Real valued functions in Pointfree Topology

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– joint work with Tomasz Kubiak (Poznan) and Jorge Picado (Coimbra)
“The aim of these notes is to show how various facts in classical topology connected with the real numbers have their counterparts, if not actually their logical antecedents, in pointfree topology, that is, in the setting of frames and their homomorphisms.

... the treatment here will specifically concentrate on the pointfree version of continuous real functions which arises from it.”

B. Banaschewski,

The real numbers in pointfree topology,

Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.
“The set $C(X)$ of all continuous, real-valued functions on a topological space $X$ will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection $\mathbb{R}^X$ of all functions from $X$ into the set $\mathbb{R}$ of real numbers. [...]”

In fact, it is clear that $\mathbb{R}^X$ is a commutative ring with unity element (provided that $X$ is non empty). [...]”

Therefore $C(X)$ is a commutative ring, a subring of $\mathbb{R}^X$.”

L. Gillman and M. Jerison,
Rings of Continuous Functions
Motivation: Katětov-Tong Theorem

Urysohn’s Lemma.

Let $X$ be a topological space. TFAE:

1. $X$ is normal.
2. For every disjoint closed sets $F$ and $G$, there exists a continuous $h : X \rightarrow [0, 1]$ such that $h(F) = \{0\}$ and $h(G) = \{1\}$.
3. For every closed set $F$ and open set $U$ such that $F \subseteq U$, there exists a continuous $h : X \rightarrow \mathbb{R}$ such that $\chi_F \leq h \leq \chi_U$.

Question

Let $X$ be a topological space and let $f, g : X \rightarrow \mathbb{R}$ be such that $f \in \text{USC}(X)$, $g \in \text{LSC}(X)$ and $f \leq g$.

Does there exists a continuous $h \in C(X)$ such that $f \leq h \leq g$?
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Does there exists a continuous $h \in C(X)$ such that $f \leq h \leq g$?

**Answer**

Yes, if $X$ is **METRIC** [Hahn, 1917]

Yes, if $X$ is **PARACOMPACT** [Dieudonné, 1944]

Yes, if $X$ is **NORMAL** [Katětov-Tong, 1948]
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Motivation: Katětov-Tong Theorem

Katětov-Tong Insertion Theorem.

Let $X$ be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

1. $X$ is normal.
2. For every $f \in \text{USC}(X)$ and every $g \in \text{LSC}(X)$ with $f \leq g$, there exists a continuous $h \in C(X)$ such that $f \leq h \leq g$.

---

M. Katětov,

*On real-valued functions in topological spaces,*


H. Tong,

*Some characterizations of normal and perfectly normal spaces,*

Stone Insertion Theorem.

Let $X$ be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

1. $X$ is extremally disconnected (any two disjoint open sets in $X$ have disjoint closures).
2. For every $f \in LSC(X)$ and every $g \in USC(X)$ with $f \leq g$, there exists a continuous $h \in C(X)$ such that $f \leq h \leq g$.

M.H. Stone,

*Boundedness properties in function-lattices*,

Dowker Insertion Theorem.

Let $X$ be a topological space and let $f, g : X \rightarrow \mathbb{R}$. TFAE:

1. $X$ is normal and countably paracompact.
2. For every $f \in \text{USC}(X)$ and every $g \in \text{LSC}(X)$ with $f < g$, there exists a continuous $h \in \text{C}(X)$ such that $f < h < g$.

C. H. Dowker,

*On countably paracompact spaces*,

Motivation: Michael Insertion Theorem

Michael Insertion Theorem.

Let $X$ be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

1. $X$ is perfectly normal (every two disjoint closed sets can be precisely separated by a continuous real valued function).
2. For every $f \in \text{USC}(X)$ and every $g \in \text{LSC}(X)$ with $f \leq g$, there exists a continuous $h \in C(X)$ such that $f \leq h \leq g$ and $f(x) < h(x) < g(x)$ whenever $f(x) < g(x)$.

E. Michael,

*Continuous selections I,*,

Motivation: Kubiak Insertion Theorem

A topological space $X$ is **completely normal** if for every pair of subsets $A$ and $B$ of $X$ which are separated (i.e. $\overline{A} \cap B = \emptyset = A \cap \overline{B}$) there are disjoint open sets containing $A$ and $B$ respectively.
A topological space $X$ is **completely normal** if for every pair of subsets $A$ and $B$ of $X$ which are separated (i.e. $\overline{A} \cap B = \emptyset = A \cap \overline{B}$) there are disjoint open sets containing $A$ and $B$ respectively.
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(A standard exercise is to show that this is equivalent to hereditary normality.)
Motivation: Kubiak Insertion Theorem

Kubiak Insertion Theorem.

Let $X$ be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

(1) $X$ is completely normal.

(2) If $\overline{A} \subseteq B$ and $A \subseteq \overset{\circ}{B}$, then there exists an open set $U$ such that $A \subseteq U \subseteq \overline{U} \subseteq B$.

(3) If $f^- \leq g$ and $f \leq g^\circ$, then there exists a lower semicontinuous $h : X \to \mathbb{R}$ such that $f \leq h \leq h^- \leq g$

(where $f^-$ denotes the upper regularization of $f$ and $g^\circ$ denotes the lower regularization of $g$).
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T. Kubiak,

*A strengthening of the Katětov-Tong insertion theorem*,

“The aim of these notes is to show how various facts in classical topology connected with the real numbers have their counterparts, if not actually their logical antecedents, in pointfree topology, that is, in the setting of frames and their homomorphisms.

