Localic analogues of general insertion and extension theorems for real-valued functions

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**Motivation**

**Topological Extension Theorem (Mrówka).**

Let $X$ be a topological space, $S \subseteq X$ and $f : S \to \mathbb{R}$ be a bounded continuous function. TFAE:

1. $f$ has a continuous extension to the whole of $X$.
2. If $r > s \in \mathbb{Q}$, then $[f \geq r]$ and $[f \leq s]$ are completely separated in $X$.

($A$ and $B$ are said to be completely separated in $X$ if there is a continuous $f : X \to [0, 1]$ such that $f = 0$ on $A$ and $f = 1$ on $B$).

*S. Mrówka*

On some approximation theorems

*Nieuw Archief voor Wiskunde, (3) 16 (1968) 94–111.*
Motivation

Topological Insertion Theorem (Blair-Lane).

Let $X$ be a topological space and let $f, g : X \to \mathbb{R}$. TFAE:

1. There exists a continuous $h : X \to \mathbb{R}$ such that $f \leq h \leq g$.
2. If $r > s \in \mathbb{Q}$, then $[f \geq r]$ and $[g \leq s]$ are completely separated.

R.L. Blair
Extensions of Lebesgue sets and of real valued functions

E.P. Lane
Insertion of a continuous function
Background: the sublocale lattice $S(L)$

Frm locale $L$, subobject lattice: is a CO-FRAME

$S L = \text{the dual FRAME}$

$\forall c(a_i) = c(\bigvee_{i \in I} a_i)$

$c(a) \land c(b) = c(a \land b)$

$\forall o(a_i) = o(\bigvee_{i \in I} a_i)$

$o(a) \lor o(b) = o(a \land b)$

for each $a \in L$

\[
\begin{align*}
&c(a) : \text{closed} \\
o(a) : \text{open}
\end{align*}
\]

complemented

subframe $cL := \{c(a) : a \in L\} \simeq L$

subframe $oL := \langle\{o(a) : a \in L\}\rangle$

(the geometric motivation reads backwards)
Let $L$ be a frame and $S \subset L$ a sublocale.

The **closure** of $S$:

$$
\overline{S} = \bigvee \{ c(a) : c(a) \leq S \} = c(\wedge S) = \uparrow \wedge S
$$

$$
\overline{o(a)} = c(a^*)
$$

The **interior** of $S$:

$$
S^\circ = \bigwedge \{ o(a) : S \leq o(a) \}.
$$

$$
c(a)^\circ = o(a^*)
$$
Background: the frame of reals \( \mathcal{L}(\mathbb{R}) \)

\[ \mathcal{L}(\mathbb{R}) = \text{FRM} \langle (p, q) \mid p, q \in \mathbb{Q} \rangle \]

(R1) \( (p, q) \land (r, s) = (p \lor r, q \land s) \)

(R2) \( p \leq r < q \leq s \Rightarrow (p, q) \lor (r, s) = (p, s) \)

(R3) \( (p, q) = \bigvee \{(r, s) \mid p < r < s < q\} \)

(R4) \( \bigvee \{(p, q) \mid p, q \in \mathbb{Q}\} = 1 \).

\[ (-, q) := \bigvee_{p \in \mathbb{Q}} (p, q) \]

\[ (p, -) := \bigvee_{q \in \mathbb{Q}} (p, q) \]

\[ \mathcal{L}_l(\mathbb{R}) = \langle (-, q) \mid q \in \mathbb{Q} \rangle \]

\[ \mathcal{L}_u(\mathbb{R}) = \langle (p, -) \mid p \in \mathbb{Q} \rangle \]
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Background: localic real-valued functions

- $f : L(\mathbb{R}) \to SL$ general $F(L)$
- $f : L(\mathbb{R}) \to SL$ usc $USC(L)$ s.t. $f(L_{l}(\mathbb{R})) \subseteq cL$
- $f : L(\mathbb{R}) \to SL$ lsc $LSC(L)$ s.t. $f(L_{u}(\mathbb{R})) \subseteq cL$
- $f : L(\mathbb{R}) \to SL$ continuous $C(L)$ s.t. $f(L(\mathbb{R})) \subseteq cL$

$f \leq g \equiv f(p, -) \leq g(p, -) \quad \forall p \in \mathbb{Q} \iff f(-, q) \geq g(-, q) \quad \forall q \in \mathbb{Q}$
A collection of sublocales \( \mathcal{C} = \{S_r : r \in \mathbb{Q}\} \subseteq SL \) is a **scale** on \( SL \) if

- \( S_r \lor S_s^* = 1 \) whenever \( r < s \).
- \( \bigvee \mathcal{C} = 1 = \bigvee \mathcal{C}^* \).

