The RTAL-Family of Rules for Bankruptcy Problems: A Characterization*

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Abstract

We characterize a family of bankruptcy rules on the basis of the consistency and additivity in a limited domain: "restricted additivity". The family contains the constrained equal awards, the constrained equal losses and the reverse Talmud rules. Each member of the family is characterized using consistency and additivity in broader domains than restricted additivity. We also show that the reverse Talmud rule is the only rule satisfying consistency, restricted additivity and half-claim boundedness.

Keywords: Bankruptcy problems; reverse Talmud rule; axiomatic analysis.

JEL classification: C71; D63; D74

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1 Introduction

A bankruptcy problem consists of a set of individuals with claims to a divisible resource, the estate, which is insufficient to satisfy all their claims. The typical example is the problem of dividing the liquidation value of a bankrupt firm among its creditors, but many rationing problems concerned with the division of a resource according to individuals’ rights, or with the division of a cost according to individuals’ liabilities are mathematically equivalent. Starting with the seminal paper by O’Neill (1982), the literature proposes “rules” that associate each bankruptcy problem with an allocation, i.e. a division of the estate such that every creditor receives a non-negative part of the estate and no creditor obtains more than his/her claim. For an extensive review on this topic the reader may consult Moulin (2002) and Thomson (2003).

Given an allocation, the award made to each creditor can be thought of as a loss that he/she bears with respect to his/her claim (i.e. as the difference between the claim and the award). This dual viewpoint was first taken into account by the celebrated Talmud rule (see Aumann and Maschler (1985)). The Talmud rule applies the CEA rule if the estate is at most half of the total claim and the CEL rule otherwise. The CEA rule divides the estate equally among the creditors, provided that nobody obtains more than their claim, whereas the CEL rule divides the total loss equally, provided that no creditor ends up with a negative award. The application of the Talmud rule to some rationing problems when fair allocations are pursued seems quite appealing. For instance, consider the regulation of fisheries in the wake of overfishing of certain species in the EU. Often the total allowable catch (TAC) of a species has to be divided among countries whose demands over that resource are based on historical rights (see Gallastegui et al. (2002)). Considering the quotas assigned by the Talmud rule, it is found that if the TAC is small enough then the countries with the greatest rights may receive the same share as the others. This egalitarian allocation can be justified on the grounds that those countries with greater historical rights are more responsible for the overexploitation of the species in question than those with smaller historical rights. Furthermore, if the TAC is large enough, and once all countries have received half of their claims, the countries with the greatest rights...
rights get higher quotas since they bear larger losses.

Still, it is not obvious why it should be desirable to switch from the CEA to the CEL rule at the point where the estate is equal to half of the total claim. It turns out that this criterion treats agents fairly in the sense that either they all bear losses greater than or equal to their awards, or vice versa. However, there may be situations in which favoring agents with small (or large) claims is desirable and ratios other than $1/2$ should be allowed. Based on this idea, Moreno-Ternero and Villar (2006) present a family of rules called the TAL-family that takes the ratios in the unit interval as switching points between the CEA and the CEL rules. A ratio greater than $1/2$ indicates that the rule favors creditors with relatively small claims, whereas a ratio lower than $1/2$ indicates that the rule favors creditors with relatively large claims. But for some rationing problems other rules may be appealing, as suggested below.

Consider a common project whose cost has to be covered by contributions of the agents participating in it and assume that there is a maximum contribution (tax) that can be levied on the agents (see Young, (1987) and (1988)). We argue that for problems of this type the reverse Talmud rule, introduced by Chun et al. (2001), seems to be quite suitable. The reverse Talmud rule applies the CEL rule if the estate is at most half of the total claim and the CEA rule otherwise. In the rationing problem mentioned, if the cost of the project is small the agents with highest capacity bear the full cost. However, if the cost of the project is high enough all taxpayers contribute. Furthermore, once the agents with lowest capacity have paid out in full, the contributions of the agents with highest capacity increase as the cost of the project increases.

Concerning the half ratio that switches between the CEL rule and the CEA rule, a reason similar to that given for the Talmud rule can be alleged in the case of the reverse Talmud rule. The family of reverse Talmud rules, introduced by van den Brink et al. (2013) and referred to as the RTAL-family, takes the ratios in the unit interval as switching points between the CEL and the CEA rules. In the RTAL-family, contrary to what happens with the rules in the TAL-family, a ratio higher than $1/2$ implies that the rule benefits creditors with relatively large claims, whereas a ratio lower than $1/2$ implies that the rule benefits creditors with relatively small claims.
One important line of research in bankruptcy problems is the axiomatic characterization of rules. This exercise provides normative arguments for the suitability of rules. The aim of this paper is to characterize the RTAL-family and each of its members.

In the characterization of the RTAL-family two axioms, consistency and restricted additivity, are used. Consistency has played a significant role in the analysis of many allocation problems. In our model, roughly speaking, it requires that what a rule assigns to each creditor in a group should coincide with what it assigns to each creditor in any subgroup.

The use of the additivity axiom also seems relevant in this setting. For instance, assume that a conglomerate of firms goes into bankruptcy. Often the law does not permit a joint procedure, i.e. a single court filing, judgment and list of creditors; hence the liquidation of each firm proceeds separately without connection to any other procedure involving the rest of the firms. However in France, under certain circumstances, the law allows the estates of the different firms of a conglomerate to be consolidated (see Gumpelson (2012)). In these cases the allocation yielded by a joint procedure might not be equivalent to the sum of the allocations given by the individual procedures. Thus, the additivity axiom arises as a convincing requirement for bankruptcy rules in these contexts.

According to this axiom, for two (or more) bankruptcy problems with the same set of creditors, one might expect that if these problems were jointly solved then the award for each creditor should amount to the sum of the awards obtained as the solutions of each individual problem. Unfortunately, as Thomson (2003) points out and Bergantiños and Méndez-Naya (2001) prove with an example, no bankruptcy rule satisfies additivity. The literature offers two ways of overcoming this negative result: One consists of restricting the domain where the rule satisfies additivity and additivity, and the other of somehow weakening the additivity axiom so that it becomes valid for the entire domain. The first approach is followed by Bergantiños and Méndez-Naya (2001)\(^1\), while Alcalde et al. (2014)\(^2\) adopt the second.

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\(^1\)Here additivity is applied to problems where the highest claim coincides with the estate and creditors are equally ordered. In this domain the Ibn Ezra rule is characterized using additivity and equal treatment of equals.

\(^2\)The authors introduce a new version of additivity which, together with equal treatment of equals and continuity, serves to characterize the minimal overlap rule (see O’Neill, 1982).
This paper follows the first approach. The idea is to identify additive rules in domains that allow this to be done. Therefore, a non-modified definition of additivity is applied to a limited domain. The domain chosen to characterize the RTAL-family is defined as the intersection of the domains in which the CEA and the CEL rules are additive, and is referred to as “restricted additivity”. Thus, a natural question to ask is whether there are any additional consistent rules that satisfy additivity in this domain. It turns out that only the rules in the RTAL-family do so (see Theorem 1).

