Monotone normality, quasi-metrizable spaces and the role of the $T_1$ axiom

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Why monotone normality?

Separation axioms for metric spaces

Monotone normality without $T_1$ and quasidmetrizable spaces

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Monotone normality without $T_1$

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Regularity

$U$

$\bigcup$

$\overline{G(x, U)}$

$\bigcup$

$G(x, U)$

$\bigcup$

$x$

$(X, d)$

$U$ open

$d(x, X \setminus U)$

$G(x, U) = ?$

Introduction
**Why monotone normality?**

Separation axioms for metric spaces

\[
(X, d)
\]

**Regularity**

\[
U \cup \emptyset = G(x, U) \cup G(y, X \setminus \{x\}) = \emptyset
\]

\[
G(x, U) = B\left(x, \frac{d(x, X \setminus U)}{2}\right)
\]

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\[ (X, d) \]

\( U \) open

\[ d(x, X \setminus U) \]

\[ G(x, U) = B\left(x, \frac{d(x, X \setminus U)}{2}\right) \]

But we have more!
**Why monotone normality?**

Separation axioms for metric spaces

**Monotone Regularity**

\[ G(x, U) = B\left(x, \frac{d(x, X \setminus U)}{2}\right) \]

But we have more! (1) If \( x \in U \subseteq V \) then \( G(x, U) \subseteq G(x, V) \)
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Monotone normality

Regularity

If $x \in U \subseteq V$ then $G(x, U) \subseteq G(x, V)$

(2) If $x \neq y$ then $G(x, X \setminus \{y\}) \cap G(y, X \setminus \{x\}) = \emptyset$

But we have more! (1) If $x \in U \subseteq V$ then $G(x, U) \subseteq G(x, V)$
Metric spaces are normal

Separation axioms for metric spaces

Let $(X, \tau)$ be a metric space. If $U$ is open and $F$ is closed, then

\[ \Delta(x, U) = ? \]

Normality

If $F_h \subseteq F_i$ and $U_h \subseteq U_i$ then

\[ \Delta(F_h, U_h) \leq \Delta(F_i, U_i) \]
Metric spaces are normal

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Monotone normality without $T_1$

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$(X, \tau)$

Normality

$U$

$\Delta(F, U)$

$\Delta(F, U)$

$F$

Here again we have more

If $F_h \subseteq F_i$ and $U_h \subseteq U_i$ then $\Delta(F_h, U_h) \leq \Delta(F_i, U_i)$

$\Delta(x, U) = ?$

$U$ open

$G(x, U)$

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\[ \Delta(x, U) = ? \]

Normality

\[
\begin{align*}
U & \quad \text{open} \\
\Delta(F, U) & \quad \text{closed} \\
F & \quad \text{closed}
\end{align*}
\]

Here again we have

If \( F_h \subseteq F_i \) and \( U_h \subseteq U_i \) then \( \Delta(F_h, U_h) \leq \Delta(F_i, U_i) \)

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Normality

\[
\Delta(F, U) = \bigcup_{x \in F} G(x, U)
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\[(X, \tau)\]

\[U \text{ open}\]

\[F \text{ closed}\]

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Here again we have more!

Normalization

\[U\]

\[\bigcup_{I} \Delta(F, U)\]

\[\bigcup_{I} \Delta(F, U)\]

\[F\]
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Here again we have more!

If $F_1 \subseteq F_2$ and $U_1 \subseteq U_2$ then $\Delta(F_1, U_1) \leq \Delta(F_2, U_2)$
**Monotonization of normality**

Let $X$ be a topological space with topology $\mathcal{O}X$ (and $\mathcal{C}X$ being the family of all closed sets of $X$), let

$$P = \{(F, U) \in \mathcal{C}X \times \mathcal{O}X : F \subseteq U\} \quad \text{and} \quad Q = \mathcal{O}X.$$ 

Both $P$ and $Q$ carry natural orderings. Namely, $\leq_Q$ is the usual inclusion and $P$ is ordered by componentwise inclusion $\leq_P$, i.e.,

$$(F_1, U_1) \leq_P (F_2, U_2) \iff F_1 \subseteq F_2 \text{ and } U_1 \subseteq U_2.$$
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Definition

A space $X$ is **monotonically normal** if there exists a monotone $\Delta : P \rightarrow Q$ such that

1. $\Delta(F, U) \subseteq \overline{F} \subseteq U$ for all $(F, U) \in P$;  
2. if $(F_1, U_1) \leq_P (F_2, U_2)$ then $\Delta(F_1, U_1) \leq_Q \Delta(F_2, U_2)$.  