If \( \mathcal{C} = \{S_r : r \in \mathbb{Q}\} \subseteq SL \) is a scale on \( SL \) then there exists a unique \( f \in F(L) \) such that for all \( r \in \mathbb{Q} \)

(i) \( f(r, -) = \bigvee_{s > r} S_s \), \( f(-, r) = \bigvee_{s < r} S_s^* \) and

(ii) \( f(r, -) \leq S_r \leq f(-, r)^* \).

\( f \) is the localic real valued function generated by \( \mathcal{C} \).

Given \( f \in F(L) \), both \( \{f(r, -) : r \in \mathbb{Q}\} \) and \( \{f(-, r)^* : r \in \mathbb{Q}\} \) are scales that generate \( f \).
Proposition

Let $f, g \in F(L)$ be generated be the scales $\mathcal{C} = \{S_r : r \in Q\}$ and $\mathcal{D} = \{T_r : r \in Q\}$, respectively. Then:

$$f \leq g \quad \text{if and only if} \quad S_r \leq T_s \quad \text{whenever} \quad r > s.$$
Background: Katětov relation

Let \((M, \leq)\) be a complete lattice. A binary relation \(\varrho\) on \(M\) is a Katětov relation if and only if for all \(x, y, z, x_1, x_2, y_1, y_2 \in M\) the following hold:

(P1) \(x \varrho y \Rightarrow x \leq y\).

(P2) \(x_2 \leq x_1 \varrho y_1 \leq y_2 \Rightarrow x_2 \varrho y_2\).

(P3) \(x_1 \varrho y\) and \(x_2 \varrho y \Rightarrow (x_1 \lor x_2) \varrho y\).

(P4) \(x \varrho y_1\) and \(x \varrho y_2 \Rightarrow x \varrho (y_1 \land y_2)\).

(P5) \(x \varrho y \Rightarrow x \varrho z \varrho y\) for some \(z \in M\). (Interpolation Property)

(Such a relation has various names in the literature: quasi-proximity relation, subordination. . .)

M. Katětov

On real-valued functions in topological spaces

Katětov lemmas

Lemma (Katětov)
Let $\varrho$ be a Katětov relation on $M$ and $A, B \subset M$ countable such that

$$(\bigvee A) \varrho b \quad \text{and} \quad a \varrho (\bigwedge B)$$

for all $a \in A, b \in B$, then there is a $c \in M$ such that $a \varrho c \varrho b$ for all $a \in A$ and $b \in B$.

Lemma (Katětov)
Let $\varrho$ be an Katětov relation on $M$ and $\{a_r\}_{r \in \mathbb{Q}}, \{b_r\}_{r \in \mathbb{Q}} \subset M$ such that

$$r > s \implies a_r \leq a_s, \quad b_r \leq b_s \quad \text{and} \quad a_r \varrho b_s.$$

Then there is $\{c_r\}_{r \in \mathbb{Q}} \subseteq K$ such that

$$r > p > q > s \implies a_r \varrho c_p \varrho c_q \varrho b_s.$$
Katětov relations on $SL$

We are particularly interested in considering Katětov relations on the frame $SL$.

Given a frame $L$, a Katětov relation $\varrho$ in $SL$ is said to be strong, if

$$S \varrho T \implies S^\circ \leq T \text{ and } S \leq \overline{T}.$$ 

Examples

Given $S, T \in SL$ we write

(1) $S \prec T \iff S^\circ \leq f(-, 1)^* \leq f(0, -) \leq \overline{T}$ for some $f \in C(L)$.

$\prec$ is a strong Katětov relation.

(2) $S \sqsubset T \iff S^\circ \leq \overline{T}$.

$\sqsubset$ is a strong Katětov relation iff $L$ is normal.
The insertion result

Theorem

Let $L$ be a frame. Let $f, g \in F(L)$ be two localic real functions on $L$. If there exists a strong Katětov relation $\varrho$ on $SL$ such that

$$ f(r, -) \varrho g(s, -) \text{ whenever } r > s, $$

then there exists an $h \in C(L)$ such that $f \leq h \leq g$.

