The Properties of Simple vs. Absolute Majority Rule:
cases where absences and abstentions are important*

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Abstract. Little attention has been paid to the differences between absolute majority rule and simple majority rule, which differ in their treatment of absences and “votes to abstain.” This paper fills that gap by undertaking a probabilistic analysis of the two voting rules assuming two alternatives and a quorum requirement for simple majority rule. The rules are compared in both a modified sincere setting and a strategic setting using five criteria: 1) the Pareto criterion, 2) the BT criterion (Buchanan and Tullock 1962), 3) the Expected Social Gain criterion, 4) the Responsiveness criterion, and 5) a modified version of Rae’s criterion. In the sincere setting, we find that simple majority rule (with and without a quorum) outperforms absolute majority rule under most conditions for four out of the five criteria. In the strategic setting, we find that the voting rules perform much more similarly.

1 Introduction

How should an assembly that wants to make decisions using majority rule treat absences and “votes to abstain?” Two treatments have been used in practice. Absolute majority rule passes a proposal if more than half the eligible members vote yea. Implicitly, it treats absences and “votes to abstain” the same as votes against the proposal. Simple majority rule, in contrast, compares the number of yeas to the number of nays. In effect, it disregards absences and “votes to abstain” in the tally.¹ Although Felsenthal and Machover (1997) compare the responsiveness of these two voting rules, Dougherty and Edward (2004) make some comparisons based on the BT criterion, and Freixas and Zwicker (2009) compare the two voting rules using properties related to May’s theorem, no published work has extensively compared these two procedures. In fact, as Felsenthal and Machover (2001a, b) point out, public choice scholars rarely distinguish the two voting rules² and frequently misreport which rule is used in practice.

¹ These definitions are based on Riker (1982: 44-5). Sen (1979: 71 & 181) makes a similar distinction but uses different nomenclature.

² Notable exceptions include Fishburn 1973; Sen 1979; Riker 1982; Freixas and Zwicker 2003, and Zwicker 2009.
At first glance, it seems strange to treat absences and “votes to abstain” the same as votes against a proposal. It would appear more natural to ignore them in the tally. Given this observation, it is not clear why impartial institutional framers would want absolute majority rule.

One reason may be that absolute majority rule determines the preponderance of the body, thereby making decisions more legitimate. This concern may be partially addressed by requiring a quorum for simple majority rule. Another possibility is that absolute majority rule is less likely to pass proposals. In some contexts, institutional framers want to favor the status quo and this is why they prefer absolute majority rule to simple majority rule. ³

This paper determines whether absolute majority rule or simple majority rule (with various quorum requirements) is more desirable by comparing their ability to meet five criteria (separately): 1) the Pareto criterion, 2) the BT criterion (Buchanan and Tullock 1962), 3) the Expected Social Gain criterion, 4) the Responsiveness criterion, and 5) a modified version of Rae’s criterion. This is done assuming two alternatives (a proposal and a status quo) and various probabilities of individual preferences.

We conduct our analysis using two frameworks. The first is a modified version of a sincere voting model that allows for absences and “votes to abstain.” In this framework, we find that simple majority rule (with or without a quorum) always outperforms absolute majority rule in terms of the Pareto and Rae criteria and that it generally outperforms absolute majority rule in terms of the Expected Social Gain and Responsiveness criteria. The BT criterion generally favors absolute majority rule but there are surprising exceptions for small populations with many indifferent voters.

³ For example, this might be the case for votes of no confidence or changes in procedural rules.
The second framework determines the probability of adhering to each of the criteria in a complete information game. In this framework, we find more mixed results. With the exception of the BT criterion, absolute majority rule tends to perform better relative to simple majority rule than it does under the sincere voting framework. However, simple majority rule still performs at least as well as absolute majority rule in terms of the Pareto criterion and in many cases the voting rules perform equivalently. The latter is particularly true if there is a quorum requirement of exactly one member more than half of the body.

2 Definitions

One way to compare two voting rules is to establish a set of criteria and see whether the voting rules and the criteria are consistent with an unrestricted domain (Arrow 1951; May 1952; Sen 1979). The impossibility theorems associated with this approach suggest that most voting rules violate common criteria for a variety of preference orders. An alternative approach, the one used here, determines the probability that each voting rule adheres to a criterion or set of criteria (Rae 1969; Caplin and Nalebuff 1988; and Dougherty and Edward 2004). This allows comparisons between voting rules that violate a criterion for one or more voter profiles and helps to determine whether the differences between the voting rules are large.

To compare these two voting rules, let N be the number of individuals in a group, committee, or voting population and M be the smallest majority of those individuals; so that M = (N+1)/2 for N odd and M = (N+2)/2 for N even. Also let there be two alternatives: a proposed alternative x and the status quo q.
Although our methods can be easily adapted to model cardinal preferences, we shall assume ordinal preferences to avoid the perception that we are making interpersonal comparisons of utility. We also assume no vote trading.

Two types of majority rule have been used in practice.

Definition 1. *Absolute majority rule* (AM): alternative x defeats the status quo q, by absolute majority rule if and only if the number of yeas is at least as great as M; otherwise q is chosen.

Examples from this category include: the U.S. House of Representatives requiring a majority of its members (218) to sign a successful discharge petition; the method of ratifying Constitutional amendments in the Virginia Senate; and the votes needed for ordinary decisions in the Estonian parliament, Russian Duma, and Guatemalan congress. All of these cases determine the winner based on a predetermined threshold of affirmative votes. As a result, they treat abstentions and non-votes the same as negative votes.

Definition 2. *Simple majority rule with a quorum* (SMQ): alternative x passes by SMQ if and only if the number of yeas is strictly greater than the number of nays and at least Q individuals vote, where $1 \leq Q \leq M$; otherwise q is chosen.\(^4\)

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\(^4\) Note that ties go to the status quo under this definition. That is, if the quorum is met and the yeas equal the nays, then the status quo is chosen.
Throughout this paper we assume $Q \leq M$. This ensures that AM is less likely to pass proposals than SMQ, which is necessary for some of the proofs.\(^5\) SMQ is used for ordinary decisions in the parliament of Finland ($Q=0$), the British House of Commons ($Q = 40$, out of 650), and in both houses of the U.S. Congress ($Q=M$) – to name a few (see Rasch 1995 for more examples).

Note that if $x$ passes under AM, then with the same set of votes it also passes under SMQ. If $x$ passes under SMQ, however, then it may or may not pass under AM. Felsenthal and Machover (2001a) observed that from 1996 to 1998 the U.S. House of Representatives passed 70 bills (8\%) under SMQ ($Q=M$) that would not have passed under AM.\(^6\) Similarly, we find that the U. S. Constitutional Convention passed 121 resolutions (23\%) under SMQ ($Q=M$) that would not have passed under AM. If everyone votes and there are no indifferent voters, the two procedures select equivalently.

To compare the properties of SMQ and AM we consider five criteria. Each criterion comes from one of the many research traditions reported in Mueller (2003). Other criteria such as transitivity (Condorcet 1785), independence of irrelevant alternatives (Arrow 1951), disjoint equality (Fishburn 1978), and elimination of indifferent individuals (Ng 1981) are limited to settings with three or more alternatives, while criteria such as anonymity (May 1952), continuity (Young 1975), and positive association (Blin and Satterthwaite 1978) are always satisfied by both voting rules. Criteria such as neutrality (May 1952) seem to favor one rule trivially.\(^7\)

\(^5\) Cases where $Q > M$ will be examined in future work.

\(^6\) In making this statement, and the statement of the next sentence, we implicitly assume that voting behavior would not be affected by the voting rule used. We address the possibility of endogenous voting behavior in section 4.

\(^7\) Another criterion, consistency (Young 1974), favors AM over SMQ ($Q > 0$) somewhat trivially. However, consistency is a technical property that does not effectively measure social welfare or the
Definition 3. *Pareto Criterion.* For any two alternatives y and z, y is Pareto preferred to z if and only if it makes at least one individual better off than z and no individual worse off than z.

We adopt the Pareto criterion because it is the most widely used welfare criterion in political science and economics (Mueller 2003).

Definition 4. *BT Criterion.* Proposal x is BT preferred to the status quo q if and only if it is Pareto preferred to q; otherwise q is BT preferred to x.

The BT criterion differs from the Pareto criterion by judging all indeterminate cases from the Pareto criterion as cases which should favor the status quo. It was used by Buchanan and Tullock (1962) in their seminal work on constitutional formation, as well as by Head (1974), Rogowski (1974: 47), and Tsebelis (1990: 104). Consistent with the contractarian tradition, Buchanan and Tullock argued that new institutions should not be adopted unless they made at least one person better off and no one else worse off (Buchanan 1975; Sen 1979: 25).

Definition 5. *The Expected Social Gain Criterion.* Expected Social Gain (ESG) is the expected number of individuals in favor of the adopted alternative minus the expected effect of social decisions on individual welfare like the criteria we adopt here.

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8 The first three authors refer to this criterion as the Pareto criterion. Tsebelis refers to it as an efficiency criterion. We call it the BT criterion in honor of Buchanan and Tullock.
number of individuals opposed to the adopted alternative. The Expected Social Gain criterion recommends the voting rule that maximizes $\text{ESG} / N$.

The Expected Social Gain criterion might be considered the ordinal version of Utilitarianism (for the latter see Harsanyi 1975; and Mueller 2003: 605). However, the Expected Social Gain criterion does not make interpersonal comparisons of utility. Unlike the other four criteria, the Expected Social Gain criterion judges the benefits to society as a whole, rather than making social judgements based on the individual. In this sense, it may be considered the polar opposite of the Pareto and BT criteria.

Our next definition is based on an individual being pivotal. An individual is pivotal if he/she can change the assembly’s choice by changing his/her action.

Definition 6. **Responsiveness Criterion.** Responsiveness is the probability that an arbitrary individual is pivotal. The responsiveness criterion recommends the voting rule that maximizes responsiveness.

Responsiveness has been widely analyzed in the literature (Mueller 2003: 304-5). As a normative criterion, it indicates a connection between individual preferences and social outcomes. For example, an individual who attends a town meeting may find the meeting more democratic if it is more responsive to changes in his/her preferences. The responsiveness
criterion is a more complete version of positive responsiveness (May 1952), and it is at the core of the Banzhaf power index (Freixas 2005a).  

Definition 7. **The Modified Rae Criterion**: The Rae index (our terminology) is the sum of
1) the probability that an individual votes in favor of the proposal but society chooses the
status quo, and 2) the probability that an individual votes in favor of the status quo but
society chooses the proposal. The Modified Rae criterion recommends the voting rule
that minimizes the Rae index.

In a seminal work, Rae (1969) used a simpler version of this criterion to compare various k-
majority rules, but he did not compare AM to SMQ, nor allow for indifference and non-voters
(also see Taylor 1969). The Modified Rae criterion differs from the Pareto and BT criteria in

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9 It has long been assumed that the Responsiveness and Rae criteria are equivalent (see Dubey
and Shapley 1979). However, it is possible for AM to outperform SMQ in terms of Responsiveness and
for SMQ to outperform AM in terms of the Modified Rae criterion. To see this, we use the notation of
section 3 and consider N = 3, p_{1,1} = .40, p_{0,1} = .40, and p_{1,1} = .20 in the sincere framework (Table 5 and
Table 6). Such cases suggest that the two criteria are not equivalent. Similar cases can be generated
using Rae’s original definition.