Furthermore, we characterize each member of the RTAL-family using additivity in broader domains than restricted additivity (see Theorem 2). We also provide an additional characterization of the reverse Talmud rule in which consistency and restricted additivity are maintained and a new axiom is added. (see Theorem 3). Aumann and Maschler (1985) argue that in bankruptcy problems "the half-way point is a psychological watershed. If you get more than half your claim, your mind focuses on the full debt, and your concern is with the size of your loss". Based on this observation, we define an axiom called half-claim boundedness which requires that no creditor should receive more than half of his/her claim if someone else receives less than half of his/hers.

The paper concludes with some related literature: First, duality and self-duality are discussed in the wake of Thomson (2008) and van den Brink et al. (2012). This last paper characterizes each member of the RTAL-family. Each of their characterizations contained in it singles out a member of the family but these authors do not identify a set of axioms that only all members of the RTAL-family satisfy. Finally, following Young (1987), who gives a parametric representation of the Talmud rule, we find that minimizing the sum of the discrepancies between awards and losses with respect to the average award and the average loss respectively gives the reverse Talmud rule.

The rest of the paper is organized as follows. Section 2 introduces the model, classes of bankruptcy problems, rules and axioms. Section 3 contains the characterizations. Section 4 contains some related literature. Two proofs are relegated to the Appendix.


2 Bankruptcy problems

This section, divided in three subsections, describes the model, some classes
of problems and the rules and axioms used to characterize the RTAL-family
and each of its members.

2.1 The model

There is an infinite set of potential creditors, indexed by the natural number
\( \mathbb{N} \). Each given bankruptcy problem involves a finite number of creditors. Let
\( \mathcal{N} \) denote the class of non-empty finite subsets of \( \mathbb{N} \). Given \( N \in \mathcal{N} \) and \( i \in N \),
let \( c_i \) be creditor \( i \)'s claim and \( c = (c_i)_{i \in N} \) the claims vector and let \( E \) be the
estate to be divided among the creditors in \( N \). A bankruptcy problem (or just
a problem) is a pair \((c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+\), where \( N \in \mathcal{N} \), such that \( \sum_{i \in N} c_i \geq E \).
Let \( \mathcal{B}^N \) denote the class of all problems with set of creditors \( N \). An allocation
for \((c, E) \in \mathcal{B}^N \) is a vector \( x \in \mathbb{R}^N \) such that it satisfies the non-negativity and
claim boundedness conditions, \( i.e. \ 0 \leq x_i \leq c_i \), and the efficiency condition
\( \sum_{i \in N} x_i = E \).\(^3\) Let \( X(c, E) \) be the set of allocations of \((c, E) \). A bankruptcy
rule (or just a rule) is a function defined on \( \bigcup_{N \in \mathcal{N}} \mathcal{B}^N \) that associates an
allocation in \( X(c, E) \) with each \( N \in \mathcal{N} \) and each \((c, E) \in \mathcal{B}^N \). In this paper,
the generic notation for a rule is \( \phi \). For each \( N' \subset N \) we denote by \( c_{N'} \) the
claims vector of creditors in \( N' \). Similarly, \( \phi_{N'}(c, E) = \phi(c, E)_{i \in N'} \).

Given a problem \((c, E) \in \mathcal{B}^N \), we denote by \( C = \sum_{i \in N} c_i \) and \( L = C - E \) the
total claim and total loss respectively. The equal awards function divides the estate equally among the creditors, \( i.e. \) for each \( N \in \mathcal{N} \), each
\((c, E) \in \mathcal{B}^N \), \( EA(c, E) = (\frac{E}{|N|}, ..., \frac{E}{|N|}) \) and the equal losses function divides
the total loss equally among the creditors, \( i.e. \) for each \( N \in \mathcal{N} \), each \((c, E) \in \mathcal{B}^N \), \( EL(c, E) = c - (\frac{L}{|N|}, ..., \frac{L}{|N|}) \). Neither is a rule since they do not necessarily
select allocations.

2.2 Classes of problems

First of all, for each set of creditors, each claims vector and each parameter
\( \lambda \) in the unit interval we specify classes of problems by defining a lower and
an upper bound for the estate.

\(^3\)The notation \( x \leq y \) means that for each \( i \in N \), \( x_i \leq y_i \).
Let $\lambda \in [0, 1]$ and $(N, c)$ be a set of creditors and a claims vector. Then
\[
E^\lambda(N, c) = \lambda C - \lambda |N| \min_{i \in N} c_i \quad \text{and} \quad E(N, c) = E^\lambda(N, c) + |N| \min_{i \in N} c_i = \lambda C + (1 - \lambda) |N| \min_{i \in N} c_i
\]
We say that problem $(c, E) \in B^N$ belongs to the class of problems $B^N(\lambda)$ if $E \in [E^\lambda(N, c), E^\lambda(N, c)]$.
Let $B(\lambda) = \cup_{N \in N} B^N(\lambda)$.

For each $\lambda \in [0, 1]$ and each $(N, c)$ it is straightforward that $E^\lambda(N, c) \leq E(N, c)$. Since $E^\lambda(N, c)$ is increasing with $\lambda$, $E^1(N, c) = C - |N| \min_{i \in N} c_i$ is its highest value. Since $E^\lambda(N, c)$ is decreasing with $\lambda$, $E^0(N, c) = |N| \min_{i \in N} c_i$ is its lowest value. The two values define a new class of problems.

We say that a problem $(c, E) \in B^N$ belongs to the class of problems $B^N_0$ if $E \in [E^1(N, c), E^0(N, c)]$.
Let $B_0 = \cup_{N \in N} B^N_0$.

Note that $E \in [E^1(N, c), E^0(N, c)]$ (or $C - |N| \min_{i \in N} c_i \leq E \leq |N| \min_{i \in N} c_i$) is equivalent to $\frac{E}{|N|} \leq \min_{i \in N} c_i \leq \frac{C}{|N|}$. The last two inequalities imply that $(c, E) \in B^N_0$ if and only if $EA(c, E)$ and $EL(c, E)$ are allocations.

Next, we show some features of the class $B_0$ used in the proofs of the main results of this paper. In particular, we prove that there are claims vectors that, independently of the size of the estate, cannot yield problems in $B_0$. This takes place when the difference between claims is large enough. However, if a two-creditor problem does not belong to $B_0$, then by adding a positive real number to each claim and to the estate it is possible to generate a new problem that belongs to $B_0$.

**Lemma 1** Let $N=\{1, 2\}$ and $(c_1, c_2)$ be a claims vector with $c_2 \geq c_1$ and let $\delta \geq c_2 - c_1$.

(i) If $c_2 \leq 3c_1$, then there exists $E \in \mathbb{R}_+$ such that $((c_1, c_2), E) \in B_0$.

(ii) Let $E \leq c_1 + c_2$. Then $((c_1 + \delta, c_2 + \delta), E + \delta) \in B_0$. 7
Proof.

(i) \(E^1(\{1, 2\}, (c_1, c_2)) = c_2 - c_1 \leq 2c_1 = E^0(\{1, 2\}, (c_1, c_2))\). Hence, there exists \(E \in \mathbb{R}_+\) such that \(E \in [E^1(\{1, 2\}, (c_1, c_2)), E^0(\{1, 2\}, (c_1, c_2)]\).

