Notes on

Chapter 1: Monopoly II

Microeconomic Theory IV

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1.5. Price discrimination

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(i) Definition

“There exists price discrimination when different units of the same good are sold at different prices either to the same consumer or to different consumers”.

Discussion

- Differences in quality: passenger transport, cultural or sporting events etc.

- A single price may be discriminatory and different prices not. We say that there is no price discrimination when the difference between the prices paid by two consumers for a unit of the good exactly responds to the difference in the cost of providing them with the good.
(ii) The incentive to discriminate prices

At the profit-maximizing level of output marginal revenue equals marginal cost, \( r'(x^m) = C'(x^m) \); i.e., an infinitesimal change in the level of output changes revenue and cost equally. That is:

\[
p(x^m) + x^m p'(x^m) = C'(x^m)
\]  

(1)

The monopolist would be reluctant to sell more units if it does not have to reduce the price. Therefore, there are incentives to try to capture a higher proportion of the consumer surplus → incentives to discriminate prices.

(iii) Conditions

Two conditions are needed for a firm to be able to discriminate prices:

a) The firm must be able to classify consumers (which depends on information).
b) The firm must be capable of preventing the resell of the good (which depends on the possibilities of arbitrage and on transaction costs).

The simplest case occurs when a firm receives an exogenous sign (age, location, occupation, etc.) which allows it to classify consumers into different groups.

It is more difficult to classify according to an endogenous category (e.g., quantity purchased or the time of purchase). In that case the monopolist must establish prices in such a way that consumers classify themselves in the correct categories.

(iv) *Types of price discrimination* (Pigou, 1920)

1) **First-degree price discrimination or perfect discrimination.**

The seller charges a different price for each unit equal to the maximum willingness to pay for that unit. This requires *full information* concerning consumer preferences and no arbitrage. The monopolist succeeds in extracting the complete consumer surplus.

2) **Second-degree price discrimination** (or nonlinear pricing).

Prices differ depending on the number of units of the good but not across consumers. Each consumer faces the same price catalogue but prices depend on the quantity purchased (or on another variable, e.g., product quality). Examples: volume discounts. Self selection.

3) **Third-degree price discrimination.**

Different prices are charged to different consumers but each consumer pays a constant amount (the same price) for each unit. The firm receives an exogenous sign which allows it to classify consumers into different groups. This is the most frequent type of price discrimination. Examples: discounts for students, senior citizens, etc. Identification.

Another way of classifying price discrimination is to distinguish between direct price discrimination and indirect price discrimination. Second-degree price discrimination is a
case of indirect discrimination (consumers face a unique price schedule and they classify themselves by their choices) while first-degree price discrimination and third-degree price discrimination would be direct discrimination. In the case of third-degree price discrimination the firm gives different price menus for consumers belonging to different groups or markets.

(v) Examples

It is more difficult to find real markets where there is no price discrimination than markets where such discrimination exists. Although it is often not possible to distinguish clearly what type of price discrimination exists it is an interesting exercise to think about what type of price discrimination is been practiced in the following cases.

- Two-part tariffs: telephone, Internet, electricity, cable television, etc.
- Different electricity rates for industrial use and domestic use.
- Discounts in museums, magazine subscriptions, cultural and sporting events, for children, young people or senior citizens.
- Volume discount in public transport.
- Quality differences: different prices depending on the quality of the product in cultural or sporting events, passenger transport (trains, etc.).
- Discounts for repeated buying.
- 2x1, 3x2, etc. in supermarkets, etc.
- Home-service food, tele-shopping etc.
(vi) The model

We study the three types of price discrimination by using a very simple model. Assume that there are two potential consumers with quasi-linear utility functions: $u_i(x_i) + y_i, \ i = 1, 2.$

$u_i(0) = 0, \ i = 1, 2.$

$u_i(x_i):$ maximum willingness to pay of consumer $i = 1, 2.$

$u_i'(x_i):$ marginal willingness to pay of consumer $i = 1, 2.$

We say that the consumer 2 is a high-demand consumer and that the consumer 1 is a low-demand consumer if the following is satisfied:

$u_2(x) > u_1(x) \ \forall x$

$u_2'(x) > u_1'(x) \ \forall x$

Thus, consumer 2 is a high-demand consumer and consumer 1 is a low-demand consumer if both the maximum willingness to pay and the marginal willingness to pay are higher for consumer 2 than for consumer 1 for any quantity of the good.

The comparison between consumers of maximum willingness to pay and marginal willingness to pay only makes sense for the same level of output. Moreover, the comparison has to be made for any level of output.
The marginal cost of the monopolist is assumed to be constant (and there are no fixed costs) \( c > 0 \). In an equivalent way, we can see the cost function as: \( C(x) = c.x = c.(x_1 + x_2) \).
1.6. *First-degree price discrimination or perfect price discrimination*

(i) Definition and context.

(ii) The case of a single consumer.

(iii) Observations. Is the quantity supplied by the monopolist efficient?

(iv) The case of two consumers.

(v) Does the monopolist supply efficient outputs to consumers? The monopolist supplies a higher quantity to the high-demand consumer (proof).

(vi) What would happen if the monopolist were not able to identify consumers?

(i) *Definition and context*

The seller charges a different price for each unit of product and equals the maximum willingness to pay for that unit.

This requires *full information* on consumer preferences and *no arbitrage* of any kind. In particular, the monopolist needs to be able to identify consumers when they buy the good. (Classic example: a village doctor).

