The symplectic geometry of b-manifolds

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Outline

1. Motivation and examples
2. The dual approach: b-symplectic forms and Moser trick
3. Symplectic and Poisson geometry of the critical hypersurface
4. Extension theorem
5. Related projects
"Geometries" in this talk

1 Structures which are symplectic on an open dense set and "explode" in a controlled way along an hypersurface. $\mapsto$ b-symplectic manifolds

2 Cosymplectic manifolds (in the sense of Paulette Libermann) $\mapsto$ symplectic mapping torus

We are going to see that these objects are related. To go from one to the other we will have to make a small "Poisson" detour.
Given an oriented surface $S$ (compact or not) with a distinguished union of curves $Z$, we want to modify the volume form on $S$ by making it “explode” when we get close to $Z$. We want this “blow up” process to be controlled.
What does “controlled” mean here?
Consider the Lie algebra $\mathfrak{g}$ of the affine group in dimension 2. It is a model for noncommutative Lie algebras in dimension 2 and in a basis $e_1, e_2$ the brackets are,

$$[e_1, e_2] = e_2$$

We can naturally write this Lie algebra structure (bilinear) as the Poisson structure

$$\Pi = y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$
Motivation: b-structures in dimension 2

This Poisson structure is “dual” to the 2-form

\[ \omega = \frac{1}{y} dx \wedge dy \]

In this example \( \mathcal{Z} \) is the \( x \)-axis:

In this example \( \mathcal{Z} \) is formed by symplectic leaves of dimension 0 (points on the line). The upper and lower half-planes are symplectic leaves of dimension 2.

**Dimension 2**

b-Poisson structures in dimension 2 were studied by Olga Radko.
Definition of b-Poisson manifolds

**Definition**

Let \((M^{2n}, \Pi)\) be an oriented Poisson manifold such that the map

\[ p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM) \]

is transverse to the zero section, then \(Z = \{ p \in M | (\Pi(p))^n = 0 \}\) is a hypersurface and we say that \(\Pi\) is a **b-Poisson structure** on \((M, Z)\).

1. \(\Pi\) defines a symplectic structure almost everywhere.
2. On the critical set we have an induced Poisson structure which defines a regular codimension 1 symplectic foliation.
Higher dimensions: Some compact examples.

- Let $(R, \pi_R)$ be a Radko compact surface and let $(S, \pi)$ be a compact symplectic surface, then $(R \times S_1, \pi_R + \pi)$ is a b-Poisson manifold of dimension 4.

- Other product structures to get higher dimensions.

- We can perturb this product structure to obtain a non-product one. For instance, $S^2$ with critical surface $Z$. Consider the Poisson structure $\Pi_1 = h \frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$ and the two torus $\mathbb{T}^2$ with Poisson structure $\Pi_2 = \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}$. Consider,

$$\hat{\Pi} = h \frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) + \Pi_2.$$

Then $(S^2 \times \mathbb{T}^2, \hat{\Pi})$ is a b-Poisson manifold.

Moser’s ideas

Via a path theorem, we can control perturbations that produce equivalent Poisson structures.
Take \((N, \pi)\) be a regular Poisson manifold with dimension \(2n + 1\) and rank \(2n\) and let \(X\) be a Poisson vector field. Now consider the product \(S^1 \times N\) with the bivector field

\[
\Pi = f(\theta) \frac{\partial}{\partial \theta} \wedge X + \pi.
\]

This is a b-Poisson manifold as long as,

1. the function \(f\) vanishes linearly.
2. The vector field \(X\) is transverse to the symplectic leaves of \(N\).

We then have as many copies of \(N\) as zeroes of \(f\).
Consider a manifolds with boundary:
A vector field in a point of the boundary has to be tangent to the boundary.

Observe that for any point $p \in \partial(H^n_+)$, the tangent bundle at $p$ is generated by $T_p(H^n_+) = \langle y_1 \frac{\partial}{\partial y_1}_p, y_2 \frac{\partial}{\partial y_2}_p, \ldots, y_n \frac{\partial}{\partial y_n}_p \rangle$.