(ii) \(E^1(\{1, 2\}, (c_1 + \delta, c_2 + \delta)) = c_2 - c_1\) and \(E^0(\{1, 2\}, (c_1 + \delta, c_2 + \delta)) = 2c_1 + 2\delta\). Since \(\delta \geq c_2 - c_1\), it follows that \(c_2 - c_1 \leq E + \delta \leq 2c_1 + 2\delta\). Hence, \((c_1 + \delta, c_2 + \delta), E + \delta) \in \mathcal{B}_0\).

Example 1 Let \(N = \{1, 2\}\) and \((c_1, c_2) = (2, 8)\).

(i) Let \(E \in [0, 10]\). Then problem \(((c_1, c_2), E) = ((2, 8), E)\) does not belong to \(\mathcal{B}_0\) because \(c_2 = 8 > 6 = 3c_1\).

(ii) Let \(E = 2\) and let \(\delta = c_2 - c_1 = 6\). It can be checked that problem \(((c_1 + \delta, c_2 + \delta), E + \delta) = ((8, 14), 8)\) belongs to \(\mathcal{B}_0\).

It is immediately apparent that problems where all creditors have the same claim belong to \(\mathcal{B}_0\). Next, we show that starting from a two-creditor problem which does not belong to \(\mathcal{B}_0\) it is possible to build a new problem which do belong to \(\mathcal{B}_0\).

Lemma 2 Let \(N = \{1, 2\}\), and \(((c_1, c_2), E) \notin \mathcal{B}_0\) with \(c_2 > c_1\) and let \(x\) be an allocation such that \(0 < x_1 < c_1\). Then there exists \(m \in \mathbb{N}\) such that \((c_i^{m+1}, E^{m+1}) \in \mathcal{B}_0\) where \((c_i^{m+1}, E^{m+1})\) is an \((m + 1)\)-creditor problem with \(c_i^{m+1} = c_1\) if \(i \neq m + 1\), \(c_{m+1}^{m+1} = c_2\) and \(E^{m+1} = mx_1 + x_2\).

Proof. It needs to be proven that there exists an integer \(m\) such that the estate \((mx_1 + x_2) \in [E^1(\{1, 2, ..., m + 1\}, c^{m+1}), E^0(\{1, 2, ..., m + 1\}, c^{m+1})]\). This is equivalent to \(c_2 - c_1 \leq mx_1 + x_2 \leq (m + 1)c_1\). Since \(0 < x_1 < c_1\), it is possible to find \(m \in \mathbb{N}\) such that the two conditions are satisfied.

Example 2 Let \(N = \{1, 2\}\) and \((c_1, c_2) = (2, 8)\) and let \(x = (\frac{1}{2}, \frac{3}{2})\). Consider the set of creditors \(\{1, 2, ..., 11\}\) and \((c', E')\) where \(c'_i = 2\) if \(i \neq 11\), \(c'_{11} = 8\) and \(E' = (5 + \frac{3}{2})\). Then \(E^1(\{1, 2, ..., 11\}, c') = 6\), \(E^0(\{1, 2, ..., 11\}, c') = 22\) and \(E' = (5 + \frac{3}{2}) \in [6, 22]\). Hence, problem \((c', 5 + \frac{3}{2}) \in \mathcal{B}_0\).
2.3 Rules and axioms

Next, we present several well-known rules that are used in the paper.

- The proportional rule divides the estate among the creditors proportionally to their claims.

  **Proportional rule, PR:** For each \( N \in \mathcal{N} \), each \((c, E) \in B^N\), and each \( i \in N \),

  \[
  PR_i(c, E) = \frac{E}{C} c_i.
  \]

- The constrained equal awards rule divides the estate equally among the creditors under the constraint that no creditor receives more than his/her claim.

  **Constrained equal awards rule, CEA:** For each \( N \in \mathcal{N} \), each \((c, E) \in B^N\), and each \( i \in N \),

  \[
  CEA_i(c, E) = \min\{\beta, c_i\} \quad \text{where} \quad \beta \in \mathbb{R}_+ \quad \text{solves} \quad \sum_{i \in N} \min\{\beta, c_i\} = E.
  \]

- The constrained equal losses rule divides the total loss equally among the creditors under the constraint that no creditor receives a negative amount.

  **Constrained equal losses rule, CEL:** For each \( N \in \mathcal{N} \), each \((c, E) \in B^N\), and each \( i \in N \),

  \[
  CEL_i(c, E) = \max\{0, c_i - \beta\} \quad \text{where} \quad \beta \in \mathbb{R}_+ \quad \text{solves} \quad \sum_{i \in N} \max\{0, c_i - \beta\} = E.
  \]

- The Talmud rule combines the CEA and the CEL rules switching their application to problems at the point in which the estate is equal to half of the total claim.

  **Talmud rule, T:** For each \( N \in \mathcal{N} \), each \((c, E) \in B^N\), and each \( i \in N \),

  \[
  T_i(c, E) = \begin{cases} 
  CEA_i(\frac{1}{2}c, E) & \text{if } E \leq \frac{C}{2} \\
  \frac{1}{2}c_i + CEL_i(\frac{1}{2}c, E - \frac{C}{2}) & \text{otherwise}.
  \end{cases}
  \]
Any member of the Talmud family of rules combines the CEA and the CEL rules switching its application at the point in which the estate is equal to $\lambda C$. The CEA rule is obtained for $\lambda = 1$, the CEL rule for $\lambda = 0$, and the Talmud rule for $\lambda = 1/2$.

The **TAL-family of rules** consists of all rules with the following form:
For any $\lambda \in [0, 1]$, each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$T_i^\lambda(c, E) = \begin{cases} 
CEA_i(\lambda c, E) & \text{if } E \leq \lambda C \\
\lambda c_i + CEL_i(\lambda c, E - \lambda C) & \text{otherwise.}
\end{cases}$$

The reverse Talmud rule exchanges the roles played by the CEA and the CEL rules in the Talmud rule.

**Reverse Talmud rule, RT:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$RT_i(c, E) = \begin{cases} 
CEL_i\left(1 - \frac{c}{2}, E\right) & \text{if } E \leq \frac{c}{2} \\
\frac{1}{2} c_i + CEA_i\left(1 - \frac{c}{2}, E - \frac{c}{2}\right) & \text{otherwise.}
\end{cases}$$

Simple algebraic manipulations of the formula above provide an alternative interpretation of the rule. Consider two creditors $i$ and $j$ with $c_j > c_i$ such that $RT_i(c, E) \in (0, c_i)$ and $RT_j(c, E) \in (0, c_j)$. In this case, the definition of the rule implies that $RT_i(c, E) = \frac{c_i}{2} + \beta$ and $RT_j(c, E) = \frac{c_j}{2} + \beta$ where $\beta \in \mathbb{R}$. Hence,

$$RT_j(c, E) - RT_i(c, E) = (c_j - RT_j(c, E)) - (c_i - RT_i(c, E)).$$

If the equality above is not satisfied, then $RT_i(c, E) \in \{0, c_i\}$. Thus, the reverse Talmud rule respects a **bilateral fairness principle**: For each pair of creditors the difference between their awards is equal to the difference between their losses whenever the constrains of non-negativity and claim boundedness are satisfied.

Any member of the reverse Talmud family of rules combines the CEL and the CEA rules but switching its application at the point in which the estate is equal to $\lambda C$. The CEL rule is obtained for $\lambda = 1$, the CEA rule for $\lambda = 0$, and the reverse Talmud rule for $\lambda = 1/2$. 