(ii) *The case of a single consumer*

The monopolist supplies a price-quantity bundle \((r^*, x^*)\) which maximizes profits. The monopolist proposes a “take it or leave it” choice: \(
\begin{cases} 
(r^*, x^*) \\
(0, 0)
\end{cases}
\). The consumer either pays \(r^*\) for \(x^*\) units or does not receive the good. The maximization problem of the monopolist is:

\[
\max_{r,x} r - cx \\
\text{s.a. } u(x) \ge r
\]  

(1)
Constraint (1) can be equivalently written as $u(x) - r \geq 0$: the consumer has to obtain a non-negative surplus from good $x$. This type of constraint is known as participation restriction or individual rationality restriction.

Given that the monopolist wishes to maximize profits it will choose the highest possible tariff $r$ and, therefore, condition (1) will be satisfied as equality: $r = u(x)$. The problem thus consists of:

$$\max_x u(x) - cx$$

$$\frac{d\Pi}{dx} = u'(x) - c = 0 \implies u'(x^*) = c$$

$$\frac{d^2\Pi}{dx^2} = u''(x) < 0$$

Given this level of output the tariff will be: $r^* = u(x^*)$.

(iii) Observations

a) Is the quantity supplied by the monopolist efficient?

The monopolist produces a Pareto-efficient output, $x^* = x^*$, given that it supplies a quantity such that the marginal willingness to pay equals the marginal cost. (Review the problem of maximizing social welfare and compare with the problem we have just solved). However, the monopolist obtains the entire social surplus.
b) The monopolist produces the same quantity that it would produce if it behaved as a perfectly competitive firm. If it took price as a parameter then its output decision would be \( p(x) = c \) but given that utility is quasi-linear then \( p(x) = u'(x) \) and consequently \( u'(x) = c \). However, the distribution of trade gains would be just the opposite.

c) We might obtain the same results by using a two-part tariff.

d) We would obtain the same result if the monopolist sold each unit to the consumer at a different price equal to his/her maximum willingness to pay for that unit. Assume that we break production down into \( n \) equal portions of size \( \Delta x \) so as \( x = n \Delta x \).
The maximum willingness to pay for the first unit (of size $\Delta x$) is given by:

$$u(0) + m = u(\Delta x) + m - p_1 \rightarrow u(0) = u(\Delta x) - p_1$$

The maximum willingness to pay for the second unit is:

$$u(\Delta x) + m - p_1 = u(2\Delta x) + m - p_1 - p_2 \rightarrow u(\Delta x) = u(2\Delta x) - p_2$$

And so on. We would obtain the following sequence of equations:

$$u(0) = u(\Delta x) - p_1$$
$$u(\Delta x) = u(2\Delta x) - p_2$$
$$u(2\Delta x) = u(3\Delta x) - p_3$$

..............................................................
$$u((n-1)\Delta x) = u(n\Delta x) - p_n$$

Adding and taking into account that $u(0) = 0$ we get $u(n\Delta x) = \sum_{i=1}^{n} p_i$. When the size of the units becomes infinitesimal, we obtain that proposing a “take it or leave it” choice to the consumer is equivalent to selling him/her each (infinitesimal) unit at a price equal to the marginal willingness to pay for it.

$$u(x^*) = \int_{0}^{x^*} \frac{u'(z)}{p(z)} dz$$
(iv) The case of two consumers

The monopolist supplies consumer $i$, $i=1,2$, with a price-output bundle $(r_i^*, x_i^*)$ in order to maximize profits. The monopolist gives consumer $i$, $i=1,2$, a “take it or leave it” choice: $(r_i^*, x_i^*)$. Consumer $i$, $i=1,2$, either pays $r_i^*$ for $x_i^*$ units or does not receive the good. The maximization problem of the monopolist is:

$$\max_{r_1, r_2, x_1, x_2} r_1 + r_2 - c(x_1 + x_2)$$

s.t.

$$u_1(x_1) - r_1 \geq 0$$
$$u_2(x_2) - r_2 \geq 0$$

$$\Rightarrow r_1 = u_1(x_1)$$
$$r_2 = u_2(x_2)$$

Therefore, the problem becomes:

$$\max_{x_1, x_2} u_1(x_1) + u_2(x_2) - c(x_1 + x_2)$$

$$\frac{\partial \Pi}{\partial x_1} = u_1(x_1) - c = 0 \Rightarrow u_1'(x_1^*) = u_2'(x_2^*) = c$$

$$\frac{\partial \Pi}{\partial x_2} = u_2'(x_2) - c = 0$$

Given these levels of output the tariffs are: $r_1^* = u_1(x_1^*)$ and $r_2^* = u_2(x_2^*)$.

(v) Does the monopolist supply efficient outputs to consumers? The monopolist supplies a higher quantity to the high-demand consumer (proof)

The monopolist offers efficient outputs: $x_1^* = x_1^e$ and $x_2^* = x_2^e$. (Review the problem of obtaining a Pareto-efficient allocation and compare with the problem we have just solved)

We next demonstrate that the monopolist offers a higher quantity to the high-demand consumer: $x_2^* > x_1^*$. 

\[ u'_i(x^*_i) = c \]
\[ u'_2(x^*_2) = u'_1(x^*_1) < u'_2(x^*_2) \]
\[ u'_2(x^*_2) = c \]

Consumer 2 is the high-demand consumer:
\[ u'_2(x) > u'_1(x) \quad \forall x \]

Therefore, \[ u'_2(x^*_2) < u'_2(x^*_1) \] but given that function \( u_2 \) is strictly concave then \[ \frac{d(u'_2(x))}{dx} < 0 \]

and in consequence \( x^*_2 > x^*_1 \).