Monotone normality without $T_1$ 

Monotone normality and quasidmetrizable spaces
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**Definition**

A space $X$ is **monotonically normal** if there exists a monotone $\Delta : P \rightarrow Q$ such that

1. $F \subseteq \Delta(F, U) \subseteq \overline{F} \subseteq U$ for all $(F, U) \in P$;
2. if $(F_1, U_1) \leq_P (F_2, U_2)$ then $\Delta(F_1, U_1) \leq_Q \Delta(F_2, U_2)$.

$\Delta$ is called a **monotone normality operator**.
**Equivalent formulation of monotone normality**

**Theorem (Borges, Heath, Lutzer, Zenor ∼ 1970)**

Let $X$ be a topological space. The following are equivalent:

1. $X$ is monotonically normal.
2. There is an assignment of an open set $G(x, U)$ to each pair $(x, U)$ such that $U$ is an open neighborhood of $x$, in such a way that
   - (i) $x \in G(x, U) \subseteq G(x, U) \subseteq U$;
   - (ii) if $x \in U \subseteq V$, then $G(x, U) \subseteq G(x, V)$.
   - (iii) if $x \neq y$ then $G(x, X \setminus \{y\}) \cap G(y, X \setminus \{x\}) = \emptyset$.
3. There is an assignment of an open set $H(x, U)$ such that
   - (i) $x \in H(x, U) \subseteq U$;
   - (ii) if $H(x, U) \cap H(y, V) = \emptyset$, then either $x \in U$ or $y \in U$. 

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Proof.

(3) $\implies$ (2): $G(x, U) = \bigcup \{H(x, V) | x \in V \subseteq U\}$. 
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Proof.

(3)$\implies$(2): $G(x, U) = \bigcup \{H(x, V) \mid x \in V \subseteq U\}$.
(2)$\implies$(1): $\Delta(F, U) = \bigcup \{G(x, U) \mid x \in F\}$.
Equivalent formulation of monotone normality

Theorem (Borges, Heath, Lutzer, Zenor \( \sim \) 1970)

Let \( X \) be a topological space. The following are equivalent:

1. \( X \) is monotonically normal.
2. There is an assignment of an open set \( G(x, U) \) to each pair \((x, U)\) such that \( U \) is an open neighborhood of \( x \), in such a way that
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Proof.

(3)\( \implies \) (2): \( G(x, U) = \bigcup \{H(x, V) \mid x \in V \subseteq U\} \).
(2)\( \implies \) (1): \( \Delta(F, U) = \bigcup \{G(x, U) \mid x \in F\} \).
(1)\( \implies \) (3): \( H(x, U) = \Delta(\{x\}, U) \cap \Delta(X \setminus U, X \setminus \{x\}) \).
**Equivalent formulation of monotone normality**

**Theorem (Borges, Heath, Lutzer, Zenor ~ 1970)**

Let $X$ be a $T_1$ topological space. The following are equivalent:

1. $X$ is monotonically normal.
2. There is an assignment of an open set $G(x, U)$ to each pair $(x, U)$ such that $U$ is an open neighborhood of $x$, in such a way that
   - (i) $x \in G(x, U) \subseteq G(x, U) \subseteq U$;
   - (ii) if $x \in U \subseteq V$, then $G(x, U) \subseteq G(x, V)$.
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   - (i) $x \in H(x, U) \subseteq U$;
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**Proof.**

(3)$\implies$(2): $G(x, U) = \bigcup \{H(x, V) | x \in V \subseteq U\}$.

(2)$\implies$(1): $\Delta(F, U) = \bigcup \{G(x, U) | x \in F\}$.

(1)$\implies$(3): $H(x, U) = \Delta(\{x\}, U) \cap \Delta(X \setminus U, X \setminus \{x\})$. (If $X$ is $T_1$ !)
Some properties of monotonically normal $T_1$ spaces

- Metrizable spaces are monotonically normal.
Some properties of monotonically normal $T_1$ spaces

- Metrizable spaces are monotonically normal.
- Linearly ordered topological spaces are monotonically normal.
Some properties of monotonically normal $T_1$ spaces

- Metrizable spaces are monotonically normal.
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(The proof depends on the last characterization of monotone normality, hence it is only valid for $T_1$ spaces.)
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- Monotone version of Tietze’s theorem:
  
  Suppose $A$ is a closed subspace of a monotonically normal space $X$. Then there is a function $\Phi_A : C(A, [0, 1]) \rightarrow C(X, [0, 1])$ such that:
  
  1. for each $f \in C(A, [0, 1])$, $\Phi_A(f)$ extends $f$;
  2. if $f, g \in C(A, [0, 1])$ and $f \leq g$ in $A$, then $\Phi_A(f) \leq \Phi_A(g)$ in $X$.