10 An alternative way of generalizing the Rae criterion to cases with voters and non-voters is to
define it as the probability that society chooses against an individual (regardless of whether the
individual votes or does not vote). In this case, the Rae index would be trivially related to ESG by the
equation: ESG/N = 1−2R−p_{0,1}−p_{0,0}, where R is the Rae index. If we adopt this definition, then for any
fixed N and fixed preference probabilities, AM would outperform SMQ in terms of the Modified Rae
criterion if and only if it outperforms SMQ in terms of the Expected Social Gain criterion. The same is
not true with the definition adopted here. For the Modified Rae criterion, ESG/N = 1−2R−L_0−p_{0,1}−p_{0,0},
where L_0 is the expected number of non-voting losers (i.e. non-voters who strictly prefer one alternative
yet society chooses the other).

11 According to Rae’s assumption III (1969: 45) “proposals do not enter the committee’s agenda
unless at least one member supports each of them. Therefore, there will be no combination of responses
in which no member supports the proposal.” We do not adopt assumption III here because Rae himself
does not use it in his calculations, and Taylor (1969) does not use it in his proof. Our sincere formulas
its treatment of the cases where some individuals favor and other individuals oppose a proposal. It differs from the Expected Social Gain criterion by only counting voters who are against the adopted alternative. The Expected Social Gain criterion considers the preferences of both voters and non-voters.

3 Sincere Behavior

For our first model, we assume that voting is a more general version of sincere. Namely, individuals who vote will vote according to their preferences, but not voting is also possible.\footnote{12} Sincere voting is commonly used in the social choice literature to compare voting rules (Sen 1979; Riker 1982; Sarri 2003, Kaminski 2006). It is also consistent with Denzau, Riker, and Shepsle’s (1985) rational legislator who votes solely to signal constituents. And it is consistent with individuals behaving according to simple heuristics (Tversky & Kahneman 1974; Hutchinson & Gigerenzer 2005) or other non-rational theories of human behavior, such as genetics (Fowler et al. 2008; Alford et al. 2005).

We allow the additional possibility of not voting to study the effects of quorum requirements and to make our model more realistic. In our framework both indifferent individuals and individuals with a strict preference may not vote. The latter might result from individuals with a strict preference being constrained, such as being caught in traffic, or valuing non-voting tradeoffs, such as working on the campaign trail.

\footnote{12} Freixas and Zwicker (2003, 2009) and Freixas (2005a, 2005b) have also used these assumptions.
Each individual $i$ has a preference-action combination $D_i = (u, v)$ for the two alternatives, where $u$ indicates the individual’s preference ($u = 1$ iff the individual prefers $x$, $u = -1$ iff the individual prefers $q$, and $u = 0$ iff the individual is indifferent between the two alternatives) and $v$ indicates the individual’s action ($v = 1$ iff the individual votes for his/her preferred alternative and $v = 0$ iff the individual does not vote). These combinations are summarized in Table 1.

[Table 1 here]

To allow each preference-action combination to occur with different probabilities, further assume that each $D_i = (u, v)$ occurs with probability $p_{u,v}$. For example, $p_{0,1}$ is the probability that an individual is both indifferent and “votes abstain” (i.e. calls out “abstain” in an assembly). These probabilities capture uncertainty about which individuals will be in the voting population and/or what alternatives will be available, as in the case of establishing constitutional rules before actors or alternatives are fully known.

For simplicity, further assume that 1) each individual has the same $p_{u,v}$ and 2) that these probabilities are independent (both of each other and the voting rule). Although the latter is controversial in small $N$ settings, there is a tradition of assuming both properties in large $N$ and small $N$ cases (Rae 1969; Taylor 1969; Caplin and Nalebuff 1988; Young 1988).¹³

0. One way to meet these assumptions is to randomly draw an assembly from a meta-population. For example, a board of education randomly drawn from a conservative county may prefer school vouchers to the status quo of publically funded education with probabilities $p_{1,v} = .60$, $p_{0,v} = .10$, and $p_{-1,v} = .30$, while a board of education drawn from a liberal county may prefer the two alternatives with probabilities $p_{1,v} = .30$, $p_{0,v} = .10$, and $p_{-1,v} = .60$. Note, however, the welfare criteria used here are intended to judge the welfare of the assembly, not the welfare of the meta-
Comparing the probability that AM and SMQ adhere to each criterion in our sincere voting framework produces the following results.

3.1 Pareto Criterion

**Theorem 1.** Assume $p_{1,i}, p_{0,1} > 0$. Then for any $Q$ s.t. $0 \leq Q \leq M$, SMQ is more likely to select the Pareto preferred alternative than AM.

The proof of theorem 1 is straightforward. If $q$ is Pareto preferred to $x$, then both voting rules select $q$. If $x$ is Pareto preferred to $q$ and AM selects $x$, then SMQ selects $x$ as well. The theorem follows since SMQ can select Pareto preferred proposals that AM does not.\[14\]

[Table 2 here]

The numerical difference in performance between SMQ and AM is determined with the aid of a computer. As shown in Table 2, the difference in performance can be stark or unnoticeable, depending upon the distribution of the preference-action combinations and the size of the population. If the probability distribution is uniform (which may be appropriate for a priori analyses) and $Q=M$, then SMQ outperforms AM moderately in terms of the Pareto criterion. If favoring and disfavoring a proposal are equally likely and there is only a small probability of indifferent voters (as in the second set of preference probabilities), then the differences are smaller. Finally, in a case where the two procedures select differently, such as $p_{1,1} = .40, p_{0,1} = .40, p_{-1,1} = .20$, SMQ outperforms AM fairly substantially.

population from which they are drawn. Stochastic preferences might also assure independence.

\[14\] Consider, for example, three voters (a, b, and c) with $\{D_a = (1,1); D_b = (0,1); D_c = (0,0)\}$. 11
For larger populations, say $N \geq 15$, the difference in performance is unnoticeable. This is because an alternative is rarely Pareto preferred to another alternative in populations of such size.

### 3.2 BT Criterion

For large $N$, we might expect AM to outperform SMQ in terms of the BT criterion because the probability that $x$ is Pareto preferred to $q$ is small and AM is more likely to select $q$ than SMQ in large $N$ cases. However, Theorem 2 suggests that there are important exceptions.

**Theorem 2.** Assume $(2^{0(N-1)} - 1)p_{0,1} \geq (p_{0,0} + p_{1,0} + p_{-1,0} + p_{-1,1})$. Then for any $Q$ s.t. $0 \leq Q \leq M$, SMQ is more likely to observe the BT criterion than AM (see the appendix for proof).

The intuition behind Theorem 2 is that if there are a sufficiently large number of $D_i = (0, 1)$, then assemblies can select Pareto preferred proposals under SMQ that they would not select under AM. Theorem 2 states a sufficient condition for this to occur while simultaneously accounting for cases where $q$ is BT preferred.

For small populations (like $N=3$) the sufficient condition of Theorem 2 is easily met. SMQ can outperform AM for reasonable preference probabilities (see Table 3). For larger populations (say $N \geq 9$), the sufficient condition is difficult to meet and AM outperforms SMQ for the same preference probabilities.
If $N = \infty$, the status quo is BT preferred. AM chooses the status quo if $p_{1,1} < .5$, while SMQ chooses the status quo if $p_{1,1} < p_{-1,1}$ or $\text{pr(voting)} < Q/N$. This explains the output for $N = \infty$ in Table 3. It also helps to explain the behavior of the two voting rules at $N = 100$.

[Table 3 here]

### 3.3 Expected Social Gain

**Theorem 3.** Assume $p_{1,1}, p_{0,1}, p_{-1,1} > 0$ and $p_{1,0} \geq p_{-1,0}$. Then for any $Q$ s.t. $0 \leq Q \leq M$, SMQ produces greater expected social gain than AM (see the appendix for proof).

The key to understanding the proof of Theorem 3 is to consider cases where a measure passes by SMQ but is defeated by AM. Clearly yeas exceed nays in such cases, and in view of $p_{1,0} > p_{-1,0}$ we expect more individuals to favor the outcome than to oppose it. Hence, we expect SMQ to yield positive social gain, but AM to yield negative social gain.

The reasoning changes if a “silent majority” opposes the proposal (i.e. if $p_{-1,0} > p_{1,0}$, then AM can outperform SMQ). See, for example, the last row of Table 4.

[Table 4 here]

### 3.4 Responsiveness

**Theorem 4.** Assume $N = 3$ and $p_{0,1} > 0$. Then for $0 \leq Q \leq M$, SMQ is more responsive than AM.
Proof: We prove the result for \( Q = M \). The proof for \( Q < M \) is similar. Set aside individual \( i \) and let \( j \) and \( h \) refer to the second and third individuals (in either order). In what follows, we count the profiles for which \( i \) is pivotal. These are:

**AM:** \( \{D_j = (1,1), D_h = (0,1)\}; \{D_j = (1,1), D_h = (-1,1)\}; \{D_j = (1,1), D_h = (1,0)\}; \{D_j = (1,1), D_h = (0,0)\}. \)

**SMQ:** the same profiles as above and also \( \{D_j = (0,1), D_h = (0,1)\}; \{D_j = (0,1), D_h = (-1,0)\}; \{D_j = (0,1), D_h = (1,0)\}; \{D_j = (0,1), D_h = (0,0)\}. The proof follows, since all voter profiles which make \( i \) pivotal under AM also make \( i \) pivotal under SMQ. \( \square \)

For larger populations the superiority of SMQ is no longer automatic. However, one does have:

**Theorem 5.** Assume \( p_{1,1} = p_{0,1} = p_{-1,1} = 1/3 \) and \( 3 < N < \infty \). Then for any \( Q \) s.t. \( 0 \leq Q \leq M \), SMQ is more responsive than AM (see the appendix for proof).

Felsenthal and Machover (1997) also compare SMQ (\( Q=0 \)) and AM in terms of responsiveness (see their Corollary 3.5). Theorem 5 improves on their result in three ways. First, their result does not account for non-voters and quorum requirements. Second, in Felsenthal and Machover’s formulation, a voting rule is responsive only if an individual can alter the outcome of the vote by switching his/her preference from strict preference to indifference or vice versa. In our formulation, a voter can also alter the outcome of the vote by switching from
In the notation of Felsenthal and Machover (1997: 243), \( T > 0, U(T) = 1, U(T^0) = 1 \) does not imply \( T > 0, U(T) = 1, U(T^0) = 1 \).

Third, Felsenthal and Machover compare AM with \( p_{1,1} = p_{-1,1} = 1/2 \) to SMQ (Q=0) with \( p_{1,1} = p_{0,1} = p_{-1,1} = 1/3 \). It seems more natural to compare the two voting rules using the same inputs. Hence, Theorem 5 can be considered an improvement on their initial result.