... the treatment here will specifically concentrate on the pointfree version of continuous real functions which arises from it.”

Our intention in this talk is to extend this study to the case of general real valued functions (paying particular attention to the semicontinuous ones) in the setting of pointfree topology.

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The real numbers in pointfree topology,

Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.
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Pointfree topology

\[(X, \mathcal{O}X) \rightarrow (\mathcal{O}X, \subseteq)\]

\[A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)\]

\[f^{-1}\] preserves \(\bigcup\) and \(\cap\)

\[(Y, \mathcal{O}Y) \rightarrow (\mathcal{O}Y, \subseteq)\]

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Real valued functions in Pointfree Topology
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TOPOLOGY

Abstraction

POINTFREE TOPOLOGY
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Real valued functions in Pointfree Topology
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**Abstraction**
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\[ (X, \mathcal{O}X) \xrightarrow{f} (\mathcal{O}Y, \subseteq) \xleftarrow{f^{-1}} (Y, \mathcal{O}Y) \]

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\[(X, \mathcal{O}X) \rightarrow (\mathcal{O}X, \subseteq)\]

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\[(X, \mathcal{O}X) \Rightarrow (\mathcal{O}X, \subseteq)\]

\[f \downarrow \quad f^{-1} \downarrow\]

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B. Banaschewski,
*The real numbers in pointfree topology,*
Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

P. T. Johnstone,
*Stone Spaces,*

P. T. Johnstone,
*The point of pointless topology.*
The objects in \( \text{Frm} \) are *frames*, i.e.

- complete lattices \( L \) in which
  
  \[ a \land \bigvee_{i \in I} a_i = \bigvee \{ a \land a_i : i \in I \} \quad \text{for all } a \in L \text{ and } \{a_i : i \in I\} \subseteq L. \]

- Morphisms, called *frame homomorphisms*, are those maps between frames \( h \) that preserve

  - arbitrary joins,
    
    \[ h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i), \quad h(0) = 0, \]

  - finite meets,
    
    \[ h(a_1 \land a_2) = h(a_1) \land h(a_2), \quad h(1) = 1. \]
Pointfree topology

The category of frames \( \mathbf{Frm} \)

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- The objects in \( \text{Frm} \) are *frames*, i.e.
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Pointfree topology

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Pointfree topology

the category of frames \( \text{Frm} \)

Being a Heyting algebra, each frame \( L \) has the Heyting operation \( \rightarrow \) satisfying \( a \land b \leq c \) iff \( a \leq b \rightarrow c \).

The pseudocomplement of \( a \in L \) is

\[
a^* = a \rightarrow 0 = \bigvee \{ b \in L : a \land b = 0 \}.
\]

When \( a \) is complemented, \( a^* \) is its complement and we denote it by the usual notation \( \neg a \).

The set of all morphisms from \( L \) into \( M \) is denoted by

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\text{Frm}(L, M)
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Being a Heyting algebra, each frame $L$ has the Heyting operation $\to$ satisfying $a \land b \leq c$ iff $a \leq b \to c$.

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Pointfree topology

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Pointfree topology

**Motivating example:** the lattice $\mathcal{O}X$ of all open subsets of a space $X$ is a frame and if $f : X \to Y$ is a map, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism.

Consequently we have a contravariant functor

$$\begin{array}{ccc}
\text{Top} & \xrightarrow{\mathcal{O}} & \text{Frm} \\
\end{array}$$

There is a functor in the opposite direction, the **spectrum functor**

$$\begin{array}{ccc}
\text{Top} & \xleftarrow{\Sigma} & \text{Frm} \\
\end{array}$$

which assigns to each frame $L$ its spectrum $\Sigma L = \text{Frm}(L, 2 = \{0 < 1\})$, with open sets $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$.
**Motivating example:** the lattice $\mathcal{O}X$ of all open subsets of a space $X$ is a frame and if $f : X \rightarrow Y$ is a map, then $\mathcal{O}f : \mathcal{O}Y \rightarrow \mathcal{O}X$ defined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism.

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Pointfree topology

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**Motivating example:** the lattice \( \mathcal{O}X \) of all open subsets of a space \( X \) is a frame and if \( f : X \rightarrow Y \) is a map, then \( \mathcal{O}f : \mathcal{O}Y \rightarrow \mathcal{O}X \) defined by \( \mathcal{O}f(U) = f^{-1}(U) \) is a frame homomorphism.

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which assigns to each frame $L$ its spectrum $\Sigma L = \text{Frm}(L, 2 = \{0 < 1\})$, with open sets $\Sigma_a = \{\xi \in \Sigma L : \xi(a) = 1\}$. 
We have two contravariant functors:

\[
\begin{array}{ccc}
\text{Top} & \overset{\eta}{\longrightarrow} & \text{Frm} \\
\underset{\Sigma}{\longleftarrow} & \underset{\omega}{\longleftarrow} & \text{Frm}
\end{array}
\]

which form a dual adjunction.

That is, there are adjunction maps

\[
\eta_L : L \rightarrow \omega \Sigma L, \quad \eta_L(a) = \Sigma_a \quad (a \in L)
\]

and

\[
\varepsilon_X : X \rightarrow \Sigma \omega X, \quad \varepsilon_X(x) = \hat{x}, \quad \hat{x}(U) \text{ iff } x \in U \quad (x \in X)
\]

natural in \( L \) and \( X \) respectively.
We have two contravariant functors:

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which form a dual adjunction.