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(The proof depends on the last characterization of monotone normality, hence it is only valid for $T_1$ spaces.)
Why monotone normality without $T_1$ axiom?

(1) Monotone normality (with $T_1$ axiom) is hereditary, while normality is only hereditary for closed subspaces. What about monotone normality without $T_1$ axiom?
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   It is not hereditary!!
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Example
Let $(X, \tau)$ an arbitrary space and $Y = X \cup \{\infty\}$ with $\infty \notin X$.

Define on $Y$ the topology $\tau^* = \tau \cup \{Y\}$.

$X$ is an open, dense subspace of the monotonically normal non $T_1$ compact space $Y$.

If $(X, \tau)$ fails to be monotonically normal, we have the desired counterexample.
Why monotone normality without $T_1$ axiom?

(1) Heritability
Why monotone normality without $T_1$ axiom?

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(2) The Tietze-Urysohn theorem for normal spaces provides a characterization of normal spaces for arbitrary (not necessarily $T_1$) spaces.
Why monotone normality without $T_1$ axiom?

1) Heritability

2) The Tietze-Urysohn theorem for normal spaces provides a characterization of normal spaces for arbitrary (not necessarily $T_1$) spaces.

What about the monotonically normal analogue of the Tietze-Urysohn theorem?
Why monotone normality **without** $T_1$ axiom?

1. Heritability

2. Tietze-Urysohn theorem
Why monotone normality without $T_1$ axiom?

(1) Heritability

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(3) Since metrizable spaces are monotonically normal (and $T_1$) spaces, it is natural to think that quasi-metrizable spaces could also be monotonically normal (but not necessarily $T_1$).
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A first example of a quasi-metrizable (but not metrizable) space is the Sorgenfrey line, and it is indeed monotonically normal.
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A first example of a quasi-metrizable (but not metrizable) space is the Sorgenfrey line, and it is indeed monotonically normal.

However, the Sorgenfrey plane is also quasi-metrizable but not even normal.
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A first example of a quasi-metrizable (but not metrizable) space is the Sorgenfrey line, and it is indeed monotonically normal.

However, the Sorgenfrey plane is also quasi-metrizable but not even normal.

Hence it is natural to try to find which quasi-metrizable spaces are monotonically normal.
Monotone normality without $T_1$

Every topological $X$ induces, in a natural way, a partial order $\leq$ on $X$ (called the specialization order) defined by $y \leq x \iff y \in \{x\}$.

For each $x \in X$ we shall also denote $\downarrow x = \{y \in X : y \leq x\} = \overline{\{x\}}$. 
Monotone normality without $T_1$

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**Theorem**

*Let $X$ be a topological space. The following are equivalent:*

1. $X$ is monotonically normal;
2. There is an assignment of an open set $H(x, U)$ to each pair $(x, U)$ such that $U$ is an open neighborhood of $\downarrow x$, in such a way that
   - (i) $\downarrow x \in H(x, U) \subseteq \overline{H(x, U)} \subseteq U$;
   - (ii) if $x \leq y$ and $U \subseteq V$, then $H(x, U) \subseteq H(y, V)$.
   - (iii) if $\downarrow x \cap \downarrow y = \emptyset$, then $H(x, X \setminus \downarrow y) \cap H(y, X \downarrow x) = \emptyset$. 
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Monotone normality without $T_1$

As a corollary of the previous characterization, and in connection with hereditary monotone normality we have the following:
Monotone normality without $T_1$

Consequences: Heritability

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(1) Monotone normality is a *weakly hereditary* property (any closed subspace of a monotonically normal space is monotonically normal), but not hereditary.
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As a corollary of the previous characterization, and in connection with hereditary monotone normality we have the following:

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(1) Monotone normality is a weakly hereditary property (any closed subspace of a monotonically normal space is monotonically normal), but not hereditary.

(2) Monotone normality is hereditary under the assumption of the $T_1$ axiom.
Introduction Monotone normality without $T_1$

Monotone normality without $T_1$  Consequences: Heritability

As a corollary of the previous characterization, and in connection with hereditary monotone normality we have the following:

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(1) Monotone normality is a weakly hereditary property (any closed subspace of a monotonically normal space is monotonically normal), but not hereditary.

(2) Monotone normality is hereditary under the assumption of the $T_1$ axiom.

(3) A space $X$ is hereditarily monotonically normal if and only if every open subspace of $X$ is monotonically normal.
Monotone normality without $T_1$

As a second corollary of the characterization, we can conclude that the monotone version of the Tietze’s result is still valid for monotone normality in the $T_1$-free context.