When the two voting rules are compared using numerics, we find that SMQ is typically more responsive to changes in individual preferences than AM (see Table 5). However, SMQ is less responsive in special cases. For example, in the case of \( N = 100, p_{1,1} = .40, p_{0,1} = .40, \) and \( p_{-1,1} = .20 \), SMQ is less responsive than AM.

Based on numerical computations, we make two additional conjectures.

**Conjecture 1:** For \( 3 < N < \infty \), a necessary condition for AM to be more responsive than SMQ (Q ≤ M) is \( |.5 - p_{1,1}| < |p_{1,1} - p_{-1,1}| \).

**Conjecture 2:** For \( 20 < N < \infty \), a sufficient condition for AM to be more responsive than SMQ (Q ≤ M) is \( p_{1,1} > M / N \).

The last case in Table 5 supports conjecture 2.

### 3.5 Modified Rae Criterion

**Theorem 6.** Assume \( p_{1,1}, p_{-1,1} > 0, (p_{1,1} + p_{-1,1}) < 1, \) and \( N \geq 3 \). Then for any \( Q \) s.t. \( 0 \leq Q \leq M \), SMQ has a smaller Rae index than AM.

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\(^{15}\) In the notation of Felsenthal and Machover (1997: 243), \([T_a > 0, U(T) = 1, U(T^0) = -1]\) does not imply \([T_a > 0, U(T) = 1, U(T^0) = -1]\).
Recall that smaller figures are associated with smaller probabilities of society voting against an individual, hence they indicate a more desirable voting rule. Theorem 6 holds, roughly speaking, because the probability that AM will defeat a measure that an individual favors is greater than the probability that SMQ will pass a measure that an individual opposes. The proof of this theorem, a rather delicate combinatorial argument, appears in the appendix.

The strong conclusion of Theorem 6 is supported by Table 6. As the table shows, differences in performance are large for cases where the two voting rules select most differently.

The results from our sincere voting model are summarized in Table 7.

4 Strategic Behavior

The previous analysis is based on the implicit assumption that the probability of each action is fixed and exogenous to the voting rule. To consider what might occur if actions were endogenous to the voting rule, consider an extension of Palfrey & Rosenthal’s (1983) game theoretic model of the rationality of voting. In our game, there are three types of voters: type $T_1$ prefers $x$ to $q$, type $T_{-1}$ prefers $q$ to $x$, and type $T_0$ are indifferent between $x$ and $q$. There are $N$ individuals in the population, $k$ of which are members of $T_1$ and $\ell$ of which are members of $T_{-1}$. Members of $T_1$ benefit $b = 1$ if the proposal passes, 0 otherwise; members of $T_{-1}$ benefit $b = 1$ if the proposal fails, 0 otherwise. We assume that the act of voting costs $c$, where $0 < c < 1$, for all

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16 Palfrey & Rosenthal analyze the probability of voting for variants of simple majority rule, without a quorum. We conduct our analysis using AM and SMQ.
individuals regardless of their type. Furthermore, information is complete, that is each individual knows the number of individuals of each player type.

In a one shot game, all of the strategic behavior is in the decision to vote or to not vote. This means that an individual of type $T_i$ will never “vote for q” nor “vote abstain.” They will either “vote for x” or “not vote.” Keeping this in mind, we further assume that equilibrium play is quasi-symmetric among voters of type $T_\tau$, for $\tau = 1, -1, 0$. Specifically, all members of $T_1$ vote (for x) with probability $r_1$ and do not vote with a probability of $(1-r_1)$. All members of $T_{-1}$ vote (for q) with probability $r_{-1}$ and do not vote with a probability of $(1-r_{-1})$. Of course, members of $T_0$ will never vote ($r_0 = 0$). Palfrey and Rosenthal limit their study to quasi-symmetric equilibria as well, which can be justified based on the coordination problems associated with non-symmetric equilibria.\(^{17}\) Later in this section we will consider the case of non-symmetric equilibria for $N=3$.

For an individual of type $T_i$, we denote the expected payoffs for voting (resp. not voting) as $EV_i^v$ (resp. $EV_i^{nv}$). As standard in game theory, individual $i \in T_i$ will vote with probability 1 if $EV_i^v > EV_i^{nv}$, he/she will vote with probability 0 if $EV_i^v < EV_i^{nv}$, and he/she will vote with probability $r_i \in (0, 1)$ if $EV_i^v = EV_i^{nv}$. We refer to the first two cases as “pure strategies,” and the last as a “mixed strategy.”\(^{18}\)

\(^{17}\) Palfrey and Rosenthal analyze some quasi-symmetric equilibria (which they call totally quasi-symmetric equilibria), but explicitly state that most of their analysis is limited to a narrower set of equilibria where $r_i = (1 - r_{-i})$. See Palfrey and Rosenthal (1983: 16) for specifics.

\(^{18}\) For a discussion of the necessary and sufficient conditions for a mixed strategy equilibrium, see Myerson (1997: 91).
Our goal is to determine all of the quasi-symmetric Nash equilibria under AM and SMQ. We will then use this information to determine the probability of adhering to four welfare criteria for each equilibrium: Pareto, BT, ESG, and Rae. We do not analyze Responsiveness in this section because the probability of being pivotal is used to determine $r_i$.

### 4.1 Equilibria under AM

Under AM, members of $T_{-1}$ (and $T_0$) have a dominant strategy to “not vote,” because not voting has the same effect as voting against the proposal but the voter does not incur the cost. In other words, $r_{-1} = 0$ (and $r_0 = 0$). Hence, the only non-trivial strategies are among members of $T_1$ over the decision of which of them will turnout and vote. Let $i$ be a member of $T_1$. The expected values of voting and not voting for $i$ are:

$$\text{EV}_i^v = 1 \cdot \Pr[m_i \geq M - 1] + 0 \cdot \Pr[m_i < M - 1] - c,$$

$$\text{EV}_i^{nv} = 1 \cdot \Pr[m_i \geq M] + 0 \cdot \Pr[m_i < M - 1],$$

where the subscript 1 indicates that player $i$ is type $T_1$ and $m_i$ is the total number of members of $T_1$, other than player $i$, who actually vote. We note for future reference that

$$\text{EV}_i^v - \text{EV}_i^{nv} = \Pr[m_i = M - 1] - c = (r_i)^{M-1} \cdot (1-r_i)^{k-M} - c. \quad (3)$$

where $k \cdot C_{M-1}$ is $k-1$ “choose” $M-1$. In what follows we observe the conventions that $C_1 = 0$ if $s < t$; $C_0 = 1$; $\sum_a^b = 0$ if $a > b$; and $0^0 = 1$. 


4.1.1 mixed strategies. To compute the $r_i$ associated with mixed strategy equilibria, we set $EV_i^v = EV_i^{nv}$, which by (3) is equivalent to

$$k^{-1}C_{M-1} \cdot (r_i)^{M-1} (1-r_i)^{k-M} - c = 0$$

(4)

The probability that members of $T_i$ will vote for the proposal in mixed strategy equilibrium, $0 < r_i < 1$, is the solution (or solutions) to (4). We approximate these to 20 digits using Maple 11.

4.1.1 pure strategies. Pure strategy equilibria (where $r_i = 1, 0$) can also be derived using (1) and (2) by determining whether $EV_i^v > EV_i^{nv}$ if $r_i = 1$ and whether $EV_i^v < EV_i^{nv}$ if $r_i = 0$.

If $r_i = 1$, $k \neq M$, and $k > 0$, then by (3)

$$EV_i^v - EV_i^{nv} = -c < 0,$$

hence there can be no equilibria of this type for $k \neq M$.

If $r_i = 1$ and $k = M$, then

$$EV_i^v > EV_i^{nv} \iff 1^{M-1} > c,$$

which is always the case. Hence, $(r_i, r_{-i}, r_0) = (1, 0, 0)$ is in equilibrium if and only if $k = M$.

Now, if $r_i = 0$ and $M \geq 2$, then

$$EV_i^v < EV_i^{nv} \iff k^{-1}C_{M-1} \cdot 0^{M-1} \cdot 1^{k-M} < c,$$

which is always the case. Hence, $(r_i, r_{-i}, r_0) = (0, 0, 0)$ is in equilibrium for any value of $k$. The complete set of quasi-symmetric equilibria (mixed and pure) for AM and $N = 5$ are listed in Table 8.

---

19 In general, the roots of a high ordered polynomial, such as (4), cannot be determined exactly (Cheney and Kincaid 2004).

20 If $k = 0$, then $r_i$ is not well defined. If $k = 0$, $0 < \ell < N$, then $(r_{-1}, r_0) = (0, 0)$ is the only equilibrium.
4.2 Equilibria under SMQ

We now consider two types of SMQ: Q= 1 and Q= M. As before, members of T₀ have a dominant strategy to “not vote.” However, members of T₁ might vote in this setting. In what follows, we derive the formulas that will be used in a Maple program to compute the associated equilibria. The resulting equilibria are also listed in Table 8.

4.2.1 mixed strategies. To find the solely mixed strategy equilibria under SMQ, we first note the expected payoff for a player of type T₁:

\[ EV₁ = 1 \cdot \Pr[m ≥ n \land (m + n) ≥ Q - 1] + 0 \cdot \Pr[(m < n) \lor ((m + n) < Q - 1)] - c, \]  
\[ EV₁^{nv} = 1 \cdot \Pr[m > n \land (m + n) ≥ Q] + 0 \cdot \Pr[(m ≤ n) \lor ((m + n) < Q)], \]  
where n is the total number of members of T₁ who actually vote, and m is defined as in the previous subsection. We then note the expected payoffs for a player of type T₀:

\[ EV₀ = 1 \cdot \Pr[(n + 1 ≥ m) \lor (n + m < Q - 1)] - c, \]  
\[ EV₀^{nv} = 1 \cdot \Pr[(n ≥ m) \lor (n + m < Q)], \]  
where n indicates the total number of members of T₀ who actually vote, other than j, and m indicates the total number of members of T₁ who actually vote. To solve for \( r₁, r₀ \) in mixed strategy equilibrium, it suffices to set

\[ EV₁ = EV₁^{nv} \]  
\[ EV₀ = EV₀^{nv}. \]

For k > 0, (9) is equivalent to
\[ Pr[m^i \geq n \land m^i + n \geq Q - 1] - c - Pr[m^i > n \land m^i + n \geq Q] = 0 \]

\[ \Rightarrow Pr[m^i \geq n \land m^i + n = Q - 1] + Pr[m^i = n \land m^i + n \geq Q] \]

\[ + Pr[m^i > n \land m^i + n \geq Q] - Pr[m^i > n \land m^i + n \geq Q] - c = 0 \]

\[ \Rightarrow Pr[m^i \geq n \land m^i + n = Q - 1] + Pr[m^i = n \land m^i + n \geq Q] - c = 0 \]

\[ = \sum_{s=\lceil (Q-1)/2 \rceil}^{Q-1} C_s^k \xi C_{Q-1-s}^2 \eta^s (1-\eta)^{k-s} (r_1)^s (1-r_1)^{s-1} \xi^{s+1} \]

\[ + \sum_{s=\lceil Q/2 \rceil}^{\min(k-1,k-1)} C_s^2 C_{Q-s}^2 \eta^s (1-\eta)^{k-s} (r_1)^s (1-r_1)^{k-s-1} - c = 0. \]  

(11)

For \( \ell > 0 \), (10) is equivalent to

\[ EV_{-1}^v - EV_{-1}^{nv} = 0 \]

\[ = \sum_{s=\lceil (Q-1)/2 \rceil}^{Q-1} C_s^k \xi C_{Q-1-s}^2 \eta^s (1-\eta)^{k-s} (r_1)^s (1-r_1)^{s-1} \xi^{s+1} \]

\[ - \sum_{s=0}^{\min(Q-2)/2} C_s^2 C_{Q-s}^2 \eta^s (1-\eta)^{k-s} (r_1)^s (1-r_1)^{s-1} - c = 0. \]  

(12)

We approximate the numerical solutions to (11) and (12) for various (Q, k, \( \ell \), c) using Maple.

