2.4)*. Then there exists a semisimple real symmetric quadratic matrix polynomial $L(\lambda)$ with (J, P) as Jordan structure and positive definite trailing coefficient if and only if* (4.4) *and* (4.5) *hold.*

Proof. Let

$$\lambda_1 \geq \cdots \geq \lambda_f > 0 > \lambda_{f+1} \geq \cdots \geq \lambda_{2q}$$

be the real eigenvalues of J, and let $\{i_1, \ldots, i_r\} \subseteq \{f+1, \ldots, 2q\}$ be the set of indices such that λ_{i_β} is of negative type for $\beta = 1, \ldots, r$. By (4.4) there are indices $\{j_1, \ldots, j_r\} \subseteq \{f+1, \ldots, 2q\}$ such that, for $\beta = 1, \ldots, r$, λ_{j_β} is of positive type and $\lambda_{j_\beta} > \lambda_{i_\beta}$. The remaining real negative eigenvalues are of positive type. That is to say, if $\{k_1, \ldots, k_{2q-f-2r}\} = \{f+1, \ldots, 2q\} \setminus \{i_1, \ldots, i_r, j_1, \ldots, j_r\}$ then λ_{k_β} is of positive type for $\beta = 1, \ldots, 2q - f - 2r$.

Similarly, let $\{u_1, \ldots, u_t\} \subseteq \{1, \ldots, f\}$ be the set of indices such that λ_{u_β} is of positive type. It follows from condition (4.5) that there is a set of indices $\{v_1, \ldots, v_t\} \subseteq \{1, \ldots, f\}$ such that, for $\beta = 1, \ldots, t$, λ_{v_β} is of negative type and $\lambda_{v_\beta} < \lambda_{u_\beta}$. Moreover, if $\{w_1, \ldots, w_{f-2t}\} = \{1, \ldots, f\} \setminus \{u_1, \ldots, u_t, v_1, \ldots, v_t\}$ then λ_{w_β} is of negative type for $\beta = 1, \ldots, f - 2t$.

Since the number of prescribed eigenvalues of positive type equals the number of negative type, we conclude that $2q - f - 2r = f - 2t$. Let h denote this number and let $\lambda_{2q+1}, \ldots, \lambda_{2q+2s}$ be the non-real eigenvalues of J - in conjugate pairs. Assume, without loss of generality, that $\lambda_{2q+2j-1} = \mu_j + i\nu_j$ and $\lambda_{2q+2j} = \mu_j - i\nu_j$, $j = 1, \ldots, s$, and define

$$\begin{array}{ll} a_\beta = (\lambda - \lambda_{j_\beta})(\lambda - \lambda_{i_\beta}), & \beta = 1, \ldots, r, \\ a_\beta = (\lambda - \lambda_{u_\beta})(\lambda - \lambda_{v_\beta}), & \beta = 1, \ldots, t, \\ a_\beta = -(\lambda - \lambda_{k_\beta})(\lambda - \lambda_{w_\beta}), & \beta = 1, \ldots, h, \\ a_\beta = \lambda^2 - 2\mu_\beta\lambda + \mu_\beta^2 + \nu_\beta^2, & \beta = 1, \ldots, s, \end{array} \tag{4.6}$$

and $L(\lambda) = \mathrm{Diag}[a_1(\lambda), \ldots, a_n(\lambda)]$. Clearly, this is a semisimple real symmetric quadratic matrix polynomial with the desired eigenvalues. Using Theotem 4, it is easily verified that $L(\lambda)$ also has the desired sign characteristic. □

By combining the previous results we can provide a characterization of the sign characteristics of all semisimple real symmetric quadratic matrix polynomials with both positive definite leading and trailing coefficients:

Theorem 15. *Let $J, P \in \mathbb{R}^{2n\times 2n}$ be matrices of the form* (2.3) *and* (2.4)*. Then there exists a semisimple real symmetric quadratic matrix polynomial $L(\lambda)$ with (J, P) as Jordan structure, positive definite leading coefficient and positive semidefinite trailing coefficient if and only if conditions* (4.4), (4.5)*, and the following conditions hold:*

$$n_-(\alpha) = p_-(\alpha) \ \textit{for} \ \alpha < \lambda_{\min} \quad \textit{and} \quad n_+(\alpha) = p_+(\alpha) \ \textit{for} \ \alpha > \lambda_{\max}. \tag{4.7}$$

Proof. We already know that conditions (4.4) and (4.5) are necessary. The necessity of (4.7) follows from Theorem 6 of [18].