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The **RTAL-family of rules** consists of all rules with the following form: For any \(\lambda \in [0, 1]\), each \(N \in \mathbb{N}\), each \((c, E) \in B^N\), and each \(i \in N\),

\[
RT_i^\lambda(c, E) = \begin{cases} 
CEL_i(\lambda c, E) & \text{if } E \leq \lambda C \\
\lambda c_i + CEA_i((1 - \lambda)c, E - \lambda C) & \text{otherwise.}
\end{cases}
\]

Note that the CEA and CEL rules are the only rules that the TAL-family and the RTAL-family have in common. The TAL-family belongs to the class of ICI rules (Thomson, 2008). The name ICI reflects the fact that the award of each agent is Increasing for small \(E\), then Constant in \(E\), and finally is Increasing in \(E\). In contrast, the RTAL-family belongs to the class of CIC rules, since the award of each creditor is Constant for small \(E\), then Increasing in \(E\), and finally Constant in \(E\). In a detailed analysis of these two families, Thomson (2008) shows that the RTAL-family is included in the class of CIC rules.\(^4\) To make this point clearer, we use Figure 4 in Thomson (2008) to depict in Figure 1 the paths of awards of four members of the RTAL-family associated with the following values of the parameter \(\lambda = 0, 1/3, 1/2, 1\).

\(^4\)To be precise, in Theorem 2 of this paper, the author characterizes the collection of consistent CIC rules. This collection and the RTAL-family coincide.
Some well known axioms that are used in the following section are presented below.

- We start with the requirement that two creditors with equal claims should receive equal awards.

**Equal treatment of equals**: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, if $c_i = c_j$, then $\phi_i(c, E) = \phi_j(c, E)$.

- When the estate increases, it is required that each creditor should receive at least as much as he/she got initially.

**Resource monotonicity**: For each $N \in \mathcal{N}$, each $(c, E), (c, E') \in \mathcal{B}^N$, if $E \leq E'$, then $\phi(c, E) \leq \phi(c, E')$.

- Consider now, a problem and an allocation given by rule $\phi$. Assume that some creditors depart with their awards and the situation of the remaining creditors is reassessed. It seems desirable that the remaining creditors should receive the same awards as they received initially. Conversely, it can be established that the manner in which a rule solves a problem for two creditors can be extended to a larger number of creditors.

**Consistency**: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, each $N' \subset N$, if $x = \phi(c, E)$, then $x_{N'} = \phi(c_{N'}, \sum_{i \in N} \phi_i(c, E))$.

**Converse consistency**: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $x \in X(c, E)$, if for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = \phi(c_{N'}, \sum_{i \in N'} x_i)$, then $x = \phi(c, E)$.

The following lemma states a joint implication of the last two axioms.\(^5\)

[Thomson, 2007] **Elevator Lemma**: If a rule $\phi$ is consistent and coincides with a conversely consistent rule $\phi'$ in the class of 2-creditor problems, then $\phi$ and $\phi'$ coincide in general.

- The next axiom indicates that no creditor should receive an award amounting to half of his/her claim when someone else receives less than half of his/hers. The axiom can also be interpreted as follows: if

\(^5\)See Thomson (2011) for a detailed study of consistency and converse consistency.
the total loss is greater (or smaller) than the estate, then no creditor receives an award greater (or smaller) than his/her loss.

**Half-claim boundedness:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, either $\phi(c, E) \leq \frac{1}{2}c$ or $\phi(c, E) \geq \frac{1}{2}c$.

Note that this axiom implies that if the estate is half of the total claim, then each creditor receives half of his/her claim.\(^6\)

- Finally, we present the additivity axiom, which is the centerpiece in the characterizations of the rules of the RTAL-family in this paper.

**Additivity:** For each $N \in \mathcal{N}$, each $(c, E), (c', E') \in \mathcal{B}^N$, $\phi(c + c', E + E') = \phi(c, E) + \phi(c', E')$.

As mentioned in the introduction it has been shown that there are no additive rules in bankruptcy problems. We overcome this problem by restricting the domain where the rule satisfies additivity. That is, we do not restrict the domain where the rule is applied but we do restrict the domain where the rule satisfies additivity. The versions of additivity used in this paper are the following:

**Additivity in $\mathcal{B}(\lambda)$:** For each $N \in \mathcal{N}$ and each $(c, E), (c', E') \in \mathcal{B}^N(\lambda)$, $\phi(c + c', E + E') = \phi(c, E) + \phi(c', E')$.

**Restricted additivity:** For each $N \in \mathcal{N}$ and each $(c, E), (c', E') \in \mathcal{B}_0^N$, $\phi(c + c', E + E') = \phi(c, E) + \phi(c', E')$.

Additivity in $\mathcal{B}(\lambda)$, $\lambda \in [0, 1]$, implies restricted additivity. Furthermore, it is straightforward that for each $\lambda \in [0, 1]$ and each $N \in \mathcal{N}$, $\mathcal{B}^N(\lambda)$ and $\mathcal{B}_0^N$ are closed under addition.

### 3 Characterizations

The following characterizations involve axioms introduced in the previous section. First of all, in Theorem 1 we characterize the RTAL-family with consistency and restricted additivity. The proof is based on Lemma 3 and

\(^6\)In Chun *et al.* (2001) this property, called *midpoint property*, is defined as follows: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, if $E \leq \frac{1}{2}c$ then $\phi(c, E) = \frac{1}{2}c$. The authors analyze rules that satisfy midpoint property and resource monotonicity (called *estate-monotonicity* in that paper). The two axioms imply half-claim boundedness.
Proposition 1 which investigates the behavior of the rules in the class of two-creditor problems. Since the rules are consistent, it suffices to understand their behavior in the class of the two-creditor problems.

**Lemma 3** Let \( \lambda \in [0, 1] \). Then \( RT^\lambda \) satisfies additivity in \( B(\lambda) \).

**Proof.** Let \( F^\lambda, \lambda \in [0, 1] \), be a function defined as follows: For each \( N \in \mathcal{N} \), each \( (c, E) \in B^N \), and each \( i \in N \),

\[
F^\lambda_i(c, E) = \begin{cases} 
EL_i(\lambda c, E) & \text{if } E \leq \lambda C \\
\lambda c_i + EA_i((1 - \lambda)c, E - \lambda C) & \text{otherwise.}
\end{cases}
\]

It can be checked that \( F^\lambda_i(c, E) = \lambda c_i + \beta \), where \( \beta \) solves \( \lambda C + |N| \beta = E \). If \( F^\lambda(c, E) \) is an allocation, then \( F^\lambda(c, E) = RT^\lambda(c, E) \). This coincidence holds in the class \( B^N(\lambda) \) since \( F^\lambda(c, E) \) is an allocation whenever \( 0 \leq \lambda \min_{i \in N} c_i + \beta \leq \min_{i \in N} c_i \). It can be checked that these conditions imply that \( (c, E) \in B^N(\lambda) \).

Since for each \( N \in \mathcal{N} \), each \( (c, E), (c', E') \in B^N \) it holds that \( F^\lambda(c + c', E + E') = F^\lambda(c, E) + F^\lambda(c', E') \) and \( B^N(\lambda) \) is closed under addition, we conclude that \( RT^\lambda \) satisfies additivity in \( B(\lambda) \).