Melrose proved that there exists a vector bundle (the b-tangent bundle, $bT(M)$) with sections the set of vector fields tangent to $Z$.

With this idea Melrose constructed the b-cotangent bundle of this surface. This was the starting point of b-calculus for differential calculus on manifolds with boundary.
b-Poisson structures on b-tangent bundles and a Darboux theorem

Consider the Poisson structure

\[ \Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \]

It can also be interpreted as a section of \( \Lambda^2(bT(M)) \).

We also have a “Liouville” one-form interpretation for this Poisson structure.

**Darboux theorem for our manifolds**

We can prove a b-Darboux theorem which tells us that locally all Poisson b-manifolds look like this model.

**The b-category**

Thus, the moral is that b-Poisson manifolds lie between the symplectic and Poisson world and we can get some interesting results that we do not get for general Poisson manifolds.
We may associate to a $b$-Poisson $\Pi$, a dual form. Denote by $^b\Omega^k(M)$ the space of sections of the bundle $\Lambda^k(^bT^*M)$.

**Definition**

We say that a closed $b$-de Rham 2-form $\omega$ is $b$-symplectic if $\omega_p$ is of maximal rank as an element of $\Lambda^2(^bT_p^*M)$ for all $p \in M$.

**Moser trick for $b$-symplectic forms**

Like in the symplectic case we can prove a Moser trick for $b$-symplectic structures.
Denote by $b\Omega^k(M)$ the space of sections of the bundle $\Lambda^k(bT^*M)$.

The usual space of DeRham $k$-forms sits inside this space but in a somewhat nontrivial way: given $\mu \in \Omega^k(M)$, we interpret it as a section of $\Lambda^k(bT^*M)$ by the convention

$$\mu_p \in \Lambda^k(T^*_pM) = \Lambda^k(bT^*_pM) \text{ at } p \in M \setminus Z$$

$$\mu_p = (\iota^* \mu)_p \in \Lambda^k(T^*_pZ) \subset \Lambda^k(bT^*_pM) \text{ at } p \in Z$$

where $\iota : Z \hookrightarrow M$ is the inclusion map.

Every $b$-DeRham $k$-form can be written as

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M)$$

where $f$ is a defining function for the hypersurface.
We define the exterior $d$ De Rham operator on $^b\Omega(M)$ by setting

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$ 

And we get,

$$0 \rightarrow ^b\Omega^0(M) \xrightarrow{d} ^b\Omega^1(M) \xrightarrow{d} ^b\Omega^2(M) \xrightarrow{d} \ldots \rightarrow 0$$

Define b-cohomology as the cohomology of this complex,

We obtain

**Theorem (Mazzeo-Melrose)**

The $b$-cohomology groups of a compact $M$ are computable by

$$^bH^*(M) \cong H^*(M) \oplus H^{*-1}(\mathbb{Z}).$$

In particular for compact b-symplectic $M$ we obtain, $^bH^2(M) \neq 0$. 

For $b$-forms we can prove b-Poincaré theorem and get local, relative and global Moser’s theorems for $b$-Poisson manifolds.

**Theorem (Guillemin-Miranda-Pires)**

Let $\omega_i \in b \Omega^2(U), i = 0, 1$ be $b$-symplectic forms in a neighbourhood of $Z$ extending a given Poisson structure on $Z$ then $\omega_0$ and $\omega_1$ are $b$-symplectomorphic in a neighbourhood of $Z$. 
Another Moser’s theorem (global version)

Theorem (Guillemin-Miranda-Pires)

Let $\omega_i \in b \Omega^2(M)$, $i = 0, 1$ be $b$-symplectic forms on $M$ with modular vector fields, $v_i$. Suppose:

1. $v = v_1|_Z = v_2|_Z$.
2. $[\omega_0] = [\omega_1]$.
3. $\omega_t = (1 - t)\omega_0 + t\omega_1$ is $b$-symplectic for all $t \in [0, 1]$.