In order to prove the sufficiency of (4.4), (4.5) and (4.7), notice that the first two conditions allow us to define n quadratic polynomials $a_i(\lambda)$ as in (4.6). But it follows from (4.7) that $h = 0$. Hence $L(\lambda) = \mathrm{Diag}(a_1(\lambda), \ldots, a_n(\lambda))$ is monic and its trailing coefficient is diagonal with positive elements. □

5. Quadratic matrix polynomials with $L_0, L_2 > 0$. The matrix polynomials constructed in the main theorems of the previous sections are diagonal matrices. There is however a procedure that, starting from those diagonal matrices, allows us to construct families of non-diagonal real symmetric quadratic matrix polynomials with the same Jordan structure (J, P) and different eigenvectors.

5.1. The case $L_2 > 0$. Let us start with the case when the leading coefficient is prescribed to be positive definite The procedure to construct broad families of symmetric, quadratic matrix polynomials with $L_2 > 0$ and prescribed spectral data is based on the following observation: Assume that (J, P) is a given Jordan structure that satisfies condition (4.1), and let

$$\mathcal{P} = \{\Theta \in \mathcal{O}_n | H(\Theta) \text{ is positive definite}\}$$

where, as before, $\mathcal{O}_n$ is the orthogonal group of order n and $H(\Theta)$ is the matrix of (2.18) for the given J and P. Since these two matrices are fixed, the eigenvalues of $H(\Theta)$ depend continuously on Θ and so $\mathcal{P}$ is an open set of $\mathcal{O}_n$ with the usual relative topology. In order to obtain matrices of $\mathcal{P}$ for a given Jordan structure (J, P), we can use the proof of Theorem 13 to construct a monic diagonal real symmetric quadratic matrix polynomial with that Jordan structure. We will see that from that diagonal matrix we can extract an orthogonal matrix Θ such that $H(\Theta) > 0$. Since $\mathcal{P}$ is open, all matrices in a small enough neighborhood of Θ in $\mathcal{O}_n$ will be in $\mathcal{P}$. Any of those matrices will allow us to construct a quadratic matrix polynomial $L(\lambda)$ which is real, symmetric and with (J, P) as spectral data.

Let $D(\lambda)$ be a monic diagonal matrix with (J, P) as spectral data obtained by the procedure described in the proof of Theorem 13. Using (2.10) we know that, for $D(\lambda)$, there is a matrix of eigenvectors X such that $XJPX^T = I_n$. We also know (see (2.18)) that $XJPX^T = X_1 H(\Theta) X_1^T$ where X_1 is the submatrix of X corresponding to the eigenvectors of eigenvalues of positive type and the real parts of the eigenvectors of non-real complex conjugate eigenvalues, and $H(\Theta)$ is given by (3.3).

Let us show explicit matrices Θ and X_1 such that $I_n = X_1 H(\Theta) X_1^T$: with the notation of the proof of Theorem 13 and (4.3), we define $R_+ = \mathrm{Diag}(\lambda_{i_1}, \ldots, \lambda_{i_q})$, $R_- = \mathrm{Diag}(\lambda_{j_1}, \ldots, \lambda_{j_q})$ and

$$\Theta = \mathrm{Diag}(I_q, -I_s).$$

This is an orthogonal matrix and the corresponding $H(\Theta)$ is diagonal and positive definite. Let $H(\Theta) = \mathrm{Diag}(h_1, \ldots, h_n)$ and take

$$X_1 = \mathrm{Diag}\left(\frac{1}{\sqrt{h_1}}, \ldots, \frac{1}{\sqrt{h_n}}\right),$$

then $X_1 H(\Theta) X_1^T = I_n$. If we put $X_1 = \begin{bmatrix} X_+ & U \end{bmatrix}$ and use (2.15) to define X, then (X, J, PX^T) is a real self-adjoint Jordan triple that uniquely defines a real symmetric quadratic matrix polynomial with (J, P) as Jordan structure and monic leading coefficient. It can be seen that, actually, (X, J, PX^T) is a real self-adjoint Jordan triple of $D(\lambda)$ and so, X is a basic matrix of eigenvectors for $D(\lambda)$.