That is, there are adjunction maps

\[
\eta_L : L \rightarrow \mathcal{O}\Sigma L, \quad \eta_L(a) = \Sigma_a \quad (a \in L)
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and

\[
\varepsilon_X : X \rightarrow \Sigma\mathcal{O}X, \quad \varepsilon_X(x) = \hat{x}, \quad \hat{x}(U) \text{ iff } x \in U \quad (x \in X)
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natural in \(L\) and \(X\) respectively.
We have two contravariant functors:

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natural in \(L\) and \(X\) respectively.
Frames $L$ for which $\eta_L$ is an isomorphism are called **spatial**, and $\eta_L$ is then the reflection map from $L$ to spatial frames.

On the other hand, spaces for which $\varepsilon_X$ is an homeomorphism are called **sober**, and by general principles, the full subcategory $\text{Sob}$ of $\text{Top}$ given by these spaces is then dually equivalent to the full subcategory $\text{SpFrm}$ of $\text{Frm}$ given by the spatial frames.

Note that we also have a natural equivalence

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The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs \((u, v)\) for the given relations \(u = v\).

So, in the context of pointfree topology the frame of reals may be introduced independent of any notion of real number:

The *frame of reals* is the frame \(\mathcal{L}(\mathbb{R})\) generated by all ordered pairs \((p, q)\), where \(p, q \in \mathbb{Q}\), subject to the following relations:

\begin{align*}
(R1) \quad & (p, q) \land (r, s) = (p \lor r, q \land s) \\
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The spectrum of $\mathcal{L}(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}$ of extended reals endowed with the euclidean topology.

Consequently, the space $\mathbb{R}$ could be defined as $\Sigma \mathcal{L}(\mathbb{R})$ since the latter construct requires no previous knowledge of $\mathbb{R}$.

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A **continuous real function** on \( L \) is a frame homomorphism \( \mathcal{L}(\mathbb{R}) \to L \).
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A continuous real function on $L$ is a frame homomorphism $\mathcal{L}(\mathbb{R}) \to L$. 
We shall denote by $c(L)$ the set of all continuous real functions on $L$:

$$c(L) = \text{Frm}(\mathcal{L}(\mathbb{R}), L)$$

**Algebraic operations**

Let $\langle p, q \rangle = \{ r \in \mathbb{Q} : p < r < q \}$, let $\diamond \in \{ +, \cdot, \max, \min \}$, and let

$$\langle r, s \rangle \diamond \langle t, u \rangle = \{ x \diamond y : x \in \langle r, s \rangle \text{ and } y \in \langle t, u \rangle \}.$$

Given $f_1, f_2, f \in c(L)$ and $r \in \mathbb{Q}$, we define

$$(f_1 \diamond f_2)(p, q) = \bigvee \{ f_1(r, s) \land f_2(t, u) : \langle r, s \rangle \diamond \langle t, u \rangle \subseteq \langle p, q \rangle \},$$

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### Algebraic operations

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These operations satisfy all the lattice-ordered ring axioms in $\mathbb{Q}$ so that $(c(L), +, \cdot, \leq)$ becomes a lattice-ordered ring with unit 1.

We also have the following descriptions of the partial order:

\[
\begin{align*}
    f_1 \leq f_2 & \iff f_1(p, -) \leq f_2(p, -) \text{ for all } p \in \mathbb{Q} \hfill \\
    & \iff f_2(-, q) \leq f_1(-, q) \text{ for all } q \in \mathbb{Q} \hfill \\
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Semicontinuous real functions

Let $L_l(\mathbb{R})$ and $L_u(\mathbb{R})$ denote the subframes generated by elements:

$(-, q) := \bigvee_{p \in \mathbb{Q}} (p, q)$ and $(p, -) := \bigvee_{q \in \mathbb{Q}} (p, q)$.

One is tempted to follow the lines of the previous definition:

**Definition**

1. An upper semicontinuous real function on $L$ is a frame homomorphism $L_l(\mathbb{R}) \rightarrow L$.
2. A lower semicontinuous real function on $L$ is a frame homomorphism $L_u(\mathbb{R}) \rightarrow L$.

Y.-M. Li and G.-J. Wang,

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Semicontinuity

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(1) An upper semicontinuous real function on $L$ is a frame homomorphism $\mathcal{L}_l(\mathbb{R}) \to L$.

(2) A lower semicontinuous real function on $L$ is a frame homomorphism $\mathcal{L}_u(\mathbb{R}) \to L$.