Note that additivity in \( B(\lambda), \lambda \in [0, 1] \), implies restricted additivity, hence each member of the RTAL-family satisfies restricted additivity.

The next two lemmas are used in the proof of Proposition 1. The proofs of Lemma 5 and Proposition 1 are relegated to the Appendix.

**Lemma 4** Restricted additivity and consistency imply equal treatment of equals.

**Proof.** Let \( \phi \) be a rule that satisfies restricted additivity and consistency. By consistency of \( \phi \), it suffices to prove that \( \phi \) satisfies equal treatment of equals in the class of two-creditor problems. Let \( N = \{1, 2\} \) and \( (c_1, c_2) = (c_0, c_0) \). Recall that problems where all creditors have the same claim belong to \( B_0 \). Restricted additivity of \( \phi \) together with non-negativity and claim boundedness imply

\[
\phi((c_0, c_0), E) = \phi((E_2, E_2), E) + \phi((E_0 - E_0, c_0 - E_2), 0) = (E_2, E_2).
\]

**Lemma 5** Let \( \phi \) be a rule that satisfies restricted additivity and consistency. Then there exists \( \mu \in [0, 1] \) such that \( \phi_2((c_1, c_2), E) - \phi_1((c_1, c_2), E) = \mu(c_2 - c_1) \) for each \( ((c_1, c_2), E) \in B^{1,2} \) with \( c_2 > c_1 \) and \( \phi_1((c_1, c_2), E) \in (0, c_1) \).
**Proposition 1** Let $\phi$ be a rule that satisfies restricted additivity and consistency. Then in the class of two-creditor problems $\phi$ coincides with a member of the RTAL-family.

Chun (1999) shows that if a rule satisfies consistency and resource monotonicity, then it satisfies converse consistency. Since each member of the RTAL-family satisfies consistency and resource monotonicity (see Thomson (2008)), it also satisfies converse consistency. This fact and Proposition 1 lead to the first main result of this paper.

**Theorem 1** A rule $\phi$ satisfies restricted additivity and consistency if and only if it is a member of the RTAL-family.

**Proof.** Let $\phi$ be a rule that satisfies restricted additivity and consistency. By Proposition 1, rule $\phi$ coincides with a member of the RTAL-family in the class of two-creditor problems. Since each member of this family satisfies converse consistency, by applying the Elevator Lemma, we conclude that $\phi$ is a member of the RTAL-family.

Each member of the RTAL-family satisfies consistency and, as a consequence of Lemma 3, restricted additivity.

Next, we proceed with the characterization of each member of the RTAL-family.

Note that for each $\lambda \in [0,1]$, a rule $RT^\lambda$ and a class of problems $B(\lambda)$ can be defined. The following theorem shows that $RT^\lambda$ is the only consistent rule satisfying additivity in $B(\lambda)$.

**Theorem 2** Let $\lambda \in [0,1]$. Then $RT^\lambda$ is the only consistent rule that satisfies additivity in $B(\lambda)$.

**Proof.** By Lemma 3, $RT^\lambda$ satisfies additivity in $B(\lambda)$. Since $B_0 \subset B(\lambda)$, if a rule satisfies additivity in $B(\lambda)$, then it satisfies restricted additivity. Let $\phi$ be a consistent rule satisfying additivity in $B(\lambda)$ (and therefore restricted additivity). By Theorem 1, $\phi$ is a member of the RTAL-family. It remains to be proven that in this family, $RT^\lambda$ is the only rule satisfying additivity in $B(\lambda)$. To complete the proof we distinguish between two cases:

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1. Let $RT^\lambda$ be a member of the RTAL-family with $\lambda > \lambda$. Let $N = \{1, 2\}$ and consider $c^1 = (c^1_1, c^1_2) = (2, 4)$ and $c^2 = (c^2_1, c^2_2) = (2, 2)$. Let $(c^1, 2\lambda), (c^2, 2) \in B(\lambda)$. Then $RT^\lambda_1(c^1 + c^2, 2 + 2\lambda) = \frac{2 + 2\lambda - 2\lambda}{2} < 1 = RT^\lambda_1(c^1, 2\lambda) + RT^\lambda_1(c^2, 2) = 0 + 1$. Hence, $RT^\lambda$ does not satisfy additivity in $B(\lambda)$.

2. Let $RT^\lambda$ be a member of the RTAL-family with $\lambda < \lambda$. Let $N = \{1, 2\}$ and consider $c^1 = (c^1_1, c^1_2) = (2, 4)$ and $c^2 = (c^2_1, c^2_2) = (2, 2)$. Let $(c^1, 4 + 2\lambda), (c^2, 2) \in B(\lambda)$. Then $RT^\lambda_1(c^1 + c^2, 6 + 2\lambda) = \frac{6 + 2\lambda - 2\lambda}{2} > 3 = RT^\lambda_1(c^1, 2\lambda) + RT^\lambda_1(c^2, 2) = 2 + 1$. Hence, $RT^\lambda$ does not satisfy additivity in $B(\lambda)$.

By Theorem 1, if $\phi$ satisfies restricted additivity and consistency, then $\phi$ is a member of the RTAL-family. Since half-claim boundedness singles out a rule in this family, the following theorem immediately follows.

**Theorem 3** The reverse Talmud rule is the only rule that satisfies restricted additivity, consistency and half-claim boundedness.

In order to show the logical independence of the axioms used in the characterizations above we introduce a family of rules, $\{\varphi^\lambda\}_{\lambda \in [0, 1]}$ defined as follows: For each $N \in \mathcal{N}$, each $(c, E) \in B^N$ and each $i \in N$,

$$
\varphi^\lambda_i(c, E) = \begin{cases} 
RT^\lambda_i(c, E) & \text{if } E \geq \sigma^\lambda(c) \\
PR_i(\sigma^\lambda(c), E) & \text{otherwise}
\end{cases}
$$

where $\sigma^\lambda(c) = \lambda \sum_{i \in N} (c_i - \min_{i \in N} c_i)$.

For $\lambda = 0$, a rule $\varphi^0$, is defined as follows: For each $N \in \mathcal{N}$, each $(c, E) \in B^N$ and each $i \in N$,

$$
\varphi^0_i(c, E) = \begin{cases} 
CEA_i(c, E) & \text{if } E \leq \lfloor N \rfloor \min_{i \in N} c_i \\
CEA_i(c, \lfloor N \rfloor \min_{i \in N} c_i) + PR_i(\lfloor c - CEA(c, \lfloor N \rfloor \min_{i \in N} c_i) \rfloor, E - \lfloor N \rfloor \min_{i \in N} c_i) & \text{otherwise.}
\end{cases}
$$

Note that each $\varphi^\lambda$, $\lambda \in [0, 1]$, is a continuous rule that satisfies resource monotonicity but is not consistent. The following example shows that $\varphi^{1/2}$ is not a consistent rule.
Example 3 Let $N = \{1, 2, 3\}$ and $((c_1, c_2, c_3), E) = ((2, 4, 6), 1)$. Then $\sigma^\frac{1}{2}(2, 4, 6) = 3$ and $RT((2, 4, 6), 3) = (0, 1, 2)$.