Now, since $\mathcal{P}$ is open, any orthogonal matrix $\widetilde{\Theta}$ close enough to $\Theta = \mathrm{Diag}(I_q, -I_s)$ will be in $\mathcal{P}$; that is to say, $H(\widetilde{\Theta}) > 0$. Take any non-singular matrix $\widetilde{X_1}$ and use (2.15)- (2.16) to construct $\widetilde{X} = \widetilde{X_1} \begin{bmatrix} I_{q+s} & \widetilde{\Theta} \end{bmatrix} Q$. By (2.17), $\widetilde{X} JP \widetilde{X}^T$ is positive definite and so is $L_2 = (\widetilde{X} JP \widetilde{X}^T)^{-1}$. Furthermore, $(\widetilde{X}, J, P\widetilde{X}^T)$ is a real self-adjoint

Jordan triple of a real symmetric quadratic matrix polynomial with positive definite leading coefficient L_2. The remaining coefficients can be obtained via (2.10) or (2.12).

There is still a point that should be clarified: How do we obtain an orthogonal matrix $\widetilde{\Theta}$ close enough to Θ so that $H(\widetilde{\Theta}) > 0$?

In practice, one can proceed as follows: For any given matrix $A \in \mathbb{R}^{n\times n}$, the Gram-Schmidt orthogonalization process produces an $n \times n$ orthonormal matrix Q and an $n \times n$ upper triangular matrix R such that $A = QR$ (a QR factorization of A). It turns out that the Q and R depend continuously on A. Furthermore, this orthogonalization process can be implemented through Householder or Givens transformations. Then, if $A = \Theta + E$ is a sufficiently small perturbation of $\Theta = \mathrm{Diag}(I_q, -I_s)$, and $A = \widetilde{\Theta}R$ is a QR factorization of A then $\widetilde{\Theta}$ is in $\mathcal{P}$. Notice that E can be taken such that $\Theta + E$ is invertible and so its QR factorization is unique up to the signs in the diagonal elements of R. This means that the QR factorization of $\Theta + E$ must be implemented in such a way that the signs of the diagonal elements of $\widetilde{\Theta}$ and Θ coincide in order to be close to one-another.

In summary:

1. For the given admissible positive spectral data (J, P), reorder the diagonal elements in R_+ and R_- in such a way that $R_+ - R_- > 0$. According to Theorem 13 this is always possible provided that the prescribed Jordan structure satisfies condition (4.1).
2. Put $\Theta = \mathrm{Diag}(I_q, -I_s)$ and take any small matrix E such that $\Theta + E$ is invertible and if $\Theta + E = \widetilde{\Theta}R$ is a QR factorization of $\Theta + E$ then $H(\widetilde{\Theta}) > 0$ and $\widetilde{\Theta}$ is close enough to Θ to ensure that $\widetilde{\Theta} \in \mathcal{P}$.
3. Take any $n \times n$ non-singular matrix X_1 and define $X = X_1 \begin{bmatrix} I_{q+s} & \widetilde{\Theta} \end{bmatrix} Q$, where Q is the matrix of (2.16).
4. Define $L_2 = (XJPX^T)^{-1}$ and obtain L_1 and L_0 using (2.10) or (2.12).

Item 2 above is optional in the sense that with $\Theta = \mathrm{Diag}(I_q, -I_s)$, X_1 arbitrary and X as in item 3 (with $\widetilde{\Theta}$ replaced by Θ), (X, J, PX^T) is, in general, a real self-adjoint Jordan triple of a non-diagonal real symmetric quadratic matrix polynomial. We illustrate this procedure in the following example.

Example 16. Consider the problem of producing real symmetric quadratic matrix polynomials with positive definite leading coefficient, and the following spectral data:

Elementary divisors	$\lambda - 1$	$\lambda - 1$	λ	$\lambda + 2$	$\lambda^2 + 1$
Sign characteristic	$+1$	$+1$	-1	-1	

This spectral data is consistent with (4.1), so there exist 3×3 real symmetric quadratic matrix polynomials with positive leading coefficient and this spectral data. Using Theorem 13, a monic diagonal matrix polynomial with these properties is

$$D(\lambda) = \mathrm{Diag}[(\lambda - 1)(\lambda + 2), (\lambda - 1)\lambda, \lambda^2 + 1]$$