(vi) **What would happen if the monopolist were not able to identify consumers?**

(This subsection serves to introduce the analysis of second-degree price discrimination).

Assume now that the monopolist is not able to identify consumers when they go to buy the good. That is, the monopolist cannot propose personalized supplies and is therefore restricted to stating a single price menu. Assume that it states a price menu by using the tariffs and quantities which are optimal under perfect price discrimination:

\[
\begin{align*}
(r^*_1, x^*_1) \\
(r^*_2, x^*_2) \\
(0, 0)
\end{align*}
\]

where \( r^*_1 = u_i(x^*_1) \) and \( r^*_2 = u_2(x^*_2) \). We can see that the high-demand consumer has incentives to buy the bundle designed for the low-demand consumer.

\[
0 = u_2(x^*_2) - r^*_2 < u_2(x^*_1) - r^*_1 = u_2(x^*_1) - u_1(x^*_1) \quad (> 0)
\]

The surplus obtained by consumer 2 if he/she buys the bundle designed for him.

The surplus that consumer 2 would obtain if he/she buys the bundle designed for consumer 1.

Incentive to engage in personal arbitrage.
1.7. **Second-degree price discrimination (or non-linear pricing)**

(Keywords: no identification, unique price menu and self selection).

(i) Definition and context.

(ii) Participation restrictions and self selection restrictions. Interpretation.

(iii) Demonstration of what constraints are satisfied with equality. Interpretation.

(iv) The profit maximization problem.

(v) Observations. Does the monopolist supply efficient quantities? The monopolist offers a lower-than-efficient quantity to the low-demand consumer (Proof).

(vi) Under what conditions does the monopolist offer the good to both consumers?

(vii) Graphic representation.

**Definition and context**

The prices differ depending on the number of units bought but not from one consumer to another.

We consider a context where the monopolist knows the preferences of the consumers (it knows the preference distribution function) but is unable to identify consumers when they go to buy the good. So the firm is obliged to establish a unique price menu and to allow consumers to self classify or self select. In this sense we can say that there is indirect price discrimination. The consumers face the same price schedule but prices depend on quantity (or some other variable, e.g. the quality of the good) bought.
(ii) Participation restrictions and self selection restrictions. Interpretation

The objective of the monopolist is to optimally design the price menu in such a way that each consumer chooses the price-quantity bundle designed for him/her.

\[
\begin{align*}
(r_1, x_1) & \rightarrow \text{Consumer 1} \\
(r_2, x_2) & \rightarrow \text{Consumer 2}
\end{align*}
\]

Restrictions for the monopolist

- Participation restrictions (or individual rationality constraints)

\[
\begin{align*}
u_i(x_i) - r_i & \geq 0 \quad (1) \\
u_2(x_2) - r_2 & \geq 0 \quad (2)
\end{align*}
\]

These restrictions guarantee that each consumer wishes to buy the good. Each consumer obtains at least as much utility by consuming the good as by not consuming. Put differently, each consumer obtains a non-negative surplus by purchasing the good.

- Self selection restrictions (or incentive compatibility constraints)

\[
\begin{align*}
u_i(x_i) - r_i & \geq u_i(x_2) - r_2 \quad (3) \\
u_2(x_2) - r_2 & \geq u_2(x_1) - r_1 \quad (4)
\end{align*}
\]

These restrictions guarantee that each consumer prefers the price-quantity bundle designed for him/her to the price-quantity bundle designed for the other consumer. Put differently, these constraints avoid personal arbitrage: each consumer gets as least as great a surplus by choosing the bundle designed for him/her as he/she does by choosing the bundle designed for the other consumer.
(iii) Demonstration of what constraints are satisfied with equality. Interpretation

We now arrange constraints according to each consumer.

\[
(1) \ y (3) \rightarrow \begin{cases} 
    r_i \leq u_i(x_i) \\
    r_i \leq u_i(x_i) - u_i(x_2) + r_2
\end{cases} \quad (1)' \\
(2) \ y (4) \rightarrow \begin{cases} 
    r_2 \leq u_2(x_2) \\
    r_2 \leq u_2(x_2) - u_2(x_1) + r_1
\end{cases} \quad (2)'),
\]

The monopolist wishes to maximize profits and will therefore choose the highest possible \( r_i \) and \( r_2 \). As a consequence, only one of the first two inequalities and only one of the second two inequalities will be binding (that is, they will be satisfied with equality). The assumption that consumer 2 is the high-demand consumer and consumer 1 the low-demand consumer \( (u_2(x) > u_1(x) \ \forall x \text{ and } u_2'(x) > u_1'(x) \ \forall x) \) is sufficient to determine what constraints are binding.

1) Demonstration that \((4)'\) is satisfied with equality and \((3)'\) with strict inequality.