Y.-M. Li and G.-J. Wang,

Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,

Semicontinuity

But things become more complicated because \((\mathbb{R}, T_l)\) fails to be sober. Indeed, the spectrum \(\Sigma L_l(\mathbb{R})\) of \(L_l(\mathbb{R})\) is homeomorphic to the space \((\mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}, T_l)\). Hence

\[
\text{Top}(X, (\mathbb{R}, T_l)) \subset \text{Top}(X, (\mathbb{R}_{-\infty}, T_l)) \simeq \text{Frm}(L_l(\mathbb{R}), O_X).
\]

The frame homomorphisms \(f \in \text{Frm}(L_l(\mathbb{R}), O_X)\) corresponding to continue maps in \(\text{Top}(X, (\mathbb{R}, T_l))\) are precisely those satisfying the additional condition:

\[
\bigvee_{q \in Q} \sigma(f(-, q)) = 1
\]

---

J. Gutiérrez García and J. Picado

On the algebraic representation of semicontinuity

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\[
C(X) = \text{USC}(X) \cap \text{LSC}(X)
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---

B. Banaschewski,

*The real numbers in pointfree topology*

Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.
Y.-M. Li and G.-J. Wang,

*Localic Katětov-Tong insertion theorem and localic Tietze extension theorem,*

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### Motivation

**Top**

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### Topology

\[ C(X) = \text{USC}(X) \cap \text{LSC}(X) \]

\[ \text{Top}(X, \mathcal{I}_l) \not\cong \text{Frm}(\mathcal{L}_l(\mathbb{R}), \mathcal{O}X) \] !!!

---

**J. Gutiérrez García and J. Picado**

**On the algebraic representation of semicontinuity**

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(Q1) How to remedy this?
Every $f : X \to \mathbb{R}$ admits lsc and usc regularizations

$\text{(Q2)}$

How can we speak about general localic real functions?
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(Q2) How can we speak about general localic real functions?
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J. Gutiérrez García, J. Picado,
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J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
The quotients in Frm (equivalently, the subobjects in the dual category Loc) have been described in several equivalent ways in the literature:

- as sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
- sublocale sets.

We follow the latter approach because, in our opinion, it has revealed to be the more intuitive and the easiest to work with:

A subset $S \subseteq L$ is a sublocale of $L$ if it satisfies the following:

(S1) For every $A \subseteq S$, $\bigwedge A \in S$,
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J. Gutiérrez García

Real valued functions in Pointfree Topology
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Since the intersection of sublocales is again a sublocale, the set \( SL \) of all sublocales is a complete lattice under inclusion.

For convenience, we shall deal with the opposite order, i.e.:

\[
S_1 \leq S_2 \iff S_1 \supseteq S_2.
\]

\((SL, \leq)\) is a frame, in which \( \{1\} \) is the top and \( L \) is the bottom.

Further, given \( \{S_i \in SL : i \in I\} \), we have

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\bigvee_{i \in I} S_i = \bigcap_{i \in I} S_i \quad \text{and} \quad \bigwedge_{i \in I} S_i = \left\{ \bigwedge A : A \subseteq \bigcup_{i \in I} S_i \right\}.
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Important examples of sublocales are the *open* and *closed* ones:

\[ o(a) = \{ a \rightarrow b : b \in L \} \quad \text{and} \quad c(a) = \uparrow a = \{ b \in L : a \leq b \}. \]

Open and closed sublocales are complemented and

\[ \neg o(a) = c(a) \quad \text{for each} \quad a \in L. \]

Also, for each \( a_i, a, b \in L \):

\[
\bigvee_{i \in I} c(a_i) = c(\bigvee_{i \in I} a_i), \quad c(a) \land c(b) = c(a \land b),
\]

\[
\bigwedge_{i \in I} o(a_i) = o(\bigvee_{i \in I} a_i) \quad \text{and} \quad o(a) \lor o(b) = o(a \land b)
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Thus, \( c : L \rightarrow SL \) is an embedding from \( L \) into \( c(L) = \{ c(a) : a \in L \} \) whereas \( o : L \rightarrow SL \) is a dual lattice embedding taking finite meets to joins and arbitrary joins to meets.
Pointfree topology

sublocales (generalized subspaces)

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sublocales (generalized subspaces)

For each sublocale $S$ the closure and the interior of $S$ are given by:

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\overline{S} = \bigvee \{c(a) : c(a) \leq S\} \quad \text{and} \quad S^\circ = \bigwedge \{o(a) : S \leq o(a)\}.
$$

In particular $\overline{o(a)} = c(a^*)$ and $c(a) = o(a^*)$.

Also, for each $S, T \in SL$:

1. $\{1\} = \{1\}, \quad L = L, \quad \overline{S} \leq S \leq \overset{\circ}{S}, \quad \overline{S} = \overline{S} \quad \text{and} \quad \overset{\circ}{S} = \overset{\circ}{S}.$

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J. Picado and A. Pultr,
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J. Picado and A. Pultr,
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In order to motivate the idea, we first recall the isomorphism

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\text{Top}(X, (\mathbb{R}, T_e)) \simeq \text{Frm}(\mathcal{L}(\mathbb{R}), \mathcal{O}X)
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Now, if we observe that the set \(\mathbb{R}^X\) is in an obvious bijection with \(\text{Top}((X, \mathcal{P}(X)), (\mathbb{R}, T))\) where \(T\) is any topology on \(\mathbb{R}\), we would, in particular, have a bijection

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Localic real-valued functions

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Therefore,

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\mathbb{R}^X \simeq \text{Frm}(\mathcal{L}(\mathbb{R}), S(\mathcal{O}X))
\]

where \( S(\mathcal{O}X) \) denotes the lattice of all subspaces of \( X \).
If we finally recall one slogan of pointfree topology that elements of the frame $SL$ are identified as \textit{generalized subspaces}, we thus arrive at the conclusion that one can think of members of $\text{Frm}(\mathcal{L}(\mathbb{R}), S(\mathcal{O}X))$ as \textit{arbitrary not necessarily continuous} real functions on $L$.