The Jordan structure corresponding to $D(\lambda)$ is

$$J = \mathrm{Diag}\left(1, 1, -2, 0, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right), \qquad P = \mathrm{Diag}(1, 1, -1, -1, -1, 1).$$

Then

$$H(\Theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \Theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Theta^T - \Theta \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Theta^T$$

and taking

$$\Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad X_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

we have

$$X_1 H(\Theta) X_1^T = I_3.$$

We can confirm that, with this selection of Θ and X_1, and with X as in item 3, (X, J, PX^T) is a real self-adjoint triple of $D(\lambda)$:

$$X = X_1 \begin{bmatrix} I_3 & \Theta \end{bmatrix} Q = \begin{bmatrix} 1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$L_2 = (XJPX^T)^{-1} = I_3, \quad \begin{bmatrix} L_0 & L_1 \end{bmatrix} = -L_2 X J^2 \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We implement now the procedure described above with a random matrix X_1 and a random small perturbation of Θ. Recall that we must check whether the original Θ and the perturbed $\widetilde{\Theta}$ have the same diagonal signs and modify it accordingly.

```
>> Tht=Th+randn(3)*10^(-5); [Tht,R]=qr(Tht); Tht
Tht =
  -0.999999999996505  -0.000001773746785   0.000001960521798
   0.000001773742907  -0.999999999996470  -0.000001978132593
   0.000001960525307  -0.000001978129115   0.999999999996122
>> Tht=Tht*diag([ -1 -1 -1]); norm(Th-Tht)
ans =
     3.301942618213596e-06
>> X1=randn(3);X=X1*[eye(3) Tht]*Q;
>> L2=(X*J*P*X')^(-1), L=-L2*X*J^2*[X;X*J]^(-1); L1=L(:,4:6),L0=L(:,1:3)
L2 =
   0.564734314916456  -0.092862337573416   0.013582910394543
  -0.092862337573416   0.331961656422359   0.026598490231148
   0.013582910394543   0.026598490231148   0.082894057875177
L1 =
   0.017935233658507  -0.062356775774305  -0.040710761385933
  -0.062356775774305  -0.255953801201674   0.054644110708787
  -0.040710761385933   0.054644110708787   0.076435459806833
L0 =
   0.491427835797938  -0.054563569267843   0.140172380028954
  -0.054563569267843  -0.035035058372216  -0.103321403943779
   0.140172380028954  -0.103321403943779  -0.147432025209495
```

Thus, a non-diagonal real symmetric quadratic matrix polynomials is obtained whose leading coefficient, L_2, is positive definite (this can be seen by computing its eigenvalues, for example). Many quadratic matrix polynomials (real and symmetric) can be constructed by different choices of $\widetilde{\Theta}$ and/or X_1; all of them with same prescribed spectral data and positive definite leading coefficient. □

5.2. The case $L_2 > 0$ and $L_0 > 0$. The prescribed matrices J and P in the previous example do not satisfy conditions (4.4) and (4.5) and so L_0 cannot be constructed to be positive semidefinite. It will be shown that, given the hypotheses of Theorem 15, a procedure similar to that of the previous section, can be designed to produce a family of semisimple symmetric quadratic matrix polynomials with both $L_2 > 0$ and $L_0 > 0$, and with the same prescribed spectral data.

First of all, we extend Theorem 3 to cover the case when the trailing coefficient is also positive definite. Recall that if (X, J, PX^T) is a self-adjoint Jordan triple and $0 \notin \sigma(L)$, then this triple defines a unique self-adjoint matrix polynomial, $L(\lambda)$, with trailing coefficient (cf. (2.11))

$$L_0 = -P_1(X)^{-1} = -(XJ^{-1}PX^T)^{-1}.$$

The inverse of J in (2.3) is

$$J^{-1} = \mathrm{Diag}\left(R_+^{-1}, R_-^{-1}, \begin{bmatrix} \widetilde{M} & \widetilde{N} \\ -\widetilde{N} & \widetilde{M} \end{bmatrix}\right)$$

where (with the notation of Section 2.1)

$$\widetilde{M} = \mathrm{Diag}\left(\frac{\mu_1}{|\beta_1|^2}, \ldots, \frac{\mu_{2s-1}}{|\beta_{2s-1}|^2}\right), \quad \widetilde{N} = \mathrm{Diag}\left(\frac{\nu_1}{|\beta_1|^2}, \ldots, \frac{\nu_{2s-1}}{|\beta_{2s-1}|^2}\right).$$