Assume that \((3)'\) is satisfied with equality and, therefore, that \( r_2 = u_2(x_2) \). Then \((4)' \rightarrow r_2 \leq r_2 - u_2(x_i) + r_i \rightarrow r_i \geq u_2(x_i) \). Given that consumer 2 is the high-demand consumer \( u_2(x) > u_1(x) \ \forall x \) then \( r_i \geq u_2(x_i) > u_1(x_i) \). That is, \( r_i > u_1(x_i) \) which means that restriction \((1)'\) would not be satisfied which is a contradiction. (The fact that the participation constraint of the high-demand consumer is satisfied with equality is not compatible with the fact that the low-demand consumer buys the good). As a conclusion, \((3)'\) is not binding and \((4)'\) is satisfied with equality:

\[r_2 = u_2(x_2) - u_2(x_1) + r_1 \quad (5)\]
2) **Demonstration that (1)’ is satisfied with equality and (2)’ with strict inequality**

Assume that condition (2)’ is satisfied with equality and, therefore, that
\[ r_1 = u_1(x_1) - u_1(x_2) + r_2. \]

By substituting \( r_2 \) from condition (5) we get:
\[
\mathcal{X}'_i = u_i(x_1) - u_i(x_2) + \frac{u_1(x_2) - u_2(x_1)}{r_2}
\]
which implies
\[
u_2(x_2) - u_2(x_1) = u_i(x_2) - u_i(x_1)
\]
\[
\int_{t_1}^{t_2} u_2'(t) dt = \int_{t_1}^{t_2} u_i'(t) dt
\]
\[
\int_{t_1}^{t_2} [u_2'(t) - u_i'(t)] dt = 0
\]
But this contradicts the assumption that consumer 2 is the high-demand consumer, \( u_2(x) > u_i(x) \) \( \forall x \). Therefore, (2)’ is not binding and (1)’ is satisfied with equality:
\[
r_1 = u_i(x_1) \quad (6)
\]

**Interpretation**

The monopolist charges consumer 1 a tariff equal to his maximum willingness to pay given that the low-demand consumer has no incentive to engage in personal arbitrage. Given that the high demand consumer has incentive to engage in personal arbitrage (and to mimic the low-demand consumer) the monopolist charges him/her the maximum price that induces him/her to choose the bundle designed for him/her (the amount of money that just leaves him/her indifferent between his/her bundle and that designed for the low-demand consumer).
We now show (in a different more intuitive way) why the monopolist must provide a positive surplus to the high-demand consumer. Consider the self selection constraint for the high-demand consumer:

\[ u_2(x_2) - r_2 \geq u_2(x_1) - r_1 \quad (4) \]

Note that the right side of this constraint is positive conditional on the low-demand consumer’s wishing to buy the good. That is, if we choose the maximum value for \( r_1 \) condition (4) would be:

\[ u_2(x_2) - r_2 \geq u_2(x_1) - u_1(x_1) > 0 \]

given that consumer 2 is the high-demand consumer (which implies that the participation restriction of consumer 2 cannot be satisfied with equality). But given that the monopolist must allow the high-demand consumer to obtain a positive surplus, it decides to leave the consumer with the minimum possible surplus, just that amount such that the high-demand consumer is indifferent between his/her bundle and the bundle designed for consumer 1. That is, rearranging restriction (5):

\[ u_2(x_2) - r_2 = u_2(x_1) - u_1(x_1) > 0 \]

Given that the low-demand consumer has no incentive to engage in personal arbitrage the monopolist charges him/her the maximum that he/she is willing to pay \( r_1 = u_1(x_1) \).
(iv) The profit maximization problem

\[
\begin{align*}
\max_{r_1, x_1, r_2, x_2} & \quad r_1 + r_2 - c.(x_1 + x_2) \\
\text{s.a} & \quad u_1(x_1) - r_1 \geq 0 \quad (1) \\
\text{s.a} & \quad u_2(x_2) - r_2 \geq 0 \quad (2) \\
u_1(x_1) - r_1 \geq u_1(x_2) - r_2 \quad (3) \\
u_2(x_2) - r_2 \geq u_2(x_1) - r_1 \quad (4)
\end{align*}
\]

By substituting we get:

\[
\begin{align*}
\max_{x_1, x_2} & \quad \pi(x_1, x_2) = u_1(x_1) + u_2(x_2) - [u_2(x_2) - u_1(x_1)] - c.(x_1 + x_2) \\
\frac{\partial \pi}{\partial x_1} & = u_1'(\tilde{x}_1) - c - [u_2'(\tilde{x}_1) - u_1'(\tilde{x}_1)] = 0 \quad (7) \\
\frac{\partial \pi}{\partial x_2} & = u_2'(\tilde{x}_2) - c = 0 \quad (8)
\end{align*}
\]

The tariffs are given by:

\[
\begin{align*}
\tilde{r}_1 & = u_1'(\tilde{x}_1) \\
\tilde{r}_2 & = u_2'(\tilde{x}_2) - [u_2(\tilde{x}_1) - u_1(\tilde{x}_1)]
\end{align*}
\]

(v) Observations

1) The monopolist provides the high-demand consumer with the efficient quantity and leaves him/her with a positive surplus.