Then $\omega_0$ and $\omega_1$ are $b$-symplectomorphic.
Radko classified b-Poisson structures on compact oriented surfaces giving a list of invariants:

- Geometrical: The topology of $\mathcal{S}$ and the curves $\gamma_i$ where $\Pi$ vanishes.
- Dynamical: The periods of the “modular vector field” along $\gamma_i$.
- Measure: The regularized Liouville volume of $\mathcal{S}$. 
Modular vector field

Definition

Given a Poisson manifold \((M, \Pi)\) and a volume form \(\Omega\), the modular vector field \(X^\Omega_{\Pi}\) associated to the pair \((\Pi, \Omega)\) is the derivation given by the mapping

\[ f \mapsto \frac{L_{u_f} \Omega}{\Omega} \]

1. \(L_{X^\Omega_{\Pi}}(\Pi) = 0\) and \(L_{X^\Omega_{\Pi}}(\Omega) = 0\).
2. \(X^{H\Omega} = X^\Omega - u_{\log(H)}\). \(\Rightarrow\) its first cohomology class in Poisson cohomology does not depend on \(\Omega\).
3. Examples of unimodular (vanishing modular class) Poisson manifolds: symplectic mflds.
4. In the case \(\{x, y\} = y\) and the modular vector field is \(\frac{\partial}{\partial x}\).
Modular vector fields

As a consequence of Darboux theorem, we obtain the following,

**Modular vector field for Darboux form**

The modular vector field of a local b-Poisson manifold with local normal form,

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i},$$

with respect to the volume form $\Omega = \sum_i dx_i \wedge dy_i$ is,

$$X^\Omega = \frac{\partial}{\partial x_1}.$$

As a consequence, we obtain. The modular vector field of a Poisson b-manifold $(M, \omega_\Pi)$ is tangent to the vanishing set $Z$ of the $2n$-vector field $\Pi^n$ and is transverse to the symplectic leaves inside $Z$. 

Consider $h$ a function vanishing linearly on the curves $\gamma_1, \ldots, \gamma_n$ and not vanishing elsewhere.

Let $V^\epsilon_h(\Pi) = \int_{|h| > \epsilon} \omega_\Pi$

The limit

$$V(h) = \lim_{\epsilon \to 0} V^\epsilon_h(\Pi)$$

exists and is independent of the function. This limit is known as regularized Liouville volume.

Using Moser theorem for $b$-forms we can give another proof of Radko’s theorem:

**Theorem (Radko)**

The set of curves, modular periods and regularized Liouville volume completely determines, up to Poisson diffeomorphisms, the $b$-Poisson structure on a compact oriented surface $S$. 

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Another applications of this global Moser theorem

**Theorem (Guillemin-Miranda-Pires)**

Let $\omega_t \in b\Omega^2(M)$, with $0 \leq t \leq 1$, be a family of $b$-symplectic forms varying smoothly with $t$. If $\omega_t|_Z$ and the cohomology class $[\omega_t] \in bH^2(M)_DeRham$ are independent of $t$, then all the $\omega_t$'s are symplectomorphic.
Induced Poisson structures

Given a b-Poisson structure $\Pi$ on $M^{2n}$ we get an induced Poisson structure on $Z$ (critical set) which is a regular Poisson structure with symplectic leaves of codimension 1.

We can look for converse results.
Given a Poisson manifold $Z$ with codimension 1 symplectic foliation $\mathcal{L}$, we want to answer the following questions:

1. Does $(Z, \Pi_L)$ extend to a b-Poisson structure on a neighbourhood of $Z$ in $M$?
2. If so to what extent is this structure unique?
3. Global results à la Radko?

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Global constructions for higher dimensions: Going backwards...