For any $n \times n$ orthogonal matrix Θ, define

$$\begin{aligned} H_0(\Theta) &= \begin{bmatrix} I_n & \Theta \end{bmatrix} \begin{bmatrix} R_+^{-1} & 0 & 0 & 0 \\ 0 & \widetilde{M} & 0 & \widetilde{N} \\ 0 & 0 & -R_-^{-1} & 0 \\ 0 & \widetilde{N} & 0 & -\widetilde{M} \end{bmatrix} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix} \\ &= \begin{bmatrix} R_+^{-1} & 0 \\ 0 & \widetilde{M} \end{bmatrix} + \Theta \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{N} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{N} \end{bmatrix} \Theta^T - \Theta \begin{bmatrix} R_-^{-1} & 0 \\ 0 & \widetilde{M} \end{bmatrix} \Theta^T. \end{aligned} \tag{5.1}$$

Then we have:

THEOREM 17. *Given a real self-adjoint Jordan structure (J, P), there is a semisimple, real, symmetric quadratic matrix polynomial with (J, P) as Jordan structure and positive definite leading and trailing coefficients if and only if 0 is not an eigenvalue of J and there exists an orthogonal matrix Θ such that $H(\Theta)$ of (2.18) and $H_0(\Theta)$ of (5.1) are positive and negative definite, respectively.*

In this case, if $X_1 \in \mathbb{R}^{n\times n}$ is any non-singular matrix and X is defined by (2.15), then (X, J, PX^T) is a self-adjoint Jordan triple which uniquely defines a semisimple real symmetric quadratic matrix polynomials with (J, P) as Jordan structure and positive definite leading and trailing coefficients.

Proof. Assume that $L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0$ is a real symmetric matrix polynomial with (J, P) as Jordan structure and $L_0, L_2 > 0$. Then $0 \notin \sigma(L) = \sigma(J)$ and there is a full rank matrix $X \in \mathbb{R}^{n\times 2n}$ such that (X, J, PX^T) is a self-adjoint Jordan triple of $L(\lambda)$, $XJPX^T > 0$ and $XJ^{-1}PX^T < 0$. Then (see Section 2.3) there is a non-singular matrix $X_1 \in \mathbb{R}^{n\times n}$ and an $n\times n$ orthogonal matrix Θ such that $X = X_1 \begin{bmatrix} I_{q+s} & \Theta \end{bmatrix} Q$ where Q is the permutation matrix of (2.16). It follows from the results of Section 2.3 (cf. (2.17)) that $XJPX^T = X_1 H(\Theta) X_1^T$. Then a simple computation shows that

$$XJ^{-1}PX^T = X_1 H_0(\Theta) X_1^T.$$

Since X_1 is non-singular, $H(\Theta) > 0$ and $H_0(\Theta) < 0$.

Conversely, if 0 is not an eigenvalue of J, X_1 is an arbitrary non-singular matrix, Θ orthogonal and $X = X_1 \left[\begin{array}{cc} I_{q+s} & \Theta \end{array} \right] Q$ then (X, J, PX^T) is a self-adjoint Jordan triple that defines a real symmetric quadratic matrix polynomial whose leading and trailing coefficients are, respectively,

$$L_2 = (XJPX^T)^{-1} = (X_1 H(\Theta) X_1^T)^{-1}, \quad L_0 = -(XJ^{-1}PX^T)^{-1} = -(X_1 H_0(\Theta) X_1^T)^{-1}.$$

If $H(\Theta) > 0$ and $H_0(\Theta) < 0$ then L_0 and L_2 are positive definite.

The second part of the Theorem follows readily from the results of Sections 2.2 and 2.3. □

Assume now that matrices $J, P \in \mathbb{R}^{2n \times 2n}$ of the form (2.3) and (2.4) are given satisfying conditions (4.4), (4.5), (4.7), and 0 is not an eigenvalue of J. By Theorem 15 there is a monic diagonal quadratic matrix $D(\lambda)$ with positive definite trailing coefficient. Using the notation of the proof of Theorem 14,

$$D(\lambda) = \mathrm{Diag}[D_1(\lambda), D_2(\lambda), D_3(\lambda)]$$

where $D_1(\lambda),\ D_2(\lambda),\ D_3(\lambda)$ are diagonal matrices of scalar quadratics:

$$\begin{array}{l} D_1(\lambda) = \mathrm{Diag}[(\lambda - \lambda_{j_1})(\lambda - \lambda_{i_1}), \cdots, (\lambda - \lambda_{j_r})(\lambda - \lambda_{i_r})], \\ D_2(\lambda) = \mathrm{Diag}[(\lambda - \lambda_{u_1})(\lambda - \lambda_{v_1}), \cdots, (\lambda - \lambda_{u_{q-r}})(\lambda - \lambda_{i_{q-r}})], \\ D_3(\lambda) = \mathrm{Diag}[\lambda^2 - 2\mu_1 \lambda + \mu_1^2 + \nu_1^2, \cdots, \lambda^2 - 2\mu_s \lambda + \mu_s^2 + \nu_s^2], \end{array}$$

with $\lambda_{j_k} > \lambda_{i_k}$ for $k = 1, \ldots, r$, $\lambda_{u_k} > \lambda_{v_k}$ for $k = 1, \ldots, q - r$, λ_{j_k} and λ_{u_k} are of positive type and λ_{i_k} and λ_{v_k} are of negative type.

We can rearrange the diagonal real eigenvalues of J so that

$$R_+ = \mathrm{Diag}[\lambda_{j_1}, \ldots, \lambda_{j_r}, \lambda_{u_1}, \ldots, \lambda_{u_{q-r}}], \quad R_- = \mathrm{Diag}[\lambda_{i_1}, \ldots, \lambda_{i_r}, \lambda_{v_1}, \ldots, \lambda_{v_{q-r}}].$$

Then $R_+ - R_- > 0$ and $R_+^{-1} - R_-^{-1} < 0$. And if we take $\Theta = \mathrm{Diag}(I_q, -I_s)$, an easy computation shows that

$$H(\Theta) = \left[\begin{array}{cc} R_+ - R_- & 0 \\ 0 & 2N \end{array} \right] > 0$$

and

$$H_0(\Theta) = \left[\begin{array}{cc} R_+^{-1} - R_-^{-1} & 0 \\ 0 & -2\widetilde{N} \end{array} \right] < 0.$$

Let $X_1 \in \mathbb{R}^{n \times n}$ be an arbitrary non-singular matrix and construct X using (2.15). Then (X, J, PX^T) is a real self-adjoint Jordan triple that uniquely determines a real symmetric quadratic matrix polynomial with positive definite leading and trailing coefficients. In particular, with

$$X_1 = \left[\begin{array}{cc} (R_+ - R_-)^{-1/2} & 0 \\ 0 & (2N)^{-1/2} \end{array} \right],$$

one can check that (X, J, PX^T) is a self-adjoint Jordan triple of $D(\lambda)$.

Since the set

$$\{\Theta \in \mathcal{O}_n | H(\Theta) > 0 \text{ and } H_0(\Theta) < 0\}$$

is open, we can use the same procedure of Section 5.1 to produce many real symmetric quadratic matrix polynomials with positive definite leading and trailing coefficients, and with (J, P) as spectral data. The final example illustrates this procedure.

EXAMPLE 18. Spectral data:

Elementary divisors	$\lambda+2$	$\lambda+1$	$\lambda-1$	$\lambda-1$	$\lambda-2$	$\lambda-2$	$\lambda^2+\lambda+1$
Sign characteristic	-1	$+1$	-1	-1	$+1$	$+1$	

A monic diagonal matrix with this spectral data and positive definite leading and trailing coefficients is:

$$D(\lambda) = \mathrm{Diag}[(\lambda+1)(\lambda+2), (\lambda-2)(\lambda-1), (\lambda-2)(\lambda-1), \lambda^2+\lambda+1].$$

Then define

$$J = \mathrm{Diag}\left(-1, 2, 2, -2, 1, 1, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right), \quad P = \mathrm{Diag}(I_3, -I_3, -1, 1) \quad \Theta = \mathrm{Diag}(I_3, -1)$$

and Q as in (2.16).