Thus the above bijection justifies to adopt the following:

**Definition**

A \textit{localic real function} on $L$ is a frame homomorphism $\mathcal{L}(\mathbb{R}) \to SL$. 
Localic real-valued functions

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Localic real-valued functions

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If we finally recall one slogan of pointfree topology that elements of the frame \( SL \) are identified as \textit{generalized subspaces}, we thus arrive at the conclusion that one can think of members of

\[ \text{Frm}(\mathcal{L}(\mathbb{R}), SL) \]

as \textit{arbitrary not necessarily continuous} real functions on \( L \).

Thus the above bijection justifies to adopt the following:

**Definition**

A \textbf{localic real function} on \( L \) is a frame homomorphism \( \mathcal{L}(\mathbb{R}) \to SL \).
We write: \( F(L) = \text{Frm}(\mathcal{L}(\mathbb{R}), SL) \).

Recall now that the map \( c : L \rightarrow SL \), associating to each \( a \in L \) the closed sublocale \( c(a) \), is an embedding.

Then for each frame \( M \) we have a further embedding

\[
\begin{array}{ccc}
\phi & \mapsto & c \circ \phi \\
\text{Frm}(M, L) & \rightarrow & \text{Frm}(M, SL)
\end{array}
\]

Hence

\[
\text{Frm}(M, L) \cong \{ f \in \text{Frm}(M, SL) : f(M) \subseteq c(L) \}
\]

In particular we have:

\[
c(L) = \text{Frm}(\mathcal{L}(\mathbb{R}), L) \cong \{ f \in \text{Frm}(\mathcal{L}(\mathbb{R}), SL) : f(\mathcal{L}(\mathbb{R})) \subseteq c(L) \}
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\[
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Localic real-valued functions

Definition

We shall say that a localic real function \( f \in F(L) \) is:

1. **continuous** if \( f(L(\mathbb{R})) \subseteq c(L) \).
2. **upper semicontinuous** if \( f(L_l(\mathbb{R})) \subseteq c(L) \).
3. **lower semicontinuous** if \( f(L_u(\mathbb{R})) \subseteq c(L) \).

We denote by \( C(L) \), \( USC(L) \), and \( LSC(L) \) the corresponding collections of members of \( F(L) \).

Of course, one has

\[
C(L) = LSC(L) \cap USC(L)
\]

J. Gutiérrez García, T Kubiak, J. Picado,
Localic real-valued functions: a general setting
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J. Gutiérrez García, T Kubiak, J. Picado,

Localic real-valued functions: a general setting

For each $\varphi \in \text{usc}(L)$ we define $f : \mathcal{L}(\mathbb{R}) \to SL$ by:

$$f(\cdot, q) := c(f(\cdot, q)) \quad \text{and} \quad \varphi(p, \cdot) := \bigvee_{r > p} \sigma(\varphi(\cdot, r)).$$

Then

$$f \in \text{USC}(L) \iff \bigvee_{q \in \mathbb{Q}} f(\cdot, q) = 1 \iff \bigvee_{p \in \mathbb{Q}} f(p, \cdot) = 1 \iff \bigvee_{q \in \mathbb{Q}} f(\cdot, q) = 1 \quad \text{and} \quad \bigvee_{p \in \mathbb{Q}} \sigma(\varphi(\cdot, p)) = 1 \iff f \in \text{usc}(L).$$

We conclude that the restriction to $\text{usc}(L)$ is also an order-isomorphism between $\text{usc}(L)$ and $\text{USC}(L)$. 
Localic real-valued functions

For each \( \varphi \in \text{usc}(L) \) we define \( f : \mathcal{L}(\mathbb{R}) \rightarrow SL \) by:

\[
    f(-, q) := c(f(-, q)) \quad \text{and} \quad \varphi(p, -) := \bigvee_{r > p} \sigma(\varphi(-, r)).
\]

Then

\[
    f \in \text{USC}(L) \iff \bigvee_{q \in \mathbb{Q}} f(-, q) = 1 = \bigvee_{p \in \mathbb{Q}} f(p, -) \iff \bigvee_{q \in \mathbb{Q}} f(-, q) = 1 \quad \text{and} \quad \bigvee_{p \in \mathbb{Q}} \sigma(\varphi(-, p)) = 1 \iff f \in \text{usc}(L).
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We conclude that the restriction to \( \text{usc}(L) \) is also an order-isomorphism between \( \text{usc}(L) \) and \( \text{USC}(L) \).
Localic real-valued functions

For each $\varphi \in \text{usc}(L)$ we define $f : \mathcal{L}(\mathbb{R}) \to SL$ by:

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Then

$$f \in \text{USC}(L) \iff \bigvee_{q \in \mathbb{Q}} f(-, q) = 1 = \bigvee_{p \in \mathbb{Q}} f(p, -)$$
$$\iff \bigvee_{q \in \mathbb{Q}} f(-, q) = 1 \quad \text{and} \quad \bigvee_{p \in \mathbb{Q}} \varphi(-, p) = 1$$
$$\iff f \in \text{usc}(L).$$

We conclude that the restriction to $\text{usc}(L)$ is also an order-isomorphism between $\text{usc}(L)$ and $\text{USC}(L)$.
For each \( \varphi \in \text{usc}(L) \) we define \( f : \mathcal{L}(\mathbb{R}) \to SL \) by:

\[
f(\mathbf{a}, q) := c(f(\mathbf{a}, q)) \quad \text{and} \quad \varphi(p, \mathbf{a}) := \bigvee_{r > p} s(\varphi(\mathbf{a}, r)).
\]