Semilocal answer

We will thicken our regular Poisson manifold $\mathcal{Z}$ and we will consider a tubular neighbourhood construction:

$$\omega = p^*(\alpha_{\mathcal{Z}}) \wedge \frac{df}{f} + p^*(\omega_{\mathcal{Z}})$$

Using $\alpha_{\mathcal{Z}}$ a defining one-form for the symplectic foliation on $\mathcal{Z}$ and $\omega_{\mathcal{Z}}$ a two form that restricts to the symplectic form on every symplectic leaf. These forms need to satisfy more constraints in order to work. So the answer is: Not always.
The $\mathcal{L}$-De Rham complex

Choose $\alpha \in \Omega^1(Z)$ and $\omega \in \Omega^2(Z)$ such that for all $L \in \mathcal{L}$ (symplectic foliation) such that for all $L \in \mathcal{L}$, $i^*_L \alpha = 0$ and $i^*_L \omega = \omega_L$.

Notice that

$$d\alpha = \alpha \wedge \beta, \beta \in \Omega^1(Z)$$

Therefore we can consider the complex

$$\Omega^k_{\mathcal{L}} = \Omega^K / \alpha \Omega^{k-1}$$

Consider $\Omega_0 = \alpha \wedge \Omega$ we get a short exact sequence of complexes

$$0 \rightarrow \Omega_0 \xrightarrow{i} \Omega \xrightarrow{j} \Omega_{\mathcal{L}} \rightarrow 0$$

By differentiation of 2 we get

$$0 = d(d\alpha) = d\beta \wedge \alpha - \beta \wedge \beta \wedge \alpha = d\beta \wedge \alpha,$$

so $d\beta$ is in $\Omega_0$, i.e., $d(j\beta) = 0$.

First obstruction class

We define the **obstruction class** $c_1(\Pi_{\mathcal{L}}) \in H^1(\Omega_{\mathcal{L}})$ to be $c_1(\Pi_{\mathcal{L}}) = [j\beta]$

Notice that $c_1(\Pi_{\mathcal{L}}) = 0$ iff we can find a closed one form for the foliation.
The $\mathcal{L}$-De Rham complex

Assume now $c_1(\Pi_\mathcal{L}) = 0$ then, we obtain $d\omega = \alpha \wedge \beta_2$.

Second obstruction class

We define the **obstruction class** $c_2(\Pi_\mathcal{L}) \in H^2(\Omega_\mathcal{L})$ to be

$$c_2(\Pi_\mathcal{L}) = [j\beta_2]$$

Main property

$c_2(\Pi_\mathcal{L}) = 0 \iff$ there exists a **closed** 2-form, $\omega$, such that $i^*_L(\omega) = \omega_L$. 
The role of these invariants

Integrability

The vanishing of $c_2(\Pi_L)$ implies that there exists a leafwise symplectic embedding of the symplectic foliation and this implies integrability of the Poisson structure in the sense of Crainic-Fernandes.

The role of these invariants

$c_1(\Pi_L) = c_2(\Pi_L) = 0 \iff$ there exists a Poisson vector field $v$ transversal to $L$.

Relation of $v$, $\omega$ and $\alpha$:

1. $i_v \alpha = 1$.
2. $i_v \omega = 0$.

The fibration is a symplectic fibration and $v$ defines an Ehresmann connection.
Dynamics of codimension-1 foliations on Poisson manifolds with vanishing invariants

Let $\beta$ satisfy $d\alpha = \beta \wedge \alpha$. Then with respect to the volume form $\alpha \wedge \omega^n$ we have $\iota(v_{\text{mod}})\omega_L = \beta_L$.

**Theorem**

A regular corank 1 Poisson manifold is unimodular iff we can choose closed defining one-form $\alpha$ for the symplectic foliation (i.e. if and only if $c_1(\Pi_L) = 0$).

**Corollary**

Foliations with vanishing $c_1(\Pi_L)$ have vanishing Godbillon-Vey class.