Proof:

(1) Apply Katětov Lemma with $a_r = f(r, -)$ and $b_r = g(r, -)$ to obtain a countable family $\{S_r\}_{r \in Q} \subset SL$ such that

$$ r > p > q > s \implies f(r, -) \varrho S_p \varrho S_q \varrho g(s, -). $$

(2) $C = \{S_r : r \in Q\}$ is a scale on $SL$ and the real-valued function $h$ generated by $C$ satisfies

$$ f \leq h \leq g \quad \text{and} \quad h \in C(L). \quad \Box $$
Katětov-Tong Theorem

Let \( S, T \in SL \) we write

\[ S \subseteq T \iff S^\circ \leq T. \]

Theorem (Localic Katětov-Tong)

Let \( L \) be a frame. Then the following are equivalent:

1. \( L \) is normal.
2. \( \subseteq \) is a strong Katětov relation
3. If \( f \in \text{USC}(L) \), \( G \in \text{LSC}(L) \), and \( f \leq g \), then there exists \( h \in C(L) \) such that \( f \leq h \leq g \).
**Localic Insertion Theorem**

Let \( S, T \in SL \) we write

\[
S \prec T \iff S^\circ \leq f(-, 1)^* \leq f(0, -) \leq \overline{T} \quad \text{for some } f \in C(L).
\]

**Definition**

Two sublocales \( S \) and \( T \) in \( L \) are said to be **completely separated** if

\[
f(s, -) \leq S \quad \text{and} \quad f(-, t) \leq T \quad \text{for some } f \in C(L).
\]

**Localic Insertion Theorem (Blair-Lane).**

Let \( L \) be a frame and let \( f, g \in F(L) \). TFAE:

1. There exists \( h \in C(L) \) such that \( f \leq h \leq g \).
2. If \( r > s \), then \( f(r, -) \prec g(s, -) \).
3. If \( r > s \), then \( f(-, r) \) and \( g(s, -) \) are completely separated.
4. If \( r > s \), then \( f(r, -)^* \) and \( g(-, s)^* \) are completely separated.
**Extension results: Localic Tietze**

Given a frame $L$, we shall denote

$$F^*(L) = \{ f \in F(L) : 0 \leq f \leq 1 \} = \{ f \in F(L) : f((-0) \lor (1,-)) = 0 \}$$

and

$$C^*(L) = \{ f \in C(L) : 0 \leq f \leq 1 \} = \{ f \in C(L) : f((-0) \lor (1,-)) = 0 \}$$

**Theorem (Localic Tietze)**

Let $L$ be a normal frame, $S$ a closed sublocale in $L$ and $f \in C^*(S)$. Then there exists an extension of $f$ to the whole $L$, i.e. there exists $\tilde{f} \in C^*(L)$ such that $c_{cS} \circ \tilde{f} = f$. 
Extension results: Localic Extension Theorem

Localic Extension Theorem (Mrówka)

Let $L$ be a frame, $S$ a complemented sublocale in $L$ and $f \in C^*(S)$. Then the following are equivalent:

1. There exists an extension of $f$ to the whole $L$, i.e. there exists $\tilde{f} \in C^*(L)$ such that $c_{cS} \circ \tilde{f} = f$.

2. If $r > s$, then $f(r, -)$ and $f(-, s)$ are completely separated in $L$. 

\begin{tikzpicture}
  \node (L) at (0,0) {$\mathcal{L}(\mathbb{R})$};
  \node (S) at (2,0) {$cS$};
  \node (L') at (2,2) {$cL$};
  \node (f) at (1,1) {$f$};
  \node (g) at (1,2) {$\tilde{f}$};

  \draw[->] (L) to (S);
  \draw[->] (L) to (f);
  \draw[->] (S) to (f);
  \draw[->] (g) to (L');
  \draw[->] (g) to (S);
  \draw[->] (S) to (L');
\end{tikzpicture}
Extension results: Localic Extension Theorem (proof)

Proof.

(1) $\implies$ (2) is the easy part.

(2) $\implies$ (1): Let $f_1$ and $g_2$ be generated, respectively, by the scales $\mathcal{C} = \{S_p : p \in \mathbb{Q}\}$ and $\mathcal{D} = \{T_q : q \in \mathbb{Q}\}$ where

$$S_p = \begin{cases} 0 (= L), & \text{if } p \geq 1; \\ f(p, -), & \text{if } 0 \leq p < 1; \\ 1 (= \{1\}), & \text{if } p < 0 \end{cases}$$

$$T_q = \begin{cases} 0 (= L), & \text{if } q \geq 0; \\ f(-, -q), & \text{if } -1 \leq q < 0; \\ 1 (= \{1\}), & \text{if } q < -1. \end{cases}$$

Then $f_1$ and $f_2 = -g_2$ belong to $C^*(L)$ and if $r > s$ then $f_2(r, -)$ and $f_1(-, s)$ are completely separated in $L$.

It follows from the Localic Insertion Theorem that there exists $h \in C(L)$ such that $f_2 \leq h \leq f_1$.

The real valued function $h \in C^*(L)$ is the desired extension of $f$. $\square$