4.2.2 partly mixed, partly pure strategies. In addition to the mixed strategy equilibria derived from (11) and (12), there are also potential mixed strategy equilibria where a) \( r_i = 1 \) & \( 0 < r_{-i} < 1 \), b) \( r_i = 0 \) & \( 0 < r_{-i} < 1 \), c) \( 0 < r_i < 1 \) & \( r_{-i} = 1 \), and d) \( 0 < r_i < 1 \) & \( r_{-i} = 0 \). It should be obvious that case (b) is not in equilibrium. To illustrate how the equilibria can be found in the other three cases, consider \( r_i = 1 \) & \( 0 < r_{-i} < 1 \). Thus we are looking for \( r_{-i}^* \) such that \( EV_{-1}^v > EV_{-1}^{nv} \) and \( EV_{-1}^v = EV_{-1}^{nv} \).
If such an equilibrium exists, it can be found by assuming \( r_1 = 1 \) and solving for \( r_1^* \) in (12), then verifying that \( \text{EV}_{-1} > \text{EV}_{-1}^{nv} \) for \((r_1, r_{-1}) = (1, r_{-1}^*)\). The remaining cases are handled similarly by first substituting the pure strategy value of \( r_{-1} \) into (11).

4.2.3 pure strategies. Cases where all players play pure strategies can be found by substituting each pure strategy combination into \( \text{EV}_t^r - \text{EV}_t^{nv} \) and verifying that these difference are consistent with the putative equilibrium. For example, suppose we want to determine whether \((r_1, r_{-1}) = (1, 0)\) is an equilibrium for \( Q = M \) and \( k > 0, \ell > 0 \). In this case, we substitute \((r_1, r_{-1}) = (1, 0)\) into the left hand sides of (11) and (12). We then note that

\[
(11a) = \begin{cases} 1, & k=Q \\ 0, & \text{otherwise.} \end{cases}
\]

where (11a) indicates the first sum in equation (11). We also note that (11b) = 0 unless \( s = 0 \) and \( s \geq \lceil Q/2 \rceil \), which cannot be the case. This implies that \( \text{EV}_t^r > \text{EV}_t^{nv} \) if and only if \( k = Q = M \).

Likewise, if \((r_1, r_{-1}) = (1, 0)\), then

\[
(12a) = \begin{cases} 1, & k=1 \leq \ell \land Q=1 \\ 0, & \text{otherwise.} \end{cases}
\]

where (12a) indicates the first sum in equation (12). If \( Q = k = M \), then (12a) = 0 and hence \( \text{EV}_{-1}^r - \text{EV}_{-1}^{nv} < 0 \). Thus for \( Q = M \), \((r_1, r_{-1}) = (1, 0)\) is an equilibrium if and only if \( k = M \). Other pure strategy equilibria can be found similarly.
4.3 Welfare Criteria

We wish to use the quasi-symmetric equilibrium values, \((r_1, r_{-1}, r_0)\) derived in the previous section, to compute the probability that each voting rule adheres to various welfare criteria. We shall see that the presence of multiple equilibria complicates the comparison.

[Table 8 here]

The set of all possible quasi-symmetric equilibria for \(N = 5\) and \(c = .25\) are presented in Table 8. The results are fairly representative of the results found for other \(N\) and \(c\). Note that for many \((k, \ell, c)\) and for each voting rule, there are several equilibria. This makes behavioral predictions somewhat difficult. When there are multiple Nash equilibria, the assumption that a Nash equilibrium will be played relies on a mechanism or process that will lead all players to expect the same equilibrium (Fudenberg and Tirole 1991: 18). Schelling’s (1960) focal point theory argues that players will coordinate on the equilibrium that is conspicuous among all other equilibria. We consider several focal point theories that might help the player coordinate on an equilibrium. Each produces fairly similar results.

4.3.1 Focal Points Dependent on Context Only. Our first focal point theory presumes that the only coordinating mechanism is within the context of the vote itself. For example, in a vote among members of the utilitarian society, the equilibrium that maximizes the sum of payoffs might be focal. In other groups, the way the game is presented may lead to a focal equilibrium. Since institutional framers must choose voting procedures without knowing the exact context

\[\text{\textsuperscript{21}}\] Numerical computations for other \(N \leq 10\) and \(c\) are available upon request.
that will arise, we apply the principle of insufficient reason and assume that *all possible equilibria are equally likely*. We then calculate the probability of adhering to each criterion for each voting rule and equilibrium separately. For a particular \((k, \ell)\), we compare each equilibrium for AM with each equilibrium for SMQ. Thus, for instance, if there are two equilibria for AM and three for SMQ for a given \((k, \ell)\), then there will be six pairwise comparisons. For each criterion, the voting rule that outperforms the other in more pairwise contests is considered the better voting rule in terms of that criterion. Results based on these assumptions are presented in Table 8. The voting rule which performs better on more pairwise comparisons is marked on the right hand side of the table along with the net number of pairwise comparisons in which the named voting rule outperforms the other. For example, in the first row of Table 8, SMQ(1) means that SMQ outperforms AM for \(\eta + 1\) pairs, while AM outperforms SMQ for only \(\eta\) pairs.

The results are mixed. For SMQ \((Q=1)\), if \(\ell \neq 0\), then SMQ \((Q=1)\) performs at least as well as AM in terms of the Pareto criterion, while AM tends to outperform SMQ \((Q=1)\) in terms of BT, ESG, and Rae. However, if \(\ell = 0\), then SMQ \((Q=1)\) performs at least as well as AM on all criteria. Meanwhile, SMQ \((Q=M)\) and AM perform almost identically on all four criteria, with three cases providing advantage to AM in terms of BT, ESG, and Rae.\(^{22}\) Institutional framers

\[^{22}\text{It can be shown that SMQ \((Q=M)\) and AM are equally likely to adhere to the Pareto criterion as follows. On the one hand, if } x \text{ is Pareto preferred to } q, \text{ then } \#(T) > 0, \#(T_x) = 0, \text{ and } \#(T_q) \geq 0. \text{ Since members of } T_q \text{ never vote, the entire population of potential voters consists exclusively of members of } T_x. \text{ In this case, } Pr[\text{pass | SMQ}] = Pr[\text{pass | AM}] \text{ and } Pr[\text{pass | SMQ} \& x \text{ Pareto}] = Pr[\text{pass | AM} \& x \text{ Pareto}]. \text{ On the other hand, if } q \text{ is Pareto preferred to } x, \text{ then } \#(T_x) = 0 \text{ and members of } T_x \text{ and } T_q \text{ will still have a dominant incentive to not vote. The measure will fail under both SMQ \((Q=M)\) and AM. Hence, } Pr[\text{fail | SMQ} \& q \text{ Pareto}] = Pr[\text{fail | AM} \& q \text{ Pareto}]. \text{ These results can be compared with those for sincere voter model, where for any } Q, \text{ SMQ is more likely to select the Pareto preferred alternative than AM and the differences can be large.}\]
who don’t have expectations about how many individuals will be of each player type, would probably prefer AM to SMQ (Q=1), but have little reason to prefer AM to SMQ (Q=M) or vice-versa. For larger costs, the differences between AM and SMQ are considerably smaller. For example, for \(c > .55\), there are no differences between AM and SMQ (Q=M), regardless of the criterion.\(^{23}\)

Many scholars believe that properties intrinsic to the payoffs themselves should help resolve the equilibrium selection problem (Harsanyi and Selten 1988; Myerson 1991; van Damme 2002). Three equilibrium selection concepts are common in the literature: payoff dominance, risk dominance, and coalition proof equilibria. These concepts can be thought of as focal point theories based on the payoffs themselves. Nevertheless, if they cannot reduce the equilibria to a singleton, then individuals may still need cues from the context of the vote to coordinate on the same equilibrium.

4.3.2 Payoff Dominance. Payoff dominance asserts that individuals will not play equilibria that are Pareto sub-optimal in terms of expected payoffs (Harsanyi and Selten 1988; Myerson 1991: 120-2).\(^{24}\) For example, if \((k, \ell) = (2, 1)\) and voting is conducted under SMQ (Q = 1), then there are three equilibria in terms of \((r_1, r_{-1})\): \((0.146, 0.854), (0.854, 0.146),\) and \((0.250, 1.000)\) -- see Table 8. Arguably the last equilibrium, \((0.250, 1.000)\), will not be played because \((0.146, 0.854)\)

\(^{23}\) For \(c = 0\), one might expect results similar to the sincere case. However, in this case, most \((k, \ell)\) produce infinitely many equilibria which is not analogous to the sincere results.

\(^{24}\) It is important not to confuse the Pareto criterion used to determine payoff dominance and the Pareto criterion we use to judge voting rules. The former is based on expected payoffs which are a function of play in the game. The latter are based on individual preferences for the alternatives, without reference to action or expected costs associated with those actions.
produces a greater expected payoff for all players (the expected payoffs are (0.000, 0.688) and (0.022, 0.728), respectively). This does not mean that \((r_1, r_{-1}) = (0.146, 0.854)\) will be played for certain because \((r_1, r_{-1}) = (0.146, 0.854)\) and \((r_1, r_{-1}) = (0.854, 0.146)\) are both Pareto optimal in terms of expected payoff. However, it does suggest that \((0.250, 1.000)\) will not be played and reduces the number of potential equilibria.

[Table 9 here]

Assume that payoff dominated equilibria will not be played and that the remaining equilibria are equally likely. Then a pairwise comparison yields the results presented in Table 9. The results are similar to those in Table 8, though the differences between AM and SMQ are generally reduced. For example, the number of pairwise comparisons where AM outperforms SMQ \((Q=1)\) is diminished.