Condition (8) implies \( u_2'(\tilde{x}_2) = c \) and, therefore, the monopolist offers the efficient quantity to the high-demand consumer \( \tilde{x}_2 = x_2^e \) (review Pareto-efficiency conditions). Moreover, the monopolist charges him/her a price (a tariff) lower than his/her maximum willingness to pay leaving him/her with a positive surplus equals to that which he/she would obtain if he/she chose the bundle designed for consumer 1. \( \tilde{r}_2 = u_2(\tilde{x}_2) - [u_2(\tilde{x}_1) - u_1(\tilde{x}_1)] \) and his/her surplus would thus be: \( u_2(\tilde{x}_2) - \tilde{r}_2 = [u_2(\tilde{x}_1) - u_1(\tilde{x}_1)] \).
2) The monopolist offers the low-demand consumer a quantity lower than the efficient quantity and leaves him/her with no surplus.

\[ \frac{\partial \Pi}{\partial x_i} = u'_i(x_i) - c - \left[ u'_2(x_i) - u'_i(x_i) \right] > 0 \]  \hspace{1cm} (7) \]

Given that consumer 2 is the high-demand consumer \([u'_2(x_i) - u'_i(x_i)] > 0\) and then from condition (7) we get \(u'_i(x_i) > c\). By definition, the efficient output satisfies \(u'_i(x'_i) = c\), and as a consequence \(u'_i(x_i) > u'_i(x'_i)\). The maximum willingness to pay is a strictly concave function:

\[
\begin{align*}
\left. \begin{array}{l}
  u'_i(x_i) > u'_i(x'_i) \\
  d(u'_i(x_i)) < 0 \\
  \rightarrow x_i < x'_i
\end{array} \right\}
\]

We next look at the intuition of this result. We interpret the marginal profit of \(x_i\) and evaluate it at different production levels.

\[
\frac{\partial \Pi}{\partial x_i} = u'_i(x_i) - c - \left[ u'_2(x_i) - u'_i(x_i) \right] > 0 \quad \text{at } x_i < x'_i
\]

Marginal profit from consumer 1: a change in the quantity supplied to this consumer implies a change in the profit obtained by the monopolist from him/her. Marginal profit from consumer 2: a change in the quantity supplied to consumer 1 implies a change in the surplus the monopolist must leave consumer 2 to avoid personal arbitrage.

\[
\frac{\partial \Pi(x'_i)}{\partial x_i} = u'_i(x'_i) - c - \left[ u'_2(x'_i) - u'_i(x'_i) \right] < 0
\]
Starting from $x_1^*$ a reduction in the quantity supplied to consumer 1 increases the profit because the surplus that the monopolist must leave consumer 2 to avoid arbitrage is reduced. An output such that $\tilde{x}_i < x_i < x_1^*$ satisfies the following:

$$\frac{\partial \Pi(x_i)}{\partial x_i} = u_i'(x_i) - c - [u_i'(x_i) - u_i'(x_i)] > 0$$

It is worthwhile for the monopolist to continue reducing $x_1$ because the increase in profits from the high-demand consumer (obtained by leaving him/her with a lower surplus) offsets the loss of profits from the low-demand consumer obtained by supplying him/her a lower quantity.

$$\frac{\partial \Pi(\tilde{x}_i)}{\partial x_i} = u_i'(\tilde{x}_i) - c - [u_i'(\tilde{x}_i) - u_i'(\tilde{x}_i)] = 0$$

In output $\tilde{x}_i$ the marginal gain, from an infinitesimal reduction in $x_1$, from the high-demand consumer by leaving him/her with lower surplus is just equal to the marginal loss from the low-demand consumer as a result of offering a lower quantity.

Moreover, the monopolist charges the low-demand consumer a price (tariff) equal to the maximum willingness to pay, thus leaving him/her with no surplus: $\tilde{r}_i = u_i(\tilde{x}_i)$.

(vi) *Under what conditions does the monopolist offer the good to both consumers?*

The monopolist will decide to offer the good to both consumers if it obtains more profits than by selling the good only to the high-demand consumer. That is, the monopolist supplies the good to both consumers if the following is satisfied:

$$\Pi(0, x_2^*) \leq \Pi(\tilde{x}_1, \tilde{x}_2)$$

$$u_2(x_2^*) - cx_2^* \leq u_1(\tilde{x}_1) - c\tilde{x}_1 + u_2(x_2^*) - [u_2(\tilde{x}_1) - u_2(\tilde{x}_1)] - cx_2^*$$

$$[u_2(\tilde{x}_1) - u_2(\tilde{x}_1)] \leq u_1(\tilde{x}_1) - c\tilde{x}_1$$
If this condition is not satisfied, the monopolist offers the good only to the high-demand consumer. Another equivalent way of looking at the problem consists of considering the marginal profit of $x_i$. If it were negative for any level of $x_i$

$$\frac{\partial \Pi(x_i)}{\partial x_i} = u'_i(x_i) - c - [u'_i(x_i) - u'_i(x_i)] < 0 \quad \forall x_i$$

then the monopolist would decide not to sell the good to the low-demand consumer given that for any level of $x_i$ it would increase profits by reducing the quantity supplied to the low-demand consumer.

(vi) Graphic representation (zero marginal cost)

![Graph showing perfect price discrimination](image)

**Perfect price discrimination**

\[
\begin{align*}
(r^*_i, x^*_i) & \quad i = 1, 2 \\
(0, 0) & \\
 u'_i(x'_i) & = u'_2(x'_2) = 0 \\
 r^*_i & = u_i(x^*_i) \equiv A \\
 r^*_2 & = u_2(x^*_2) \equiv A + B + C
\end{align*}
\]
\[
\Pi^* = u_1(x_1^*) + u_2(x_2^*) \equiv A + \frac{A + B + C}{r_1}
\]

**No identification**

Assume that the monopolist does not know the identity of the consumer and that it states a unique price menu where it maintains the price-quantity bundles which were optimal under perfect price discrimination. Consumer 2 would have incentives to engage in personal arbitrage.