The following MATLAB code implements the procedure of Section 5.1 starting with the above matrices J, P, Θ and Q and using random matrices to define X_1 and the perturbation applied to Θ. It concludes with the corresponding coefficient matrices L_2, L_1, L_0, and verification that $L_2 > 0$ and $L_0 > 0$. Different choices of matrices X_1 and $\widetilde{\Theta}$ yield (possibly) different real symmetric quadratic matrix polynomials with the same spectral data (J, P) and positive definite leading and trailing coefficients.

```
>> J=blkdiag(diag([-1 2 2 -2 1 1]),[1 -1;1 1]);P=diag([ 1 1 1 -1 -1 -1 -1 1]);
>> Th=diag([ 1 1 1 -1]);
>> Tht=Th+randn(4)*10^(-6); [Tht,R]=qr(Tht); Tht
Tht =
  -0.999999999998082  -0.000001750211010  -0.000000285651951   0.000000831370244
   0.000001750212827  -0.999999999996321  -0.000000533557936   0.000002002638884
   0.000000285651046   0.000000533558506  -0.999999999999816   0.000000034769244
   0.000000831366729   0.000002002640321   0.000000034770550   0.999999999997649
>> Tht=Tht*diag([ -1 -1 -1 -1]); norm(Th-Tht)
ans =
     2.851650141736129e-06
>> Q=[eye(3) zeros(3,5);zeros(1,7) 1;zeros(3) eye(3) zeros(3,2); zeros(1,6) 1 0];
>> X1=randn(4);X=X1*[eye(4) Tht]*Q;
>>  L2=(X*J*P*X')^(-1), L=-L2*X*J^2*[X;X*J]^(-1); L1=L(:,5:8),L0=L(:,1:4)
L2 =
   0.806078129692617  -0.047114715779428   0.715847669446776  -0.652914707844277
  -0.047114715779427   0.305242250370710  -0.810198431080349   1.165667206325252
   0.715847669446772  -0.810198431080350   5.295595009791147  -7.764360871474528
  -0.652914707844271   1.165667206325254  -7.764360871474526  11.766117910855193
L1 =
   2.124603814029641   0.164872355925945  -0.547271377584461   1.859722461853190
   0.164872355925948  -0.795263736097922   2.160985233380643  -2.992066022738981
  -0.547271377584480   2.160985233380648 -14.681332172684314  22.090992709697556
   1.859722461853216  -2.992066022738989  22.090992709697566 -33.173246389369389
L0 =
```

1.612154192699047	-0.094220353845558	1.431681912840817	-1.305814935090748
-0.094220353845555	0.610485373804359	-1.620395343425297	2.331336709710245
1.431681912840803	-1.620395343425304	10.591184113801775	-15.528721174414684
-1.305814935090727	2.331336709710254	-15.528721174414677	23.532241779620332

One can check that the eigenvalues of L_2 and L_0 are all positive. □

6. Conclusions. In this paper the inverse symmetric quadratic eigenvalue problem (ISQEP) has been considered under the generic assumption that the prescribed eigenvalues are semisimple but with the additional constraint that the sign characteristic associated with the real eigenvalues is also prescribed. The general theory imposes the basic restriction that half of the real eigenvalues (if any) must be of positive type and the other half of negative type. It has been shown that if the prescribed eigenvalues and sign characteristics satisfy this condition and no definiteness restriction is imposed on the coefficients, then there always exists a real symmetric quadratic matrix polynomial with these assigned spectral properties; i.e. the ISQEP always has a solution.

A procedure similar to that proposed in [1] has been designed for the construction of a broad family of real symmetric quadratic matrix polynomials with prescribed eigenvalues and sign characteristics.

The case when the leading and/or the trailing coefficient are prescribed to be positive definite has also been studied. First, it has been shown that all real symmetric quadratic matrix polynomials with positive definite leading coefficient are diagonalizable. This imposes an important restriction on the admissible geometric multiplicities of the eigenvalues (Section 2.5). Nevertheless, the main constraint is on the sign characteristic.

With the help of results from [18] (Theorems 11 and 12 above), where the distribution of eigenvalues with respect to their sign characteristic is studied, necessary and sufficient conditions have been given for the solution of the ISQEP when the leading and/or trailing coefficients are prescribed to be positive definite. When those conditions are satisfied, explicit monic quadratic diagonal matrix polynomials have been constructed with the desired eigenvalues and sign characteristic.

Then, with the information provided by such diagonal matrices, a procedure has been proposed to obtain many different real symmetric quadratic matrix polynomials with prescribed eigenvalues and characteristic signs, and with positive definite leading and trailing coefficients.

Additional conditions can be expected if also the middle coefficient is prescribed to be positive definite. This is a research project for the near future.

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