4.3.3 Risk Dominance. Harsanyi and Selten (1988) proposed a second equilibrium selection concept known as risk dominance. Loosely, a risk dominant equilibrium is the least risky equilibrium given that other players might attempt to play some other equilibrium (see Harsanyi and Selten 1988 for a formal definition). We do not include an analysis of risk dominant equilibria here because Harsanyi and Selten’s concept has not been axiomatically derived for cases with \(n\)-players, cases with more than two equilibria, or cases with mixed strategies (Myerson 1991: 119). In addition, the tracing procedure used to find risk dominance appears to be limited to two equilibria (van Damme, 2002: 1580-5).
4.3.4 Coalition Proof Equilibria. Coalition Proof Equilibrium (CPE) is a third equilibrium selection concept (Bernheim, et. al 1987; van Damme, 2002: 1585). A CPE is a Nash equilibrium for which no subcoalition can profitably deviate to another strategy vector that would be stable with respect to further deviations. For example, assume individuals are voting under AM and \((N, k, \ell) = (5, 5, 0)\). The outcome \((r_1, r_{-1}) = (0, 0)\) is not a CPE because pre-game communication among any three members of \(T_1\) would allow them all to agree to deviate to \(r_1' = 1\), while the two remaining members of \(T_1\) played \(r_{1''} = 0\). Since the deviation of the three players is self-enforcing, in the sense that none of the three players would want to deviate from the planned coalition’s deviation either by themselves or by sub-coalition, three members of \(T_1\) playing \(r_1' = 1\) and two members of \(T_1\) playing \(r_{1''} = 0\) is a rational deviation for members of the coalition. Hence, \((r_1, r_{-1}) = (0, 0)\) is not considered a CPE.

One limitation with CPE is that they may not exist (Bernheim, et. al 1987). In our setting, for example, if \(k > M\), then no quasi-symmetric Nash, CPE exist using AM. This is because for any \(k > M\), the coalitional deviation of \(M\) members of \(T_1\) playing \(r_1' = 1\) and \(k-M\) members of \(T_1\) playing \(r_{1''} = 0\) is always rational and self-enforçable. Hence, to properly consider CPE we extended our analysis to include all equilibria – asymmetric as well as quasi-symmetric equilibria. Since this increases the complexity of the analysis, we limit our study to \(N=3\).^25

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0. For \(N=3\), all non-trivial, self-enforcing coalitions are size 2, and thus any possible defecting coalitions would be size 1. For \(N=5\), deviating coalitions could be size 4, 3, or 2 and we would have to check whether these coalitions were self-enforcing by considering sub-coalitions of say size 1, 2, or 3 for coalitions sized 4. This makes the calculations for \(N=5\) considerably more
For $N=3$, all of the AM and SMQ ($Q=M$) equilibria are identical (this includes both asymmetric and quasi-symmetric equilibria). Furthermore, the CPE from these two voting rules are identical.

Nevertheless, the AM and SMQ ($Q=1$) equilibria differ. To compare the performance of these voting rules, we apply the following procedure. In cases with an unique equilibrium, we assume this equilibrium is the outcome. For cases with more than one equilibrium, we assume each of the CPE is equally likely. We then conduct our pairwise comparisons among the remaining equilibria as we did before. The results are listed in Table 10. Because there was always at least one CPE for cases with multiple equilibria, the number of equilibria reduced dramatically.

[Table 10 here]

With these assumptions, SMQ ($Q=1$) performs at least as well as AM in terms of the Pareto criterion. In addition, SMQ ($Q=1$) and AM perform equally as well in terms of the BT criterion. In the latter case, one voting rule outperforms the other for some $(k, \ell)$ but the other voting rule outperforms the first for other $(k, \ell)$ -- the same number in each case. For ESG, the results are mixed. In terms of the Rae criterion, AM performs at least as well as SMQ ($Q=1$). We conclude that there is no clear reason to prefer either SMQ ($Q=1$) or AM if CPE are played.

4.3.5 The Importance of Equilibrium Selection. Without knowing the exact selection criteria used in a particular setting, it is difficult to recommend one voting rule over the other. On the
one hand, if institutional framers expect individuals to coordinate only on features unique to the context of the vote or on Pareto optimal equilibria, then they may have some reason to prefer AM to SMQ (Q=1). Nevertheless, they would have little reason to prefer AM to SMQ (Q=M). On the other hand, if institutional framers expect pre-game communication and individuals to coordinate on CPE, then they would have little reason to prefer AM to SMQ (Q=1) and perhaps no reason to prefer AM to SMQ (Q=M). These results suggest that AM performs better relative to SMQ under strategic conditions than under our sincere conditions. However, the differences between the voting rules can be minor in the strategic case, because the probability of voting seems to adjust to the probability of passage.

Of course, none of these selection criteria and focal point concepts reduce the equilibria to singletons in our model, all have been criticized (van Damme 2002), and there are other equilibrium selection criteria that individuals may use (Ibid.). Hence, multiple equilibria make the notion of all individuals playing the same equilibrium a bit questionable and it is not clear that all players will resolve the problem “as if” they used the same equilibrium selection criterion. This problem is exacerbated for larger N cases, such as state or national legislatures.

5 Conclusion

Institutional framers who think individuals will behave strategically may want to think about the mechanisms that will be used to coordinate on a particular equilibrium before they conclude that SMQ or AM is the better voting rule. If institutional framers believe these mechanisms exist only within the context of the vote, and each equilibrium is equally likely, then they may have good reason to favor AM over SMQ (Q=1). AM typically performs at least as well as SMQ
(Q=1), depending upon the number of individuals with each preference, and AM and SMQ
(Q=M) typically perform similarly. However, if framers believe that coordinating mechanisms
such as coalition proof equilibria or payoff dominance will help individuals chose equilibrium
strategies, then they will have less reasons to favor either voting rule. Behavioral adjustments to
each voting rule seem to mitigate the differences between the voting rules themselves. For large
costs of voting these differences can be negligible.

Moreover, if individuals fail to find a clear focal point that all players recognize, then
different individuals of the same player type may be voting with different probabilities. In other
words, individuals may play in a variety of ways and the expected outcome may be more akin to
the sincere voting results. Since this study explicitly examines one-shot votes, trial and error
cannot be used to reach a particular equilibrium. Hence, out of equilibrium play may be a real
possibility even among purely rational voters. This observation and alternative explanations for
human behavior, like cognitive heuristics or genetics, might make institutional framers take
greater interest in the results of the sincere voting model. The sincere model suggests that SMQ
always outperforms AM in terms of the Pareto and Rae criteria and SMQ typically outperforms
AM in terms of the ESG and Responsiveness criteria. Surprisingly, however, there are some
reasons to favor AM even in the sincere case.

First, AM typically outperforms SMQ in terms of the BT criterion, and it is always more
likely to favor the status quo. Policy makers who want to protect the status quo, perhaps because

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26 There may be empirical reasons to question the strategic model as well. As is the case with the
Palfrey-Rosenthal model (1983), the strategic model presented here implies that voting turnout increases
as the costs of voting increases, which is not supported by empirical evidence (Mueller 2003).
they are evaluating voting rules for changing institutional procedures, may want to apply the BT criterion. If they do, then they should typically prefer AM over SMQ.27

Second, AM can outperform SMQ in terms of the Expected Social Gain criterion if a plurality of those present favor the measure while a plurality of those not present oppose the measure. This is because SMQ considers only the preponderance of those present. AM considers the preponderance of eligible members as a whole.28

Third, numerical investigations suggest that AM outperforms SMQ (Q=M) in terms of Responsiveness criterion if the probability that an individual votes in favor of the proposal is greater than M/N and the population is sufficiently large (Conjecture 2). In assemblies the size of national legislatures, this may occur if turnout is sufficiently large and individuals are likely to propose measures that pass. For example, of the 2,198 roll call votes in U.S. House of Representatives from 1999 to 2002, 64% passed with favorable votes from more than an absolute majority of the members.29 Institutional framers that value responsiveness and expect such conditions may have good reason to adopt AM.

27 Suppose one had a simple k-majority rule with quorum requirement of M that was equally likely to pass the proposal as AM. Call this voting rule SMKM. In a separate work, we show that if SMKM exists and p_{1,1}, p_{0,1}, p_{-1,1} > 0, then AM is less likely to select the BT-preferred alternative than SMKM under the sincere voting framework. In other words, the superiority of AM in Theorem 2 is fully due to AM’s propensity to reject proposals. It is not due to a more appropriate treatment of absent voters and “votes to abstain.”

28 Voting rules, by their very definition, are based solely on information provided by voters. Thus, it might seem more reasonable to define ESG in terms of voters alone. If this is done, SMQ always performs at least as well as AM in the sincere framework, because, as long as the quorum is met, SMQ always chooses the alternative where the number voting in favor of the proposal exceed the number voting opposed. AM may not chose such an alternative.

0. Data from <http://voteview.uh.edu/default_nomdata.htm>.
Finally, to illustrate that these results depend on the context, consider the U.S. Constitutional Convention. When delegates arrived at the Convention in 1787, they adopted a voting rule that differed from the one used in the Congress of the Confederation. Rather than voting in state blocks using an absolute majority of the states, the Convention voted in state blocks using SMQ (Q=M) (Farrand 1966: 9 & 69). Although the delegates were presumably unaware of the properties described in this paper, their move from AM to SMQ (Q=M) appears to have been reasonable. If we assume that N = 13, that states were equally likely to favor and disfavor proposals, and there was some probability of indifference, then SMQ (Q=M) would outperform AM in terms of the Pareto, Expected Social Gain, Rae, and Responsiveness criteria in the sincere voting framework (regardless of the likelihood of indifference). AM would outperform SMQ only in terms of the BT criterion in that framework. Meanwhile, numerical computations suggest that the two voting rules would perform almost identically in the strategic framework. Since it appears that members of the Convention did not want to bias decisions in favor of the status quo, their adoption of SMQ appears to have been right headed.

\[ p_{1,1} = p_{-1,1} \text{ and } N = 13 \text{ in the sincere case, numerical computations indicate that a necessary condition for SMQ (Q=M) to outperform AM in terms of the BT criterion is that } (p_{1,0} + p_{0,1} + p_{0,0}) > .78. \]

Although it is difficult to actually estimate \( p_{1,1} + p_{0,0} + p_{1,0} \) in this setting, if divided states, abstaining votes, and half of the states not present were counted as elements of this category, then \( (p_{1,0} + p_{0,1} + p_{0,0}) = .21. \) This is far less than the .78 required.

\[ \text{One of the rules delegates adopted was that passed measures could always be revisited at a later date (Farrand: 9-10, 16). This and the large proportion of Federalists in attendance suggest that most state delegations did not favor the status quo.} \]
5 Appendix

Proofs of various theorems follow. The formulas used in tables 3-7 are available upon request.

**Theorem 2.** Assume \((2^{(N-1)} - 1)p_{0,1} \geq p_{0,0} + p_{1,0} + p_{i,0} + p_{i,1})\). Then for any \(Q\) s.t. \(0 \leq Q \leq M\), SMQ is more likely to observe the BT criterion than AM.