Then

\[
f \in \text{USC}(L) \iff \bigvee_{q \in \mathbb{Q}} f(\mathbf{a}, q) = 1 = \bigvee_{p \in \mathbb{Q}} f(p, \mathbf{a})
\]

\[
\iff \bigvee_{q \in \mathbb{Q}} f(\mathbf{a}, q) = 1 \quad \text{and} \quad \bigvee_{p \in \mathbb{Q}} s(\varphi(\mathbf{a}, p)) = 1
\]

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We conclude that the restriction to \( \text{usc}(L) \) is also an order-isomorphism between \( \text{usc}(L) \) and \( \text{USC}(L) \).
Localic real-valued functions

For each \( \varphi \in \text{usc}(L) \) we define \( f : \mathcal{L}(\mathbb{R}) \to SL \) by:

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\]

Then

\[
f \in \text{USC}(L) \iff \bigvee_{q \in \mathbb{Q}} f(-, q) = 1 = \bigvee_{p \in \mathbb{Q}} f(p, -) \iff \bigvee_{q \in \mathbb{Q}} f(-, q) = 1 \quad \text{and} \quad \bigvee_{p \in \mathbb{Q}} \sigma(\varphi(-, p)) = 1 \iff f \in \text{usc}(L).
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Localic real-valued functions

the isomorphism

\[ F(L) \]

\[ \text{usc}(L) \leftrightarrow \text{USC}(L) \quad \leftrightarrow \quad \text{LSC}(L) \leftrightarrow \text{lsc}(L) \]

\[ C(L) \]

\[ c(L) \]
Localic real-valued functions

Given a complemented sublocale $S \in SL$ the characteristic function $\chi_S : \mathcal{L}(\mathbb{R}) \to SL$ is defined by

$$
\chi_S(-, q) = \begin{cases} 
0 & \text{if } q \leq 0 \\
S & \text{if } 0 < q \leq 1, \\
1 & \text{if } q > 1 
\end{cases}
$$

$$
\chi_S(p, -) = \begin{cases} 
1 & \text{if } p < 0 \\
\neg S & \text{if } 0 \leq p < 1 \\
0 & \text{if } p \geq 1.
\end{cases}
$$

Note that,

- $\chi_S \in \text{USC}(L)$ if and only if $S$ is closed.
- $\chi_S \in \text{LSC}(L)$ if and only if $S$ is open.
- $\chi_S \in \text{C}(L)$ if and only if $S$ is clopen.
Localic real-valued functions

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\end{cases}$$

Note that,

- $\chi_S \in USC(L)$ if and only if $S$ is closed.
- $\chi_S \in LSC(L)$ if and only if $S$ is open.
- $\chi_S \in C(L)$ if and only if $S$ is clopen.
Localic real-valued functions

For $f \in F(L)$ we define the lower regularization $f^\circ$:

\[
f^\circ(-, q) = \bigvee_{s < q} \neg f(s, -)
\]
and
\[
f^\circ(p, -) = \bigvee_{r > p} f(r, -).
\]

- $f^\circ \leq f$
- $f^{\circ\circ} = f^\circ$
- $f^\circ \in \text{LSC}(L)$
- $g \in \text{LSC}(L)$ and $g \leq f$ $\Rightarrow$ $g \leq f^\circ$
- $(\chi_S)^\circ = \chi_S^\circ$
Localic real-valued functions

For \( f \in \mathcal{F}(L) \) we define the *lower regularization* \( f^\circ \):

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\begin{align*}
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Localic real-valued functions

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\begin{align*}
 f^\circ(-, q) &= \bigvee_{s < q} \neg f(s, -) \\
 f^\circ(p, -) &= \bigvee_{r > p} f(r, -)
\end{align*}
\]

and

\[
 f^\circ \leq f \\
 f^{\circ \circ} = f^\circ \\
 f^\circ \in \text{LSC}(L) \\
g \in \text{LSC}(L) \text{ and } g \leq f \implies g \leq f^\circ \\
(\chi_S)^\circ = \chi_S^\circ
\]
Localic real-valued functions

For \( f \in F(L) \) we define the \textit{lower regularization} \( f^\circ \):

\[
f^\circ(-, q) = \bigvee_{s < q} \overline{f(s, -)} \quad \text{and} \quad f^\circ(p, -) = \bigvee_{r > p} f(r, -).\]

\[
f^\circ \leq f \quad f^{\circ\circ} = f^\circ \quad f^\circ \in \text{LSC}(L) \quad g \in \text{LSC}(L) \text{ and } g \leq f \Rightarrow g \leq f^\circ \quad (\chi_S^\circ = \chi_S^\circ)\]
Localic real-valued functions

For \( f \in \mathcal{F}(L) \) we define the \textit{lower regularization} \( f^\circ \):

\[
f^\circ(\neg, q) = \bigvee_{s < q} \neg f(s, \neg) \quad \text{and} \quad f^\circ(p, \neg) = \bigvee_{r > p} f(r, \neg).
\]

\[
f^\circ \leq f
\]

\[
f^{\circ \circ} = f^\circ
\]

\[
f^\circ \in \text{LSC}(L)
\]

\[
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\[
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For $f \in \mathcal{F}(L)$ we define the *lower regularization* $f^\circ$:

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f^\circ(-, q) = \bigvee_{s < q} \neg f(s, -)
\quad \text{and} \quad
f^\circ(p, -) = \bigvee_{r > p} f(r, -).
\]

- $f^\circ \leq f$
- $f^{\circ\circ} = f^\circ$
- $f^\circ \in \text{LSC}(L)$
- $g \in \text{LSC}(L)$ and $g \leq f \implies g \leq f^\circ$
- $(\chi_s)^\circ = \chi_{s^\circ}$
For $f \in \mathbb{F}(L)$ we define the *lower regularization* $f^\circ$:

$$f^\circ(-, q) = \bigvee_{s < q} -f(s, -)$$
and

$$f^\circ(p, -) = \bigvee_{r > p} f(r, -).$$

- $f^\circ \leq f$
- $f^{ oo } = f^\circ$
- $f^\circ \in \text{LSC}(L)$
- $g \in \text{LSC}(L)$ and $g \leq f \Rightarrow g \leq f^\circ$
- $\chi_S^\circ = \chi_S$
Localic real-valued functions

For \( f \in \overline{F}(L) \) we define the **upper regularization** \( f^- \):

\[
\begin{align*}
  f^-(−, q) &= \bigvee_{s < q} f(−, s) \\
  f^-(p, −) &= \bigvee_{r > p} \neg f(−, r).
\end{align*}
\]

and

\[
  f \leq f^- \quad f^- = f^- \\
  f^- \in USC(L) \\
  g \in USC(L) \text{ and } f \leq g \quad \Rightarrow \quad f^- \leq g \\
  (\chi_S)^- = \chi_S
\]
Localic real-valued functions

For $f \in \overline{F}(L)$ we define the **upper regularization** $f^-$:

$$ f^-(\cdot, q) = \bigvee_{s < q} f(\cdot, s) \quad \text{and} \quad f^-(p, \cdot) = \bigvee_{r > p} -f(\cdot, r). $$

- $f \leq f^-$
- $f^-- = f^-$
- $f^- \in USC(L)$
- $g \in USC(L)$ and $f \leq g \Rightarrow f^- \leq g$

$$(\chi_S)^- = \chi_S^-$$
Localic real-valued functions

- One can see semicontinuous functions as a particular kind of real-valued functions on the frame of congruences, with the same domain, namely $\mathcal{L}(\mathbb{R})$.

- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the frame of congruences, we can compare them.

- By considering the algebraic operations of the ring $\text{Frm}(\mathcal{L}(\mathbb{R}), SL)$, we obtain, in particular, a way of defining the sum of upper and lower semicontinuous functions.

- The class of continuous functions is precisely the intersection of the classes of lower and upper ones.

- The situation with respect to regularization is precisely the same as in the topological setting.
Localic real-valued functions

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- The situation with respect to regularization is precisely the same as in the topological setting.
Theorem (Katětov-Tong)

The following conditions on a frame $L$ are equivalent:

1. $L$ is normal.
2. For every $f \in \text{USC}(L)$ and every $g \in \text{LSC}(L)$ with $f \leq g$, there exists $h \in C(L)$ such that $f \leq h \leq g$.


Insertion theorems

Theorem (Stone)

The following conditions on a frame $L$ are equivalent:

1. $L$ is extremally disconnected.
2. $C(L) = \{f^- : f \in LSC(L)\}$.
3. $C(L) = \{g^\circ : g \in USC(L)\}$.
4. For every $f \in USC(L)$ and every $g \in LSC(L)$ with $g \leq f$, there exists $h \in C(L)$ such that $g \leq h \leq f$.

Y.-M. Li and Z.-H. Li,

*Constructive insertion theorems and extension theorems on extremally disconnected frames,*


J. Gutiérrez García and J. Picado,

*Lower and upper regularizations of frame semicontinuous real functions,*

Let \( \text{UL}(L) = \{(f, g) \in \text{USC}(L) \times \text{LSC}(L) : f \leq g\} \) with the order 
\((f_1, g_1) \leq (f_2, g_2) \iff f_2 \leq f_1 \text{ and } g_1 \leq g_2.\)

**Theorem (Monotone Katětov-Tong)**

For a frame \( L \), the following are equivalent:

1. \( L \) is monotonically normal.
2. There exists a monotone function \( \Lambda : \text{UL}(L) \rightarrow \text{C}(L) \) such that 
   \( f \leq \Lambda(f, g) \leq g \) for all \( (f, g) \in \text{UL}(L) \).

J. Gutiérrez García, T. Kubiak and J. Picado,

*Monotone insertion and monotone extension of frame homomorphisms*,

Let $\text{UL}(L) = \{(f, g) \in \text{USC}(L) \times \text{LSC}(L) : f \leq g\}$ with the order $(f_1, g_1) \leq (f_2, g_2) \iff f_2 \leq f_1$ and $g_1 \leq g_2$.

**Theorem (Monotone Katětov-Tong)**

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Insertion theorems

Strict insertion

Michael insertion theorem for perfectly normal frames...

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J. Gutiérrez García, T. Kubiak and J. Picado,
"Pointfree forms of Dowker and Michael insertion theorems",
Insertion theorems

Theorem

The following conditions on a frame $L$ are equivalent:

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2. $L$ is hereditarily normal (each sublocale of $L$ is normal).
3. Each open sublocale of $L$ is normal.
4. For every $f, g \in F(L)$, if $f^{-} \leq g$ and $f \leq g^{\circ}$, then there exists an $h \in LSC(L)$ such that $f \leq h \leq h^{-} \leq g$.