\[
\begin{align*}
(r_1^*, x_1^*) & \quad \text{Consumer 1} \\
(r_2^*, x_2^*) & \quad \text{Consumer 2} \\
(0, 0) & \\
\end{align*}
\]

**Second-degree price discrimination**

The following conditions are satisfied with equality:

\[
r_1 = u_1(x_1) \equiv A(x_1) \rightarrow \text{the monopolist charges consumer 1 the area below his/her inverse demand function.}
\]

\[
u_2(x_2) - r_2 = u_2(x_1) - r_1 = B(x_1) \rightarrow \text{the monopolist leaves consumer 2 with a surplus } B(x_1) \text{ (the minimum) in order to avoid arbitrage.}
\]

Firstly, we maintain quantities and only adjust the tariffs.

\[
\begin{align*}
(r_1^*, x_1^*) & \quad \Pi(x_1^*, x_2^*) = 2A + C \\
(r_2^*, x_2^*) & \quad \Pi(x_1^*, x_2^*) = A' + A + B + C - B' \\
(0, 0) & \quad \Pi(x_1^*, x_2^*) - \Pi(x_1^*, x_2^*) \equiv -(A - A') + (B - B') > 0
\end{align*}
\]

\[
0 = A + B + C - (A + B + C) < A + B - A = B
\]
As we are assuming that the marginal cost is zero:

$$\frac{\partial \Pi(x_i)}{\partial x_i} = u_i(x_i) - [u_2(x_i) - u_i(x_i)] = 0 \rightarrow u_i(x_i) = u_2(x_i) - u_i(x_i)$$

$$\Pi(x_1, x_2) = u_i(x_i) - \zeta x_i + u_2(x_2) - [u_2(x_i) - u_i(x_i)] - \zeta x_2 \equiv A + B + C - \tilde{B}$$
The decision to supply the good only to the high-demand consumer.

\[
\frac{\partial \Pi(x_1)}{\partial x_1} = u'_1(x_1) - c - \left[u'_2(x_1) - u'_1(x_1)\right] > 0 \quad \forall x_1
\]

\[
\begin{cases} 
B \quad r^*_2, x^*_2 \\
C \quad (0,0)
\end{cases}
\]
1.8. **Third-degree price discrimination**

(i) Definition and context.

(ii) Profit maximization. The rule of the inverse of elasticity.

(iii) A comparison of profits with the case of uniform pricing (single monopoly pricing).

(iv) Effects on social welfare.

(i) **Definition and context**

There is third-degree price discrimination when consumers belonging to different groups or submarkets are charged different prices, although each consumer pays the same price for each unit bought. This is probably the most common type of price discrimination. Examples: discounts to students, senior citizens etc.

The monopolist receives an exogenous sign which allows it to distinguish $m$ perfectly separated markets or submarkets: $\frac{\partial x}{\partial p_j} = 0$. This is a type of direct discrimination: the monopolist states different price menus for consumers belonging to different groups or markets. **Identification**: the monopolist classifies each consumer in a group.

(ii) **Profit maximization. The rule of the inverse of elasticity**

We consider the simple case of $m = 2$: the monopolist classifies consumers in two groups with inverse demand functions $p_1(x_i)$ and $p_2(x_2)$, with $p_i'(x_i) < 0$, $i = 1, 2$. The monopolist can establish different prices in the two markets but within a market it is not possible to discriminate prices. The maximization problem is:
\[
\max_{x_1, x_2} \left[ p_1(x_1)x_1 + p_2(x_2)x_2 - c(x_1 + x_2) \right]
\]

\[
\frac{\partial \Pi}{\partial x_1} = p_1(x_1) + x_1p_1'(x_1) - c = 0 \quad (1)
\]

\[
(i) \rightarrow MR_1 = MR_2 = c
\]

\[
\frac{\partial \Pi}{\partial x_2} = p_2(x_2) + x_2p_2'(x_2) - c = 0 \quad (2)
\]

\[
(i) \rightarrow p_i(x_i) + x_ip_i'(x_i) = c
\]

\[
p_i(x_i)[1 + \frac{x_ip_i'(x_i)}{p_i(x_i)}] = c
\]

\[
p_i(x_i)[1 + \frac{1}{\varepsilon_i(x_i)}] = c
\]

\[
p_i(x_i)[1 - \frac{1}{|\varepsilon_i(x_i)|}] = c
\]

\[
p_i(x_i) = \frac{c}{1 - \frac{1}{|\varepsilon_i(x_i)|}} \quad i = 1, 2.
\]

Therefore, \( p_1(x_1) > p_2(x_2) \) iff \( |\varepsilon_1(x_1)| < |\varepsilon_2(x_2)| \). As a consequence, the monopolist charges the highest price in the market with the lower price elasticity (in absolute value).

(iii) A comparison of profits with the case of uniform pricing (single monopoly pricing)

The monopolist’s profit under third-degree price discrimination is at least as high as the profit under uniform pricing. The reason is simple: under third-degree price discrimination the firm can always choose equal prices if that is the most profitable option.
(iv) Effects on social welfare

1) What is the problem?
2) Bounds of the change in social welfare.
3) Applications:
   a) Linear demand.
   b) Opening of markets.

1) What is the problem?
This section compares third-degree price discrimination and uniform pricing from a social welfare point of view. In general, a movement from uniform pricing to third-degree price discrimination benefits some agents and harms others.

Benefited by T-DPD: the monopolist and the consumers in the higher-elasticity market (given that the price is reduced by discrimination).