Let $(c_1^{RT}, c_2^{RT}, c_3^{RT}) = (0, 1, 2)$, then $\varphi^\frac{1}{2}((2, 4, 6), 1) = PR(c_1^{RT}, c_2^{RT}, c_3^{RT}), 1) = (0, \frac{1}{3}, \frac{2}{3})$.

Now, consider the set of creditors $\{2, 3\}$ and $((c_2, c_3), \varphi^\frac{1}{2}((2, 4, 6), 1) + \varphi^\frac{2}{3}((2, 4, 6), 1)) = ((4, 6), 1)$. Since $\sigma^\frac{1}{2}((4, 6), 1) = 1$, $\varphi^\frac{2}{3}((4, 6), 1) = RT((4, 6), 1) = (0, 1) \neq (\frac{1}{3}, \frac{2}{3})$.

Similar examples can be found for any other rule $\varphi^\lambda$, $\lambda \neq \frac{1}{2}$.

The logical independence of the axioms used in the characterizations follows from these facts:

(i) Additivity in $B(\lambda)$, $\lambda \in [0, 1]$, implies restricted additivity.

(ii) The proportional rule satisfies consistency and half-claim boundedness but it does not satisfy restricted additivity (or additivity in $B(\lambda)$, $\lambda \in [0, 1]$.

(iii) Rule $\varphi^\lambda$, $\lambda \in [0, 1]$, satisfies additivity in $B(\lambda)$ (and hence restricted additivity) but it does not satisfy consistency. Rule $\varphi^\frac{1}{2}$ satisfies half-claim boundedness and $RT^\frac{1}{2}$, $\lambda \neq \frac{1}{2}$, does not satisfy half-claim boundedness.

4 Discussion

In this section, we discuss some results concerning the reverse Talmud rule and the RTAL-family which are found in Thomson (2008) and van den Brink et al. (2013) and relate them to our work. Then, following Young (1978), we give a parametric representation of the reverse Talmud rule.

First, we introduce the notions of duality and self-duality. Two rules are dual if for each problem, one divides the estate as the other divides the total loss. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in B^N$, $\phi^d(c, E) = c - \phi(c, C - E)$. When a rule and its dual give the same allocations the rule is called self-dual.

In Thomson (2008) the family of consistent CIC rules is characterized (Theorem 2). This family coincides with the RTAL-family. It is also shown that the reverse Talmud rule is the only self-dual rule in the CIC family.
(Proposition 1 (b)). Similarly to half-claim boundedness, self-duality singles out the reverse Talmud rule in the RTAL-family. Hence, replacing half-claim boundedness by self-duality in Theorem 3, a new characterization of the reverse Talmud rule is easily derived. Although the two axioms are logically independent, the "natural" self-dual rules satisfy half-claim boundedness.\footnote{Since resource monotonicity and self-duality imply half-claim boundedness, a self-dual rule that violates half-claim boundedness must violate resource monotonicity. However, resource monotonicity and half-claim boundedness do not imply self-duality.}

Van den Brink et al. (2013) point out the precise duality relationship between the members of the RTAL-family. Namely, $(RT^\lambda)^d = RT^{1-\lambda}$. Hence, the CEL rule is the dual of the CEA rule. However, although the reverse Talmud and the Talmud rules interchange the roles played by the CEA and the CEL rules, the reverse Talmud rule is not the dual of the Talmud rule.

These authors use self-duality in some characterizations of the reverse Talmud rule. In their paper they introduce the following axioms:

**Weak exemption**: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, if $c_i \leq \frac{E-L}{|N|}$ then $\phi_1(c, E) = c_i$.

**Weak exclusion**: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, if $c_i \leq \frac{L-E}{|N|}$ then $\phi_1(c, E) = 0$.

These two axioms together with supermodularity and consistency also characterize the reverse Talmud rule. It is interesting to understand how these two axioms are related to the class $\mathcal{B}(\frac{1}{2})$. Note that $(c, E) \in \mathcal{B}^N(\frac{1}{2})$ if $\min_{i \in N} c_i \geq \frac{E-L}{n}$ and $\min_{i \in N} c_i \geq \frac{L-E}{n}$. Hence, weak exemption and weak exclusion imply that if a problem does not belong to $\mathcal{B}(\frac{1}{2})$, then at least one creditor must receive either 0 or his/her full claim. Additivity in $\mathcal{B}(\frac{1}{2})$ has not such implication. The implication follows if consistency is required jointly with additivity in $\mathcal{B}(\frac{1}{2})$.

Van den Brink et al. (2013) also characterize each member of the RTAL-family by parametrizing the weak exemption and weak exclusion axioms, which are defined as follows:

**$\lambda$-exemption**: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, if $\lambda c_i \leq \frac{\lambda E-(1-\lambda)L}{|N|}$ then $\phi_1(c, E) = c_i$.

**$\lambda$-exclusion**: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, if $\lambda c_i \leq \frac{\lambda L-(1-\lambda)E}{|N|}$ then $\phi_1(c, E) = 0$.\footnotetext{Since resource monotonicity and self-duality imply half-claim boundedness, a self-dual rule that violates half-claim boundedness must violate resource monotonicity. However, resource monotonicity and half-claim boundedness do not imply self-duality.}
Each member of the RTAL-family $RT^\lambda$, $\lambda \in [0,1]$, is characterized by 
$(1 - \lambda)$-exemption, $\lambda$-exclusion, supermodularity and consistency. They also point out that a rule $\phi$ satisfies $\lambda$-exemption if and only if its dual rule $\phi^d$ satisfies $(1 - \lambda)$-exclusion.

As in the case of the reverse Talmud rule, the last two axioms are related to the class $B(\lambda)$. Furthermore, it is straightforward to show that a problem $(c, E)$ belongs to $B(\lambda)$ if and only if its dual $(c, C - E)$ belongs to $B(1 - \lambda)$. Hence, rule $\phi$ satisfies additivity in $B(\lambda)$ if and only if its dual rule $\phi^d$ satisfies additivity in $B(1 - \lambda)$. Combining this fact and the result established in Theorem 2, we conclude that $(RT^\lambda)^d = RT^{1-\lambda}$ and that the reverse Talmud rule is the only self-dual rule in the RTAL-family.\(^8\) Note that with respect to restricted additivity we do have a duality implication. That is, if a rule satisfies restricted additivity, then its dual satisfies restricted additivity.

Next, we report a result obtained by Young (1987) that reads:

\[\text{[Young, 1987] Theorem 2: A symmetric, continuous allocation method is pairwise consistent}^9\ \text{if and only if its solutions minimize a symmetric, continuous, additively separable, strictly convex objective function.}\]

In that paper, a minimization problem is given whose unique solution is the recommendation of the Talmud rule. Since the reverse Talmud rule satisfies the axioms above,\(^10\) a minimization problem can be constructed whose unique solution is the recommendation of the reverse Talmud rule. In particular, for each $N \in N$, each $(c, E) \in B^N$, we propose the following problem:

\(^8\)Our analysis provides alternative proofs of two known results: The RTAL-family is closed under duality and the reverse Talmud rule is the only self-dual rule in the family.

\(^9\)Pairwise consistency requires consistency only for all pairs of creditors. Therefore, the latter implies the former.