**Proof:** SMQ is more likely to fulfill the BT criterion than AM if the difference between the probability that SMQ selects a BT preferred alternative and the probability that AM selects a BT preferred alternative is positive. Since AM and SMQ select differently only if AM defeats a proposal that SMQ passes, it suffices to show that the following difference is positive:

\[
\text{pr}(x \text{ is BT preferred, SMQ selects } x, \text{ AM selects } q) - \text{pr}(q \text{ is BT preferred, SMQ selects } x, \text{ AM selects } q)
\]

\[
= \sum_{i=1}^{M-1} \binom{N-s}{i} \cdot (p_{1,i})^i \sum_{s=Q}^{N-s} \binom{N-s}{r} \cdot (p_{0,1})^r \cdot (p_{1,0} + p_{0,0})^{N-s-r}
\]

\[
- \sum_{i=1}^{M-1} \binom{N-s}{i} \cdot (p_{1,i})^i \sum_{r=Q}^{N-s-1} \binom{N-s-1}{r} \cdot (p_{0,1})^r \cdot \sum_{a=0}^{N-s-1} \binom{N-s-1}{a} \cdot (p_{1,a} + p_{0,a})^{N-s-1-r-a} \sum_{t=\max(0, Q-s-r)}^{\min(N-s-1, Q-s-r)} \binom{N-s-1-t}{t} \cdot (p_{1,t} + p_{0,t})^{N-s-1-r-a-t} (A1)
\]

Note that in the second line of (A1) the upper limit of the summation in \(s\) ensures that the measure is defeated by AM, the upper limits of the summations in \(r\) and \(\alpha\) ensure that at least one person opposes the proposal (so \(q\) is BT preferred), the term \(t \leq \min(N-s-\alpha-r, s-1)\) ensures that \(s > t\) (so the proposal passes under SMQ), and the term \(t \geq \max(0, Q-s-r)\) ensures that a quorum is met (so the proposal passes under SMQ). Now (A1) =

\[
= \sum_{i=1}^{M-1} \binom{N-s}{i} \cdot (p_{1,i})^i \left[ (p_{0,1})^r \cdot (p_{1,0} + p_{0,0})^{N-s-r} \right]
\]

\[
- \sum_{r=Q}^{N-s-1} \binom{N-s-1}{r} \cdot (p_{0,1})^r \sum_{a=0}^{N-s-1} \binom{N-s-1}{a} \cdot (p_{1,a} + p_{0,a})^{N-s-1-r-a} \sum_{t=\max(0, Q-s-r)}^{\min(N-s-1, Q-s-r)} \binom{N-s-1-t}{t} \cdot (p_{1,t} + p_{0,t})^{N-s-1-r-a-t} (A1)
\]
Theorem 3. Assume $p_{1,1}, p_{0,1}, p_{-1,1} > 0$ and $p_{1,0} > p_{-1,0}$. Then for any $Q$ s.t. $0 < Q < M$, SMQ produces greater expected social gain than AM.

Proof: Let $s$ be the number of yeas, $t$ be the number of nays, $r$ the number voting abstain, $\sigma$ be the number of absent supporters, $\tau$ be the number of the absent opposers, and $\rho$ be the number of indifferent absentees.

$$
\text{ESG}_{\text{SMQ}} = \mathbb{E}[s+\sigma-t-\tau \mid \text{SMQ selects } x] + \mathbb{E}[t+\tau-s-\sigma \mid \text{SMQ selects } q] \\
= \mathbb{E}[s+\sigma-t-\tau \mid \text{SMQ selects } x] - \mathbb{E}[s+\sigma-t-\tau \mid \text{SMQ selects } q].
$$

Similarly, $\text{ESG}_{\text{AM}} = \mathbb{E}[s+\sigma-t-\tau \mid \text{AM selects } x] - \mathbb{E}[s+\sigma-t-\tau \mid \text{AM selects } q]$.

Therefore, $\text{ESG}_{\text{SMQ}} - \text{ESG}_{\text{AM}} =$

$$
\mathbb{E}[s+\sigma-t-\tau \mid \text{SMQ selects } x] - \mathbb{E}[s+\sigma-t-\tau \mid \text{AM selects } x] + \mathbb{E}[s+\sigma-t-\tau \mid \text{AM selects } q] - \mathbb{E}[s+\sigma-t-\tau \mid \text{SMQ selects } q] \\
= \mathbb{E}[s+\sigma-t-\tau \mid \text{SMQ selects } x \setminus \text{AM selects } x] + \mathbb{E}[s+\sigma-t-\tau \mid \text{AM selects } q \setminus \text{SMQ selects } q] \\
= \mathbb{E}[s+\sigma-t-\tau \mid \text{SMQ selects } x \cap \text{AM selects } q] + \mathbb{E}[s+\sigma-t-\tau \mid \text{AM selects } q \cap \text{SMQ selects } x] \\
= 2\mathbb{E}[s+\sigma-t-(N-s-t-\sigma-r-\rho) \mid \text{AM selects } q \cap \text{SMQ selects } x]
$$
For notational convenience, let \( L = L(s, t, r, \rho) = N^s \cdot c_{s,t,r} \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \),

Thus (A3) =

\[
2 \cdot \sum_{k=1}^{N-1} \sum_{t=0}^{N-1} \sum_{r=0}^{N-1} \sum_{\rho=0}^{N-1} \left( L \right)_{N-1-t-r-\rho} \cdot \cdot \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r 
\]

Now (A4) is positive since \( s > t \) in that sum. To complete the proof, we show that (A5) is non-negative; this follows if

\[
\sum_{\sigma=0}^{N-1-s-t-r-\rho} \cdot \cdot \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r 
\]

To further simplify notation, let \( R = N - s - t - r - \rho \) and assume that \( R \) is even. The proceeding argument can be easily adapted for \( R \) odd. Now since \( R \) is even, it follows that (A6) =

\[
\sum_{\sigma=0}^{R} \cdot \cdot \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r \cdot (p_{1,0})^t \cdot (p_{-1,0})^r 
\]
$\geq 0$, since $p_{1,0} \geq p_{-1,0}$ by assumption. □

**Theorem 5.** Assume $p_{1,1} = p_{0,1} = p_{-1,1} = 1/3$ and $3 < N < \infty$. Then for any $Q$ s.t. $0 \leq Q \leq M$, SMQ is more responsive than AM.

**Proof:** In this proof assume $N$ is odd. This implies that $M$ is even and $N-M = M-1$. The proof can be easily adapted to the case of $N$ even. Let $p = p_{1,1} = p_{0,1} = p_{-1,1}$. Therefore, $\Pr(i \text{ pivotal} \mid AM) = \sum_{t=0}^{2M-1} \binom{M-1}{t} \cdot (p_{1,1})^{t+1} \cdot (p_{-1,1} + p_{0,1})^{N-t} = \sum_{t=0}^{M-1} \binom{M-1}{t} \cdot p^t \cdot p^{N-t} - \binom{M-1}{t} \cdot (p_{0,1})^t \cdot (p_{1,1})^{N-t} - \binom{M-1}{t} \cdot (p_{1,0})^t \cdot (p_{-1,1})^{N-t} - \binom{M-1}{t} \cdot (p_{0,0})^t \cdot (p_{0,1})^{N-t}.$

Note the probability that a voter is absent is 0, hence the quorum will certainly be met. Thus, individual $i$ will be critical under SMQ if and only if, among the rest of the population, either the yeas exceed the nays by one, or there are equally many yeas and nays. Thus the

\[
\Pr(i \text{ pivotal} \mid SMQ) = \sum_{t=0}^{M-1} \binom{M-1}{t} \cdot (p_{0,1})^t \cdot (p_{1,1})^{N-t} \cdot C \cdot N \cdot (N-1)C \cdot (N-1)C \cdot p^{N-1} + \sum_{t=0}^{M-1} \binom{M-1}{t} \cdot (p_{0,1})^t \cdot (p_{0,0})^{N-t} \cdot C \cdot N \cdot (N-1)C \cdot (N-1)C \cdot p^{N-1}.
\]

We will show that $\Pr(i \text{ pivotal} \mid AM) - \Pr(i \text{ pivotal} \mid SMQ) < 0$. Now

\[
\Pr(i \text{ pivotal} \mid AM) - \Pr(i \text{ pivotal} \mid SMQ) / p^{N-1} = \sum_{t=0}^{M-3} \binom{M-1}{t} \cdot (M-1)C \cdot (M-1)C \cdot p^{N-1} - \sum_{t=0}^{M-2} \binom{M-1}{t} \cdot (M-1)C \cdot (M-1)C \cdot p^{N-1}.
\]
\[
= n^{-1} C_{M+1} \sum_{t=0}^{M-3} \frac{M-3}{t!} \sum_{i=0}^{M-2} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} - \sum_{t=1}^{M-2} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} \cdot n^{-1} C_{i+1}.
\]  
(A7)

Letting \(r = M - t\), (A7) =
\[
= n^{-1} C_{M+1} \sum_{t=0}^{M-3} \frac{M-3}{t!} \sum_{i=0}^{M-2} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} - \sum_{t=2}^{M-2} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} \cdot n^{-1} C_{i+1}.
\]  
(A8)

Letting \(t = r - 2\), (A8) =
\[
= n^{-1} C_{M+1} \sum_{t=0}^{M-3} \frac{M-3}{t!} \sum_{i=0}^{M-2} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} - \sum_{t=2}^{M-3} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} \cdot n^{-1} C_{i+1}.
\]  
(A9)

Recalling that we assume \(M\) is even, let \(J = (M-2) / 2\). Thus (A9) =
\[
= n^{-1} C_{M+1} \sum_{t=0}^{M-3} \frac{M-3}{t!} \sum_{i=0}^{M-3} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} + \sum_{i=0}^{J-1} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} + \sum_{i=0}^{J-1} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} \cdot n^{-1} C_{i+1}.
\]  
(A10)

Claim 1: (A10) < 0. This is shown by noting that
\[
= n^{-1} C_{M+1} \sum_{t=0}^{M-3} \frac{M-3}{t!} \sum_{i=0}^{M-3} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} + \sum_{i=0}^{J-1} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} + \sum_{i=0}^{J-1} n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} \cdot n^{-1} C_{i+1}.
\]  
(A11)

The proof is completed by showing that (A10) < 0, (A11) ≤ 0, and (A12) ≤ 0 in three claims.

Claim 2: (A11) ≤ 0. This is shown by first fixing \(t\) such that \(t \in [1, J-1]\), then noting that
\[
= n^{-1} C_{M+1} \cdot n^{-1} C_{i+1} \cdot n^{-1} C_{i+1} = \frac{(N-1)!}{(M-1-t)!} \frac{1}{(M-1-t)!} \left( \frac{1}{(M-1)(M-2) \ldots (t+1)} - \frac{1}{(M-1-t) \ldots (M-t)} \right)
\]  
(A13)

Now \(t > J + 1\), as \(M\) is even, implies \(2t > 2J + 1\) \(\Rightarrow\) \((2t > M - 1) \Rightarrow\) \((2t > N - M) \Rightarrow\) \((M - 1 > N - 1 - 2t)\).