Harmed by T-DPD: the consumers in the lower-elasticity market (given that the price is increased by discrimination).

Therefore, the effect on social welfare is indeterminate.

2) Bounds of the change in social welfare
Assume for the sake of simplicity that there are only two markets and we start from an aggregate utility function $u_1(x_1) + u_2(x_2) + y_1 + y_2$, where $x_1$ and $x_2$ are the consumptions of good $x$ by the two groups and $y$ is the money to be spent on other goods ($y = y_1 + y_2$). $u_1$ and $u_2$ are strictly concave. The inverse demand functions are given by $p_1(x_1) = u'_1(x_1)$ and $p_2(x_2) = u'_2(x_2)$. 
If $C(x_1, x_2)$ is the cost of supplying $x_1$ and $x_2$ we can measure the social welfare as:

$$W(x_1, x_2) = u_1(x_1) + u_2(x_2) - C(x_1, x_2)$$

Consider two configurations of output $(x_1^0, x_2^0)$ and $(x_1^1, x_2^1)$ whose prices are $(p_1^0, p_2^0)$ and $(p_1^1, p_2^1)$, respectively. Assume that the initial set of prices corresponds to uniform pricing (the monopoly single price) $p_1^0 = p_2^0 = p^0$ and that $p_1^1$ and $p_2^1$ are the prices under third-degree price discrimination. Consider the movement from $x^0$ to $x^1$. Due to the strictly concavity of $u$ we have (see Appendix):

\[
\frac{p_1(x_1^0) - p_1^0}{\Delta x_1} < u_1(x_1^1) - u_1(x_1^0) < \frac{p_1(x_1^1) - p_1^1}{\Delta x_1} \quad (1) \rightarrow \Delta u_1 < p_1^0 \Delta x_1 \rightarrow p_1^0 \Delta x_1 > \Delta u_1 > p_1^1 \Delta x_1 \quad (3)
\]

\[
\frac{p_1(x_1^1) - p_1^1}{\Delta x_1} < u_1(x_1^1) - u_1(x_1^0) < \frac{p_1(x_1^1) - p_1^0}{\Delta x_1} \quad (1)' \rightarrow \Delta u_1 > p_1^1 \Delta x_1
\]

\[
\frac{p_2(x_2^0) - p_2^0}{\Delta x_2} < u_2(x_2^1) - u_2(x_2^0) < \frac{p_2(x_2^1) - p_2^1}{\Delta x_2} \quad (2) \rightarrow \Delta u_2 < p_2^0 \Delta x_2 \rightarrow p_2^0 \Delta x_2 > \Delta u_2 > p_2^1 \Delta x_2 \quad (4)
\]

\[
\frac{p_2(x_2^1) - p_2^1}{\Delta x_2} < u_2(x_2^1) - u_2(x_2^0) < \frac{p_2(x_2^1) - p_2^0}{\Delta x_2} \quad (2)' \rightarrow \Delta u_2 > p_2^1 \Delta x_2
\]

By adding (3) and (4) we get

$$p_1^0 \Delta x_1 + p_2^0 \Delta x_2 > \Delta u_1 + \Delta u_2 > p_1^1 \Delta x_1 + p_2^1 \Delta x_2$$

where

$$\Delta u = \Delta u_1 + \Delta u_2; \Delta x_1 = x_1^1 - x_1^0; \Delta x_2 = x_2^1 - x_2^0$$

$$p_1^0 = p_1(x_1^0) = u_1(x_1^0); p_2^0 = p_2(x_2^0) = u_2(x_2^0);$$

$$p_1^1 = p_1(x_1^1) = u_1(x_1^1); p_2^1 = p_2(x_2^1) = u_2(x_2^1).$$

The change in social welfare is given by:

$$\Delta W = W(x_1^1, x_2^1) - W(x_1^0, x_2^0) = \frac{u_1(x_1^1) - u_1(x_1^0)}{\Delta u_1} + \frac{u_2(x_2^1) - u_2(x_2^0)}{\Delta u_2} - \frac{C(x_1^1, x_2^1) - C(x_1^0, x_2^0)}{\Delta C}$$

$$= \Delta u_1 + \Delta u_2 - \Delta C$$
Therefore
\[ p_1^0\Delta x_1 + p_2^0\Delta x_2 - \Delta C > \Delta W > p_1^1\Delta x_1 + p_2^1\Delta x_2 - \Delta C \]

If marginal cost is constant:
\[ \Delta C = c(x_1^1 + x_2^1) - c(x_1^0 + x_2^0) = c\Delta x_1 + c\Delta x_2 \]

Therefore the bounds of the change in social welfare become:
\[ \frac{(p_1^0 - c)\Delta x_1 + (p_2^0 - c)\Delta x_2}{\text{Upper bound}} > \Delta W > \frac{(p_1^1 - c)\Delta x_1 + (p_2^1 - c)\Delta x_2}{\text{Lower bound}} \] (5)

Given that \( p_1^0 = p_2^0 = p^0 \) the bounds of the change in social welfare are:
\[ \frac{(p^0 - c)(\Delta x_1 + \Delta x_2)}{\text{Upper bound}} > \Delta W > \frac{(p_1^1 - c)\Delta x_1 + (p_2^1 - c)\Delta x_2}{\text{Lower bound}} \] (6)

- Upper bound: this implies that a necessary condition for third-degree price discrimination to increase social welfare, \( \Delta W > 0 \), is that it should increase total output. Assume on the contrary that \( \Delta x = \Delta x_1 + \Delta x_2 \leq 0 \). Given that \( (p^0 - c) > 0 \) then (4) \( \Delta W < 0 \).