\(^10\)Young considers symmetric rules in the sense that allocations are determined by individuals’ claims, not by who they are. Clearly, the members of the RTAL-family satisfy this requirement.
\[
\begin{align*}
\min_x & \left( \sum_{i \in N} (x_i - \frac{E}{|N|})^2 + \sum_{i \in N} (c_i - x_i - \frac{L}{|N|})^2 \right) \\
\text{s.t} & \sum_{i \in N} x_i = E \\
& 0 \leq x_i \leq c_i \text{ for each } i \in N.
\end{align*}
\]

Since the set of allocations is compact, the solution is a singleton. From the first order conditions we obtain that for each \(i, j \in N\) with \(c_j > c_i\),

- either \(x_j - x_i = (c_j - x_j) - (c_i - x_i)\)
- or \(x_j - x_i < (c_j - x_j) - (c_i - x_i)\) and \(x_i = 0\)
- or \(x_j - x_i > (c_j - x_j) - (c_i - x_i)\) and \(x_i = c_i\)

Note that these are precisely the conditions derived from the definition of the reverse Talmud rule in Subsection 2.3.

The idea of minimizing the sum of the squares of discrepancies of awards and losses with respect to the average award and the average loss seems quite appealing from an egalitarian point of view. A similar criterion is used in TU games (see Ruiz et al. (1996) and Arin et al. (2008)).
References


5 APPENDIX

Lemma 5

Proof. The proof is divided in two parts:

Part 1 We prove that if $\phi$ satisfies restricted additivity, then there exists $\mu \in [0, 1]$ such that $\phi_2((c_1, c_2), E) - \phi_1((c_1, c_2), E) = \mu(c_2 - c_1)$ for each $((c_1, c_2), E) \in \mathcal{B}_0^{[1,2]}$.

Let $N = \{1, 2\}$ and $((c_1, c_2), E) \in \mathcal{B}_0^{[1,2]}$ with $(c_2 - c_1) = k > 0$ and consider problem $(c^k; k)$ where $c^k = (c_1^k, c_2^k) = (\frac{k}{2}, \frac{3k}{2})$. If $(c_1, c_2), E) \in \mathcal{B}_0^{[1,2]}$, $E \geq k$ and $c_1 \leq \frac{k}{2}$. Since $3c_1^k = c_2^k$, problem $(c^k, k)$ belongs to $\mathcal{B}_0^{[1,2]}$ for each $k \geq 0$. Since $\phi$ satisfies restricted additivity,

$$\phi((c_1, c_2), E) = \phi((c_1^k, c_2^k), k) + \phi((c_1 - \frac{k}{2}, c_2 - \frac{k}{2}), E - k).$$

By the proof of Lemma 4, rule $\phi$ satisfies equal treatment of equals in $\mathcal{B}^{[1,2]}$. Hence, $\phi((c_1 - \frac{k}{2}, c_2 - \frac{k}{2}), E - k) = (\frac{E - k}{2})$ and $\phi_2((c_1, c_2), E) - \phi_1((c_1, c_2), E) = \phi_2((c_1^k, c_2^k), k) - \phi_1((c_1^k, c_2^k), k)$.

Thus, it is enough to analyze problem $((c_1^k, c_2^k), k)$.

Let $c^l = (c_1^l, c_2^l) = (\frac{1}{2}, \frac{3}{2})$. We seek to prove that $\phi_2(c^k, k) - \phi_1(c^k, k) = \mu k$ where $\mu = \phi_2(c^l, 1) - \phi_1(c^l, 1)$ and that $\mu \in [0, 1]$.

If $\mu < 0$, $\phi_1(c^l, 1) > \frac{1}{2}$ and if $\mu > 1$, $\phi_1(c^l, 1) < 0$ which are incompatible with claim boundedness and non-negativity of $\phi$. Hence, $\mu \in [0, 1]$.

It remains to be proven that $\phi_2(c^k, k) - \phi_1(c^k, k) = \mu k$. We distinguish two cases:

Case 1 $k \in \mathbb{Q}$. Since $\phi$ satisfies restricted additivity, it is straightforward that $\phi(c^k, k) = k\phi(c^l, 1))$. Since $\phi_2(c^l, 1) - \phi_1(c^l, 1) = \mu$, $\phi_2(c^k, k) - \phi_1(c^k, k) = \mu k$.

Case 2 $k \in \mathbb{R}\setminus\mathbb{Q}$. Assume that $\phi_2(c^k, k) - \phi_1(c^k, k) = \alpha k$, $\alpha \neq \mu$. Consider $\alpha > \mu$. Since $k \in \mathbb{R}\setminus\mathbb{Q}$, there exist $\lambda \in \mathbb{Q}^+$ and $\varepsilon \in \mathbb{R}\setminus\mathbb{Q}$ such that $k = \lambda + \varepsilon$, $\varepsilon > 0$. Let $c^\lambda = (c_1^\lambda, c_2^\lambda) = (\frac{\lambda}{2}, \frac{3\lambda}{2})$ and $c^\varepsilon = (c_1^\varepsilon, c_2^\varepsilon) = (\frac{\varepsilon}{2}, \frac{3\varepsilon}{2})$. Since $\phi$ satisfies restricted additivity, $\phi(c^k, k) = \phi(c^\lambda, \lambda) + \phi(c^\varepsilon, \varepsilon)$. 

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By non-negativity and claim boundedness of $\phi$ it holds that $\phi_2(c^\varepsilon, \varepsilon) - \phi_1(c^\varepsilon, \varepsilon) \leq \varepsilon$. Combining the two facts we conclude that $\alpha k - \mu \lambda \leq \varepsilon$. Since $k > \lambda$, $\lambda(\alpha - \mu) < \varepsilon$. Given that $\varepsilon$ can be chosen as small as needed it follows $\alpha = \mu$. Consider now $\alpha < \mu$. Since $k \in \mathbb{R} \setminus \mathbb{Q}$, there exist $\lambda \in \mathbb{Q}_+$ and $\varepsilon > 0$ such that $\lambda = k + \varepsilon$. Reasoning in a similar way we obtain $\alpha = \mu$.

Note that in Part 1 of the proof we do not use the fact that rule $\phi$ satisfies consistency.

Part 2 Let $N = \{1, 2\}$ and $((c_1, c_2), E) \notin B_0$ with $c_2 > c_1$, $\phi((c_1, c_2), E) = (x, E - x)$ and $c_1 > x > 0$. Consequently, $\phi_2((c_1, c_2), E) - \phi_1((c_1, c_2), E) = E - 2x$.

Let $M_1 \cup M_2 = \{1, 2, ..., m\} \cup \{m + 1, ..., 2m\}$. From $((c_1, c_2), E)$, we define a 2m-creditor problem $(c^{2m}, mE)$ where $c^{2m}_i = c_1$ if $i \in M_1$ and $c^{2m}_i = c_2$ if $i \in M_2$. By Lemma 4, $\phi_1(c^{2m}, mE) + \phi_2(c^{2m}, mE) = E$ for each pair $(i, j)$ with $i \in M_1$ and $j \in M_2$. Since $\phi((c_1, c_2), E) = (x, E - x)$ and $\phi$ satisfies consistency, $\phi(c^{2m}, mE) = (x, ..., x, E - x, ..., E - x)$.