Thus, \(t > J + 1\) implies \((M - 1)(M - 2) \ldots (t + 1) > (N - 1 - 2t) \ldots (M - t)\). Thus, (A13) < 0.

Claim 2: (A11) ≤ 0. This is shown by first fixing \(t\) such that \(t \in [1, J-1]\), then noting that
\[ N^{-1} C_{M-1} \cdot C_i - N^{-1} C_{M+2} \cdot C_{M+3} \]
\[ = \frac{(N-1)!}{(M-t-2)!} \left( \frac{1}{(M-t-3)!} \left( \frac{1}{(M-t-2)(M-t-3)(M-t-4)} - \frac{1}{(2t+3)(2t+2)(2t+1)} \right) \right) \]  
\[ (A14) \]

Now \((t \leq J-1) \Rightarrow (2t \leq 2J-2) \Rightarrow (2t+3 \leq 2J+1 \leq M-1)\). Hence, \((M-1) \cdots (M-t-2) \cdot (M-t-1) \geq (2t+3) \cdots (t+1)\). Thus, \((A14) < 0\).

Claim 3: \((A12) \leq 0\). A simple calculation shows that \((A12) < N^{-1} C_{M-1} \sum_{l=J}^{J+2} C_l - \sum_{r=J}^{J+2} C_r \cdot N^{-1} C_t \)
\[ = \frac{(N-1)!}{(M-1)! (J+1)!} \left( 2 + 1 - \frac{2}{J+2} - \frac{1}{2} - 2 + \frac{1}{J+1} - \frac{(2J+1)(2J)(2J-1)}{(J+1)(J+2)^2} \right). \]  
\[ (A15) \]

Note that if \(J = 1\), \((A15) = 0\). We now consider \((A15)\) for \(J \geq 2\). A straightforward calculus exercise shows that for \(J \geq 2\), \(\frac{(2J+1)(2J)(2J-1)}{(J+1)(J+2)^2} \geq 1\). Hence, in this case, \((A15) =
\[ 1 + \left( \frac{1}{J+1} - \frac{2}{J+2} \right) - \frac{1}{2} - \frac{(2J+1)(2J)(2J-1)}{(J+1)(J+2)^2} < 0. \]

The proof of Theorem 6 uses the following lemma.

**Lemma 1.** If \(s > t\), then \(N^{-1} C_s \cdot (N^{-1} C_{(t-s)}) - N^{-1} C_{(s-t)} \cdot N^{-1} C_t < 0\).

**Proof:** \(N^{-1} C_s \cdot (N^{-1} C_{(t-s)}) - N^{-1} C_{(s-t)} \cdot N^{-1} C_t = \frac{(N-1)!}{(N-1-s)! (t-s)!} - \frac{(N-1)!}{(N-s)! (s-t)!} \frac{1}{t!} \) \[ = \frac{(N-1)!}{(N-s-t)!} \left( \frac{1}{(s-1)! (t-1)!} - \frac{1}{t!} \right). \]  
Since \(s > t\), the term in brackets is negative. \(\square\)

**Theorem 6.** Assume \(p_{1,1}, p_{1,1} > 0\), \(p_{1,1} + p_{1,1} < 1\), and \(N \geq 3\). Then for any \(Q\) s.t. \(0 < Q < M\), \(SMQ\) has a smaller Rae index than \(AM\).

**Proof:** We proceed by showing that \(\text{Rae(SMQ)} - \text{Rae(AM)} < 0\). \(\text{Rae(SMQ)} - \text{Rae(AM)} = \)
\[ p_{1,1} \cdot \text{Prob}[\text{SMQ selects } q] + p_{-1,1} \cdot \text{Prob}[\text{SMQ selects } x] - p_{1,-1} \cdot \text{Prob}[\text{AM selects } q] - p_{-1,-1} \cdot \text{Prob}[\text{AM selects } x] = \]
\[ p_{1,1} \cdot (\text{Prob}[\text{SMQ selects } q] - \text{Prob}[\text{AM selects } q]) + p_{-1,1} \cdot (\text{Prob}[\text{SMQ selects } x] - \text{Prob}[\text{AM selects } x]) = \]
\[ - p_{1,1} \cdot (\text{Prob}[\text{AM selects } q \setminus \text{SMQ selects } q]) + p_{-1,1} \cdot (\text{Prob}[\text{SMQ selects } x \setminus \text{AM selects } x]) = \]
\[ - p_{1,1} \cdot (\text{Prob}[\text{SMQ selects } x \cap \text{AM selects } q]) + p_{-1,1} \cdot (\text{Prob}[\text{SMQ selects } x \cap \text{AM selects } x]) = \]

\[ - p_{1,1} \left[ \sum_{s=0}^{M-2} \sum_{t=0}^{N-2-1} X_{s}^{(1)} \left( p_{1,1} \right)^{s} H_{1-s-t}^{(-1)} C_{t}^{(p_{-1,1})} \sum_{r=0}^{N-2-1} H_{1-s-t-r}^{(-1)} C_{r}^{(p_{0,1})} \left( p_{1,0} + p_{-1,0} + p_{0,0} \right)^{H_{1-s-t-r}} \right] \]

\[ + p_{-1,1} \left[ \sum_{s=0}^{M-2} \sum_{t=0}^{N-2-1} X_{s}^{(1)} \left( p_{1,1} \right)^{s} H_{1-s-t}^{(-1)} C_{t}^{(p_{-1,1})} \sum_{r=0}^{N-2-1} H_{1-s-t-r}^{(-1)} C_{r}^{(p_{0,1})} \left( p_{1,0} + p_{-1,0} + p_{0,0} \right)^{H_{1-s-t-r}} \right] \]

\[ = - \sum_{s=0}^{M-2} \sum_{t=0}^{N-2-1} X_{s}^{(1)} \left( p_{1,1} \right)^{s} H_{1-s-t}^{(-1)} C_{t}^{(p_{-1,1})} \sum_{r=0}^{N-2-1} H_{1-s-t-r}^{(-1)} C_{r}^{(p_{0,1})} \left( p_{1,0} + p_{-1,0} + p_{0,0} \right)^{H_{1-s-t-r}} \]

\[ + \sum_{s=0}^{M-2} \sum_{t=0}^{N-2-1} X_{s}^{(1)} \left( p_{1,1} \right)^{s} H_{1-s-t}^{(-1)} C_{t}^{(p_{-1,1})} \sum_{r=0}^{N-2-1} H_{1-s-t-r}^{(-1)} C_{r}^{(p_{0,1})} \left( p_{1,0} + p_{-1,0} + p_{0,0} \right)^{H_{1-s-t-r}} \]

\[ = - \sum_{s=0}^{M-2} \sum_{t=0}^{N-2-1} X_{s}^{(1)} \left( p_{1,1} \right)^{s} H_{1-s-t}^{(-1)} C_{t}^{(p_{-1,1})} \sum_{r=0}^{N-2-1} H_{1-s-t-r}^{(-1)} C_{r}^{(p_{0,1})} \left( p_{1,0} + p_{-1,0} + p_{0,0} \right)^{H_{1-s-t-r}} \]

\[ + \sum_{s=0}^{M-2} \sum_{t=0}^{N-2-1} X_{s}^{(1)} \left( p_{1,1} \right)^{s} H_{1-s-t}^{(-1)} C_{t}^{(p_{-1,1})} \sum_{r=0}^{N-2-1} H_{1-s-t-r}^{(-1)} C_{r}^{(p_{0,1})} \left( p_{1,0} + p_{-1,0} + p_{0,0} \right)^{H_{1-s-t-r}} \]

\[ < \sum_{s=0}^{M-2} \sum_{t=0}^{N-2-1} X_{s}^{(1)} \left( p_{1,1} \right)^{s} H_{1-s-t}^{(-1)} C_{t}^{(p_{-1,1})} \sum_{r=0}^{N-2-1} H_{1-s-t-r}^{(-1)} C_{r}^{(p_{0,1})} \left( p_{1,0} + p_{-1,0} + p_{0,0} \right)^{H_{1-s-t-r}} \]

\[ < 0, \text{ by lemma 1.} \]
References


Table 1. Assumed Preference-Action Combinations

<table>
<thead>
<tr>
<th>$D_i$</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>i prefers x and votes for x</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>i is indifferent and votes abstain</td>
</tr>
<tr>
<td>(−1, 1)</td>
<td>i prefers q and votes for q</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>i prefers x but does not vote</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>i is indifferent but does not vote</td>
</tr>
<tr>
<td>(−1, 0)</td>
<td>i prefers q but does not vote</td>
</tr>
</tbody>
</table>
Table 2. The probability of selecting the Pareto preferred alternative

<table>
<thead>
<tr>
<th>Preference Probabilities</th>
<th>N = 3</th>
<th>N = 10</th>
<th>N = $\infty$†</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{1,1}$, $P_{0,1}$, $P_{-1,1}$, $P_{1,0}$, $P_{0,0}$, $P_{-1,0}$</td>
<td>SMQ</td>
<td>SMQ</td>
<td>SMQ</td>
</tr>
<tr>
<td>0.167, 0.167, 0.167, 0.167, 0.167</td>
<td>Q=0 0.431</td>
<td>Q=0 0.034</td>
<td>Q=0 0.000</td>
</tr>
<tr>
<td>.27, .06, .27, .18, .04, .18</td>
<td>Q=M 0.317</td>
<td>Q=M 0.024</td>
<td>Q=M 0.000</td>
</tr>
<tr>
<td>.40, .40, .20, 0, 0, 0</td>
<td>AM 0.306</td>
<td>AM 0.018</td>
<td>AM 0.000</td>
</tr>
</tbody>
</table>

† A pair of alternatives are Pareto indeterminate if neither x is Pareto preferred to q nor q is Pareto preferred to x. If $(p_{1,1} + p_{1,0} > 0)$ and $(p_{-1,1} + p_{-1,0} > 0)$, then $Pr($Pareto indeterminancy$) \rightarrow 1$, as $N \rightarrow \infty$. Using $10^{-2}$ as a measure of convergence, Pareto indeterminancy converges to 1 in populations sized $N = 15$ for most preference probabilities.
Table 3. The probability of selecting the BT-preferred alternative

<table>
<thead>
<tr>
<th>Preference Probabilities</th>
<th>N = 3</th>
<th>N = 100</th>
<th>N = ∞</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SMQ</td>
<td>SMQ</td>
<td>SMQ</td>
</tr>
<tr>
<td>p₁, p₀, pₐ, p₁, p₀, pₐ</td>
<td>Q=0</td>
<td>Q=M</td>
<td>AM</td>
</tr>
<tr>
<td>0.167, 0.167, 0.167, 0.167</td>
<td>0.787</td>
<td>0.801</td>
<td>0.759</td>
</tr>
<tr>
<td>.27, .06, .27, .18, .04, .18</td>
<td>0.733</td>
<td>0.824</td>
<td>0.817</td>
</tr>
<tr>
<td>.40, .40, .20, 0, 0, 0</td>
<td>0.904</td>
<td>0.904</td>
<td>0.712</td>
</tr>
</tbody>
</table>
| Note: AM typically outperforms SMQ when N ≥ 9. This difference becomes more pronounced as N increases.
Table 4. Expected Social Gain per Person

<table>
<thead>
<tr>
<th>Preference Probabilities</th>
<th>N = 3</th>
<th>N = 100</th>
<th>N = ∞†</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SMQ</td>
<td>SMQ</td>
<td>SMQ</td>
</tr>
<tr>
<td></td>
<td>Q=0</td>
<td>Q=M</td>
<td>AM</td>
</tr>
<tr>
<td>p1,1; p0,1; p−1,1</td>
<td>0.241</td>
<td>0.158</td>
<td>0.093</td>
</tr>
<tr>
<td>0.167, 0.167, 0.167</td>
<td>.327</td>
<td>.241</td>
<td>.213</td>
</tr>
<tr>
<td>.27, .06, .27, .18, .04, .18</td>
<td>.376</td>
<td>.376</td>
<td>.248</td>
</tr>
<tr>
<td>.40, .40, .20, 0, 0, 0</td>
<td>.419</td>
<td>.541</td>
<td>.539</td>
</tr>
<tr>
<td>.30, .05, .20, 0, 0, .45</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Figures are expected values, calculated as (ESG / N), not probabilities.