- Lower bound: this indicates that a sufficient condition for third-degree price discrimination to increase social welfare is that the sum of the changes in output weighted by the difference between the price under discrimination and the marginal cost must be positive.

Graphically, for the case of a single market, the bounds would be:
3) Applications

a) Linear demands

Assume that the demands are given by \( x_i(p_i) = \frac{a_i}{b_i} - \frac{1}{b_i} p_i \), \( i = 1, 2 \), and that the marginal cost is zero, \( c = 0 \). The profit maximization problem under third-degree price discrimination is:

\[
\max_{p_1, p_2} p_1 x_1(p_1) + p_2 x_2(p_2)
\]

\[
\frac{\partial \Pi}{\partial p_1} = x_1(p_1) + p_1 x'_1(p_1) = 0 \rightarrow \frac{a_1}{b_1} - \frac{1}{b_1} p_1 - \frac{1}{b_1} p_i = 0 \rightarrow p_1^l = \frac{a_1}{2b_1}; \quad x_1^l = \frac{a_1}{2b_1}
\]

\[
\frac{\partial \Pi}{\partial p_2} = x_2(p_2) + p_2 x'_2(p_2) = 0 \rightarrow \frac{a_2}{b_2} - \frac{1}{b_2} p_2 - \frac{1}{b_2} p_2 = 0 \rightarrow p_2^l = \frac{a_2}{2b_2}; \quad x_2^l = \frac{a_2}{2b_2}
\]

The total output is:

\[
x^l = x_1^l + x_2^l = \frac{a_1}{2b_1} + \frac{a_2}{2b_2} = \frac{a_1 b_2 + a_2 b_1}{2b_1 b_2}
\]

Under uniform pricing:

\[
\max_p px_1(p) + px_2(p)
\]

\[
\frac{\partial \Pi}{\partial p} = x_1(p) + px'_1(p) + px'_2(p) \rightarrow \frac{a_1}{b_1} - \frac{1}{b_1} p + a_2 \frac{1}{b_2} - \frac{1}{b_2} p - \frac{1}{b_1} p - \frac{1}{b_2} p = 0
\]

\[\rightarrow p^0 = \frac{a_1 b_2 + a_2 b_1}{2(b_1 + b_2)}; \quad x_i^0 = \frac{a_i}{b_i} - \frac{1}{b_i} \left( \frac{a_1 b_2 + a_2 b_1}{2(b_1 + b_2)} \right) = \frac{2a_i b_1 + a_1 b_2 - a_2 b_i}{2b_i (b_1 + b_2)} = \frac{2a_i b_1 + a_2 b_2 - a_i b_1}{2b_i (b_1 + b_2)}
\]

The total output is:

\[
x^0 = x_1^0 + x_2^0 = \frac{2a_1 b_1 + a_1 b_2 - a_2 b_i}{2b_i (b_1 + b_2)} + \frac{2a_2 b_i + a_2 b_1 - a_i b_2}{2b_i (b_1 + b_2)} = \frac{2a_1 b_1 b_2 + a_1 (b_2)^2 - a_2 b_i b_2 + 2a_2 b_i b_1 + a_2 (b_1)^2 - a_i b_2 b_2}{2b_i b_2 (b_1 + b_2)}
\]

\[
= \frac{a_i b_1 b_2 + a_i (b_2)^2 + a_1 b_2 b_2 + a_2 (b_1)^2}{2b_i b_2 (b_1 + b_2)} = \frac{(a_i b_1 + a_1 (b_2))(b_1 + b_2)}{2b_i b_2 (b_1 + b_2)} = \frac{a_i b_2 + a_i b_1}{2b_i b_2}
\]
Therefore, total output is the same under both pricing policies. That is, $\Delta x = \Delta x_1 + \Delta x_2 = 0$, or, equivalently, $\Delta x_1 = -\Delta x_2$. The bounds would be

$$\frac{(p^0 - c)(\Delta x_1 + \Delta x_2)}{<0} > \Delta W > (p_1^1 - c)\Delta x_1 + (p_2^1 - c)\Delta x_2$$ (6)

Social welfare therefore decreases: $\Delta W < 0$.

As we show below, the above result depends crucially on the assumption that all markets are served under uniform pricing.

b) Opening of markets

Imagine that the two markets demands are like those in the graphic.

If the monopolist had to sell at a uniform price, it would have to reduce the price in market 1 by such an amount that the decrease in profits in that market would not be offset. Therefore,

$$\frac{(p_1^1 - c)(\Delta x_1 + \Delta x_2)}{=0} > \Delta W > \frac{(p_2^1 - c)\Delta x_1 + (p_2^1 - c)\Delta x_2}{>0}$$ (6)

Given that the lower bound in (4) is positive then $\Delta W > 0$. But not only does social welfare increase, in fact third-degree price discrimination Pareto dominates uniform pricing. A move
from uniform pricing to third-degree price discrimination implies an increase in the monopolist's profits, an improvement for consumers in market 2 and no change for consumers in market 1.
Appendix

If \( u \) is a strictly concave function for any \( x \) and \( y \) the following is satisfied:

\[
u(x) < u(y) + u'(y)(x - y).
\]

The tangents always remain above the function when it is strictly concave.
Basic Bibliography


Complementary Bibliography