Let $M_3 = \{1, 2, ..., m, m + 1\}$ and consider the following two $(m + 1)$-creditor problems:

(a) $(c^{m+1}, mx + E - x)$ where $c^{m+1}_i = c_1$ if $i \neq m + 1$ and $c^{m+1}_{m+1} = c_2$.

Integer $m$ has been chosen so that $(c^{m+1}, mx + E - x) \in B_0$. Lemma 2 guarantees the existence of such integer $m$. Since $\phi$ satisfies consistency, $\phi(c^{m+1}, mx + E - x) = (x, ..., x, E - x)$.

(b) $(c^\delta, (m + 1)\frac{\delta}{2})$ where $c^\delta_i = \delta = c_2 - c_1$ for $i \in M_3$. By Lemma 4, $\phi(c^\delta, (m + 1)\frac{\delta}{2}) = (\frac{\delta}{2}, ..., \frac{\delta}{2})$.

Problems $(c^{m+1}, mx + E - x)$ and $(c^\delta, (m + 1)\frac{\delta}{2})$ have the same set of creditors and belong to $B_0$, hence problem $(c^{m+1} + c^\delta, mx + E - x + (m + 1)\frac{\delta}{2})$ belongs to $B_0$ (for each $N \in \mathcal{N}$ the class $B_0^N$ is closed under addition). Since $\phi$ satisfies restricted additivity,

$\phi(c^{m+1} + c^\delta, mx + E - x + (m + 1)\frac{\delta}{2}) = (x + \delta\frac{\delta}{2}, ..., x + \delta\frac{\delta}{2}, E - x + \delta\frac{\delta}{2})$.

By Lemma 1, problem $((c_1 + \delta, c_2 + \delta), E + \delta) \in B_0$. By consistency of $\phi$, $\phi((c_1 + \delta, c_2 + \delta), E + \delta) = (x + \frac{\delta}{2}, E - x + \frac{\delta}{2})$. 

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Since \(0 < x + \frac{x}{2} < c_1 + \delta\) and \(((c_1 + \delta, c_2 + \delta), E + \delta) \in B_0\), it holds that

\[
E - 2x = \phi_2((c_1 + \delta, c_2 + \delta), E + \delta) - \phi_1((c_1 + \delta, c_2 + \delta), E + \delta) = \\
\mu(c_2 - c_1) = \phi_2((c_1, c_2), E) - \phi_1((c_1, c_2), E).
\]

Proposition 1

**Proof.** By lemma 5, there exists \(\mu \in [0, 1]\) such that \(\phi_2((c_1, c_2), E) - \phi_1((c_1, c_2), E) = \mu(c_2 - c_1)\) for each \(((c_1, c_2), E) \in B^{(1,2)}\) with \(c_2 > c_1\) and \(\phi_1((c_1, c_2), E) \in (0, c_1)\).

We need to prove that for \(((c_1, c_2), E) \in B^{(1,2)}\) with \(c_2 \geq c_1\) the following holds:

(i) If \(E \leq \mu(c_2 - c_1)\), then \(\phi_1((c_1, c_2), E) = 0\).

(ii) If \(E \geq 2c_1 + \mu(c_2 - c_1)\), then \(\phi_1((c_1, c_2), E) = c_1\).

(iii) If \(\mu(c_2 - c_1) < E < 2c_1 + \mu(c_2 - c_1)\), then \(\phi_2((c_1, c_2), E) - \phi_1((c_1, c_2), E) = \mu(c_2 - c_1)\) and consequently \(\phi_1((c_1, c_2), E) = \frac{E - \mu(c_2 - c_1)}{2}\).

1. Let \(\{1, 2, ..., m, m+1\}\) be a set of creditors and \((c^{m+1}, E)\) be an \((m+1)\)-creditor problem where \(c_i^{m+1} = c_1\) if \(i \neq m+1\) and \(c_{m+1}^{m+1} = c_2\). By Lemma 4, \(\phi(c^{m+1}, E) = (x, ..., x, E - mx)\). Since \(\phi\) satisfies consistency, \(\phi((c_1, c_2), x + E - mx) = (x, E - mx)\). If \(x \in (0, c_1)\), then by Lemma 5, \(E - mx - x = \mu(c_2 - c_1)\) contradicting \(E \leq \mu(c_2 - c_1)\). If \(x = c_1\), then integer \(m\) can be chosen so that \(E < mc_1\) which contradicts non-negativity of \(\phi\). Hence, \(x = 0\). Since \(\phi\) satisfies consistency, \(\phi((c_1, c_2), E) = (0, E)\).

2. Let \(\{1, 2, ..., m, m+1\}\) be a set of creditors and \((c^{m+1}, E + (m-1)c_1)\) be an \((m+1)\)-creditor problem where \(c_i^{m+1} = c_1\) if \(i \neq m+1\) and \(c_{m+1}^{m+1} = c_2\). By Lemma 4, \(\phi(c^{m+1}, E + (m-1)c_1) = (x, ..., x, E + (m-1)c_1 - mx)\). Since \(\phi\) satisfies consistency, \(\phi((c_1, c_2), x + E + (m-1)c_1 - mx) = (x, E + (m-1)c_1 - mx)\). If \(x \in (0, c_1)\), then by Lemma 5, \(E + (m-1)c_1 - mx - x = \mu(c_2 - c_1) \leq E - 2c_1\). Hence, \((m+1)c_1 \leq (m+1)x\) contradicting \(x < c_1\). If \(x = 0\), integer \(m\) can be chosen so that \(E + (m-1)c_1 > c_2\) which contradicts claim boundedness of \(\phi\). Hence, \(x = c_1\). Since \(\phi\) satisfies consistency, \(\phi((c_1, c_2), E) = (c_1, E - c_1)\).
3. Let \{1, 2, ..., m, m + 1\} be a set of creditors and \((e^{m+1}, E + (m - 1)\gamma)\) be an \((m + 1)\)-creditor problem where \(c_i^{m+1} = c_1\) if \(i \neq m + 1\) and \(c_{m+1}^{m+1} = c_2\). Let \(\gamma = \frac{E - \mu(c_2 - c_1)}{2}\). By Lemma 4, \(\phi(e^{m+1}, E + (m - 1)\gamma) = (x, ..., x, E + (m - 1)\gamma - mx)\). Note that \(\gamma \in (0, c_1)\). We seek to prove that \(\phi_1((c_1, c_2), E) = \gamma\). Since \(\phi\) satisfies consistency, \(\phi((c_1, c_2), x + E + (m - 1)\gamma - mx) = (x, E + (m - 1)\gamma - mx)\). If \(x = 0\), an integer \(m\) can be chosen so that \(E + (m - 1)\gamma > c_2\) which contradicts claim boundedness of \(\phi\). If \(x = c_1\), an integer \(m\) can be chosen so that \(E + (m - 1)\gamma - mx < 0\) (since \(x = c_1 > \gamma\)) which contradicts non-negativity of \(\phi\). Consequently, \(x \in (0, c_1)\). By Lemma 5, \(E + (m - 1)\gamma - mx - x = \mu(c_2 - c_1)\) which implies \(x = \gamma\) and \(((c_1, c_2), x + E + (m - 1)\gamma - mx) = ((c_1, c_2), E)\). Hence, \(\phi_1((c_1, c_2), E) = x = \gamma\).