M.J. Ferreira, J. Gutiérrez García and J. Picado

Completely normal frames and real-valued functions,

Topology and its Applications, in press.
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M.J. Ferreira, J. Gutiérrez García and J. Picado

*Completely normal frames and real-valued functions,*

*Topology and its Applications,* in press.
Each $\theta \in C L$ determines a unique sublocale $S_\theta \subseteq L$ and a unique frame quotient $c_\theta \in \text{Frm}(L, S_\theta)$.

$\tilde{H} \in C(L)$ is said to be a continuous extension of $H \in C(S_\theta)$ if and only if the following diagram commutes:

\[
\xymatrix{ \mathcal{L}(\mathbb{R}) \ar[r]^H & \nabla_{S_\theta} \ar[r] & S_\theta \ar[d]^{c_\theta} \ar[l]_{\nabla} \ar[r] & L \ar[d]^{\nabla} \ar[l]_{\nabla} \\
\tilde{H} \ar[u] & & & & }
\]

i.e. $c_\theta \circ \nabla \circ \tilde{H} = \nabla \circ H$. 

J. Gutiérrez García Real valued functions in Pointfree Topology
Extension theorems

Each $\theta \in \mathcal{C}L$ determines a unique sublocale $S_\theta \subseteq L$ and a unique frame quotient $c_\theta \in \text{Frm}(L, S_\theta)$. 

Given $H \in \mathcal{C}(S_\theta)$, $\tilde{H} \in \mathcal{C}(L)$ is said to be a continuous extension of $H \in \mathcal{C}(S_\theta)$ if and only if the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{L}(\mathbb{R}) & \xrightarrow{H} & \nabla S_\theta \\
\nabla L & \xrightarrow{\tilde{H}} & L \\
\nabla L & \xrightarrow{c_\theta} & S_\theta
\end{array}
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J. Gutiérrez García

Real valued functions in Pointfree Topology
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$$
\begin{array}{ccc}
\mathcal{L}(\mathbb{R}) & \xrightarrow{H} & \nabla S_\theta \\
\n\n\end{array}
\begin{array}{ccc}
\n & \nabla L & \xleftarrow{\nabla} & L \\
\n\end{array}
\begin{array}{ccc}
\n & \nabla S_\theta & \xrightarrow{c_\theta} & S_\theta \\
\n\end{array}
\begin{array}{ccc}
\n & \tilde{H} & \xrightarrow{c_\theta \circ \nabla \circ \tilde{H}} & \nabla \circ H \\
\end{array}
$$

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\[
\begin{array}{c}
\mathcal{L}(\mathbb{R}) & \xrightarrow{H} & \nabla S_\theta & \xrightarrow{\nabla} & S_\theta \\
\text{\textcolor{red}{\nabla}} \searrow & & \nearrow \text{\textcolor{red}{\nabla}} & & \\
\nabla L & \xrightarrow{\nabla} & L \\
\end{array}
\]

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\[
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\tilde{H} \downarrow & & \downarrow \\
\nabla L \quad & & \quad \nabla L \\
& \; \downarrow \nabla & \\
& L \quad & \quad L \\
\downarrow c_\theta & & \\
\nabla S_\theta & \xrightarrow{\nabla} & S_\theta
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J. Gutiérrez García  
Real valued functions in Pointfree Topology
Extension theorems

Theorem (Tietze)

The following conditions on a frame $L$ are equivalent:

1. $L$ is normal.
2. For each closed sublocale $S$ of $L$ and each $H \in C(S)$, there exists a continuous extension $\tilde{H} \in C(L)$ of $H$.

Theorem

The following conditions on a frame $L$ are equivalent:

1. $L$ is extremally disconnected.
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Extension theorems

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The following conditions on a frame $L$ are equivalent:

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Theorem

The following conditions on a frame $L$ are equivalent:

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Extension theorems

Also versions for monotone normality, perfect normality, ...
Extension theorems

Also versions for monotone normality, perfect normality, ...

Theorem

For a frame $L$, the following are equivalent:

1. $L$ is monotonically normal.
2. For every closed sublocale $S$ there exists an extender $\Phi_S : \bar{C}(S) \to \bar{C}(L)$ such that for each $S_1, S_2$ and $H_i \in \bar{C}(S_i)$ ($i = 1, 2$) with $\hat{H}_1 \leq \hat{H}_2$ one has $\Phi_{S_1}(H_1) \leq \Phi_{S_2}(H_2)$.

Theorem

For a frame $L$, the following are equivalent:

1. $L$ is perfectly normal.
2. For every closed sublocale $S$ and $H \in \bar{C}(S)$, there exists a continuous extension $\tilde{H} \in \bar{C}(L)$ of $H$ such that $\tilde{H}(\bigvee_{p, q \in \mathbb{Q}} (p, q)) \in S$. 

J. Gutiérrez García
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with J. Picado

- On the algebraic representation of semicontinuity

with T. Kubiak and J. Picado

- Monotone insertion and monotone extension of frame homomorphisms

- Lower and upper regularizations of frame semicontinuous real functions

- Pointfree forms of Dowker and Michael insertion theorems

- Localic real-valued functions: a general setting

with M.J. Ferreira and J. Picado

- Completely normal frames and real-valued functions
  *To appear in: Topology and its Applications.*