**The b-Poisson case**

The Poisson structure induced on the critical hypersurface of a b-Poisson structure manifold has vanishing invariants $c_1(\Pi_L)$ and $c_2(\Pi_L)$. 
Summing up,

The foliation induced by a $b$-Poisson structure on its critical hypersurface satisfies,

- we can choose the defining one-form $\alpha$ to be closed
- symplectic structure on leaves which extends to a closed 2-form $\omega$ on $M$

Given a symplectic foliation on a corank 1 regular Poisson manifold $\alpha$ and $\omega$ exists if and only if the invariants $c_1(\Pi_L)$ and $c_2(\Pi_L)$ vanish.

**Question**

Is every codimension one regular Poisson manifold with vanishing invariants the critical hypersurface of a $b$-Poisson manifold?
Theorem (Guillemin-Miranda-Pires)

If $c_1(\Pi_L)$ and $c_2(\Pi_L)$ vanish and $L$ contains a compact leaf $L$, then $M$ is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of the Poisson vector field $v$.

Integrability of the Poisson structure

This Poisson structure is integrable but the mapping torus theorem gives explicit models for the integrating symplectic groupoid.
The extension property

Theorem (Guillemin-Miranda-Pires)

Let \((M^{2n+1}, \Pi_0)\) be a compact corank-1 regular Poisson manifold with vanishing invariants then there exists an extension of \((M^{2n+1}, \Pi)\) to a \(b\)-Poisson manifold \((U, \Pi)\). The extension is unique, up to isomorphism, among the extensions such that \([v]\) is the image of the modular class under the map:

\[
H^1_{\text{Poisson}}(U) \longrightarrow H^1_{\text{Poisson}}(M^{2n+1})
\]
$M = \mathbb{T}^4$ and $Z = \mathbb{T}^3 \times \{0\}$. Consider on $Z$ the codimension 1 foliation given by $\theta_3 = a\theta_1 + b\theta_2 + k$, with rationally independent $a, b, 1 \in \mathbb{R}$. Then take

$$\alpha = \frac{a}{a^2 + b^2 + 1} \, d\theta_1 + \frac{b}{a^2 + b^2 + 1} \, d\theta_2 - \frac{1}{a^2 + b^2 + 1} \, d\theta_3,$$

$$\omega = d\theta_1 \wedge d\theta_2 + b \, d\theta_1 \wedge d\theta_3 - a \, d\theta_2 \wedge d\theta_3,$$

This structure can be extended to a neighbourhood of $Z$ in $M$. Indeed it can be extended to the whole $\mathbb{T}^4$ by considering

$$\Pi = f(\theta_4) \frac{\partial}{\partial \theta_4} \wedge X + \pi \omega.$$
Given a Poisson vector field $v$ on $(Z, \Pi_L)$ with $v$ transverse to $L$, chose $\alpha_Z \in \Omega^1(Z)$ and $\omega_Z \in \omega^2(Z)$ such that:

1. $i_v \alpha_Z = 1$.
2. $i_v \omega_Z = 0$.
3. $\alpha_Z$ is a defining one form for the symplectic foliation.
4. $\omega_Z$ restricts to the induced symplectic form on each symplectic leaf.
Now consider \( p : U \rightarrow Z \) a tubular neighbourhood of \( Z \) in \( U \) and let

\[
\omega = p^*(\alpha_Z) \wedge \frac{df}{f} + p^*(\omega_Z)
\]

with \( f \) a defining function for \( Z \).

Then one can check that this is a semi-local extension to Poisson b-manifold and that the modular vector field of \( \omega \) restricted to \( Z \) is \( v \).

Uniqueness relies strongly on a relative Moser’s theorem for our b-Poisson manifolds.
1. Integrable systems on b-symplectic manifolds.
2. Group actions on b-symplectic manifolds.
3. b-cohomology and Poisson cohomology.
4. Integrability of b-Poisson manifolds: The Lie algebroid $T^*_M$ has injective anchor map on an open dense set and therefore (Debord/Crainic-Fernandes) b-Poisson manifolds are integrable. Find explicit models.