† As N→∞, ESG/N = +/−(p1,1 + p1,0 − p−1,1 − p−1,0), where + or − is determined by whether the proposal passes or not.
Table 5. The Responsiveness Criterion

<table>
<thead>
<tr>
<th>Preference Probabilities</th>
<th>N = 3</th>
<th>N = 100</th>
<th>N = ∞†</th>
</tr>
</thead>
<tbody>
<tr>
<td>P_{1,1}, P_{0,1}, P_{-1,1}</td>
<td>SMQ</td>
<td>SMQ</td>
<td>SMQ</td>
</tr>
<tr>
<td>P_{1,0}, P_{0,0}, P_{-1,0}</td>
<td>Q=0</td>
<td>Q=M</td>
<td>AM</td>
</tr>
<tr>
<td>0.167, 0.167, 0.167, 0.167, 0.167</td>
<td>0.722, 0.472, 0.278</td>
<td>0.138, 0.098, 8.2E-15</td>
<td>0.000, 0.000, 0.000</td>
</tr>
<tr>
<td>.27, .06, .27, .18, .04, .18</td>
<td>0.606, 0.446, 0.394</td>
<td>0.108, 0.111, 3.8E-07</td>
<td>0.000, 0.000, 0.000</td>
</tr>
<tr>
<td>.40, .40, .20, .0, .0</td>
<td>0.640, 0.640, 0.480</td>
<td>0.004, 0.004, 0.009</td>
<td>0.000, 0.000, 0.000</td>
</tr>
<tr>
<td>.52, .02, .40, .02, .02</td>
<td>0.506, 0.502, 0.499</td>
<td>0.041, 0.041, 0.077</td>
<td>0.000, 0.000, 0.000</td>
</tr>
</tbody>
</table>

† The probability of being pivotal approaches zero as N approaches infinity.
Table 6. The Modified Rae Criterion

<table>
<thead>
<tr>
<th>Preference Probabilities</th>
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<th>N = ∞†</th>
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<td>SMQ</td>
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Note: Figures indicate the probability of a loss, hence smaller figures are more desirable. In all finite cases, SMQ outperforms AM.

† For N = ∞, pr(Rae) = p_{1,1} \cdot pr(defeated) + p_{-1,1} \cdot pr(passes). For instance, in the last row of the table and N = ∞, the probability that SMQ adheres to the modified Rae criterion is .40(0) + .20(1).
Table 7. Summary: Sincere Voting Model

<table>
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<tr>
<th>Pareto</th>
<th>BT</th>
<th>Expected Social Gain</th>
<th>Responsiveness</th>
<th>Rae</th>
</tr>
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<td>SMQ outperforms</td>
<td>conditional, but favors AM</td>
<td>conditional, but favors SMQ</td>
<td>conditional, but favors SMQ</td>
<td>SMQ outperforms</td>
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</table>

*Note:* SMQ outperforms AM in terms of the Pareto and Modified Rae criteria. SMQ tends to outperform AM in terms of the Expected Social Gain and Responsiveness criteria. AM tends to outperform SMQ in terms of the BT criterion.
Table 8: A Pairwise Comparison of SMQ and AM in Quasi-Symmetric Equilibrium
(N = 5, c=.25)

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<th>(r_1, r_{\ell})</th>
<th>(r_1, r_{\ell})</th>
<th>(r_1, r_{\ell})</th>
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**SMQ (Q=M)**

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</table>

*Notes*: For all equilibria, $r_0 = 0$. The probability of adhering to each criterion is calculated for each equilibrium separately. The comparisons reported are the net number of pairwise comparisons where AM outperforms SMQ (or visa versa) for a particular $(k, \ell)$. For example, AM (2) means that AM outperforms SMQ for $\eta + 2$ pairs, whereas SMQ outperforms AM for only $\eta$ pairs.
Table 9: A Comparison of SMQ and AM in Quasi-Symmetric Equilibria that are Not Payoff Dominated
(N = 5, c = .25)

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<thead>
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<th>Equil. 1</th>
<th>Equil. 2</th>
<th>Equil. 3</th>
<th>Pairwise Comparisons against AM</th>
</tr>
</thead>
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<td>Pareto</td>
<td>BT</td>
<td>ESG</td>
</tr>
<tr>
<td><strong>SMQ (Q=1)</strong></td>
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</tr>
<tr>
<td>1 0 (1.000, 0.000)</td>
<td>SMQ(1)</td>
<td>SMQ(1)</td>
<td>SMQ(1)</td>
</tr>
<tr>
<td>1 1 (0.250, 0.750)</td>
<td>same</td>
<td>AM (1)</td>
<td>AM (1)</td>
</tr>
<tr>
<td>1 2 (0.500, 0.500)</td>
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<td>AM (1)</td>
<td>AM (1)</td>
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**SMQ (Q=M)**

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</tr>
<tr>
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<td>(0.000, 0.000)</td>
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</tr>
<tr>
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<td>(0.000, 0.000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 2</td>
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<td>(1.000, 0.625)</td>
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</tr>
<tr>
<td>2 3</td>
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<td>(1.000, 0.500)</td>
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</tr>
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<td>(1.000, 0.000)</td>
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<td>(1.000, 0.000)</td>
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</tr>
<tr>
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<tr>
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<td>(0.896, 0.000)</td>
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<tr>
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<td>(0.361, 0.000)</td>
<td>(0.896, 0.000)</td>
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<td>5 0</td>
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<td>(0.286, 0.000)</td>
<td>(0.714, 0.000)</td>
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</table>

**Notes:** For all equilibria, \( r_0 = 0 \). For each \((k, \ell)\), payoff dominated equilibria are removed. The probability of adhering to each criterion is then calculated for each equilibrium separately. The comparisons reported are the net number of pairwise comparisons where AM outperforms SMQ (or visa versa) for a particular \((k, \ell)\). For example, AM (2) means that AM outperforms SMQ for \(\eta + 2\) pairs, whereas SMQ outperforms AM for only \(\eta\) pairs.
### Table 10: A Pairwise Comparison of Coalition Proof Equilibria
(N = 3, c= .25)

<table>
<thead>
<tr>
<th>k (\ell)</th>
<th>Equil. 1</th>
<th>Equil. 2</th>
<th>Equil. 3</th>
<th>Equil. 4</th>
<th>Equil. 5</th>
<th>Equil. 6</th>
<th>Pairwise Comparisons against AM</th>
</tr>
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<tbody>
<tr>
<td>SMQ (Q=1)</td>
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</tr>
<tr>
<td>1 0</td>
<td>((p_1, p_2, p_3))</td>
<td>((p_1, p_2, p_3))</td>
<td>((p_1, p_2, p_3))</td>
<td>((p_1, p_2, p_3))</td>
<td>((p_1, p_2, p_3))</td>
<td>Pareto, BT, ESG, Rae</td>
<td>SMQ(1) SMQ(1) SMQ(1) same</td>
</tr>
<tr>
<td>1 1</td>
<td>((0.250, 0.750, 0.000))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>same AM(1) AM(1) AM(1)</td>
</tr>
<tr>
<td>1 2</td>
<td>((0.500, 0.500, 0.500))</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>same AM(1) AM(1) AM(1)</td>
</tr>
<tr>
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<td>((1.000, 0.000, 0.000))</td>
<td>((0.750, 0.750, 0.250))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>same same SMQ(2) same</td>
</tr>
<tr>
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<td>((0.250, 0.250, 1.000))</td>
<td>((1.000, 0.750, 0.250))</td>
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<td>((0.146, 0.146, 0.854))</td>
<td></td>
<td></td>
<td>same SMQ(1) AM(1) AM(1)</td>
</tr>
<tr>
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<td>((0.000, 0.750, 0.750))</td>
<td>((0.500, 0.500, 0.500))</td>
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<td></td>
<td>same same SMQ(9) same</td>
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</table>

<table>
<thead>
<tr>
<th>AM &amp; SMQ (Q=M)</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>1 0</td>
<td>((0.000, 0.000, 0.000))</td>
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</tr>
<tr>
<td>1 1</td>
<td>((0.000, 0.000, 0.000))</td>
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<td></td>
</tr>
<tr>
<td>1 2</td>
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<td></td>
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<tr>
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<td>((0.000, 0.000, 0.000))</td>
<td>((1.000, 1.000, 0.000))</td>
<td>((0.250, 0.250, 0.000))</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>2 1</td>
<td>((0.000, 0.000, 0.000))</td>
<td>((1.000, 1.000, 0.000))</td>
<td>((0.250, 0.250, 0.000))</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>3 0</td>
<td>((0.000, 0.000, 0.000))</td>
<td>((1.000, 1.000, 0.000))</td>
<td>((1.000, 0.750, 0.750))</td>
<td>((0.000, 0.250, 0.250))</td>
<td>((0.146, 0.146, 0.146))</td>
<td>((0.854, 0.854, 0.854))</td>
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</tbody>
</table>

**Notes:** The table reports the probability, \(p_i\), that individual \(i\) votes. Player types are listed in the order \(T_1, T_2, T_3\) for each type that exists. For example, for \((k, \ell) = (2, 1)\), individual 1 and 2 are type \(T_1\), and individual 3 is type \(T_2\). In cases where more than one equilibrium exists, CPE have been marked in bold. The probability of adhering to each criterion is calculated for each equilibrium separately. The comparisons reported are the net number of pairwise comparisons where AM & SMQ (Q=M) outperforms SMQ (Q=1), or visa versa, for a particular \((k, \ell)\) -- among the CPE, or the sole equilibrium if a particular \((k, \ell)\) produces a unique equilibrium.

† There are two equilibria of this type differing in their permutations of \(p_1\) and \(p_2\).
‡ There are three equilibria of this type differing in their permutations of \(p_1\), \(p_2\), and \(p_3\).