Theorem

Let $X$ be a monotonically normal space. Then for each closed $A \subseteq X$ there exists a function $\Phi_A: C(A, [0, 1]) \to C(X, [0, 1])$ such that:

1. for each $f \in C(A, [0, 1])$, $\Phi_A(f)$ extends $f$;
2. if $f, g \in C(A, [0, 1])$ and $f \leq g$ in $A$, then $\Phi_A(f) \leq \Phi_A(g)$ in $X$. 
Monotone normality without $T_1$ Consequences: Tietze-type theorem

Even more, the following characterization proved in: I.S. Stares, Monotone normality and extension of functions, (1995) remain valid in the $T_1$-free context.

Theorem
A space $X$ is monotonically normal iff for each closed $A \subseteq X$ there exists a function $\Phi_A : C(A, [0, 1]) \to C(X, [0, 1])$ such that:

1. for each $f \in C(A, [0, 1])$, $\Phi_A(f)$ extends $f$;
2. if $f, g \in C(A, [0, 1])$ and $f \leq g$ in $A$, then $\Phi_A(f) \leq \Phi_A(g)$ in $X$.
3. If $A_1 \subseteq A_2$ are closed and $f_i \in C(A_i, [0, 1])$ are such that $f_2|_{A_1} \geq f_1$ and $f_2(x) = 1$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \geq \Phi_{A_1}(f_1)$.
4. If $A_1 \subseteq A_2$ are closed and $f_i \in C(A_i, [0, 1])$ are such that $f_2|_{A_1} \leq f_1$ and $f_2(x) = 0$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \leq \Phi_{A_1}(f_1)$. 
Let $X$ be a non-empty set. A map $d : X \times X \to [0, +\infty)$ is a quasi-metric if the following two conditions hold for all $x, y, z \in X$:

(QM1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

(QM2) $d(x, y) \leq d(x, z) + d(z, y)$. 

Quasi-metrizable spaces

Monotone normality without $T_1$

Monotone normality and quasidmetrizable spaces

Monotone normality, quasi-metrizable spaces and the role of the $T_1$ axiom
Quasi-metrizable spaces

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Every quasi-metric $d$ on $X$ generates a $T_0$ topology $\tau_d$ which has as a base the family of $d$-balls.

A topological space $(X, \tau)$ is said to be \textit{quasi-metrizable} if there exists a quasi-metric $d$ on $X$ such that $\tau = \tau_d$. 
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A quasi-metric space $(X, d)$ is $T_1$ iff the following is satisfied:

$$d(x, y) = 0 \quad \Rightarrow \quad x = y \quad (T_1)$$
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The specialization order $\leq_d$ on $X$ is given by

$$y \leq_d x \iff d(y, x) = 0 \iff y \in \overline{\{x\}}.$$
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It is well known that not all quasi-metrizable spaces are normal, a typical example being the Sorgenfrey plane.
Introduction

Monotone normality without $T_1$ normality and quasidmetrizable spaces

Quasi-metrizable spaces

As we have already mentioned, metrizable spaces are monotonically normal and, of course, satisfy the $T_1$-axiom.

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It is natural to think then about the question of which quasi-metrizable spaces are normal, or perhaps monotonically normal.
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It is well known that not all quasi-metrizable spaces are normal, a typical example being the Sorgenfrey plane.

It is natural to think then about the question of which quasi-metrizable spaces are normal, or perhaps monotonically normal. In this sense it could be mentioned, citing from: P.M. Gartside, **Cardinal invariants of monotonically normal spaces**, (1997)

> “Whenever a space can be explicitly and constructively shown to be normal, then it is probably monotonically normal.”
If the quasi-metric space is $T_1$ we have the following characterization:
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**Theorem**

Let $(X, d)$ be a $T_1$ quasi-metric space. The following are equivalent:

1. $(X, \tau_d)$ is monotonically normal;
2. There exists a map $h: X \times (0, +\infty) \to (0, +\infty)$ such that:
   
   (h1) $0 < h(x, \varepsilon) \leq \varepsilon$;
   
   (h2) if $\varepsilon_1 < \varepsilon_2$, then $h(x, \varepsilon_1) \leq h(x, \varepsilon_2)$;
   
   (h3) if $x \neq y$, then $B_d(x, h(x, d(x, y))) \cap B_d(y, h(y, d(y, x))) = \emptyset$.
Quasi-metrizable spaces

Characterization for $T_1$ spaces

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**Corollary**

Let $(X, d)$ be a $T_1$ quasi-metric space satisfying:

$$x \neq y \implies B_d(x, k \cdot d(x, y)) \cap B_d(y, k \cdot d(y, x)) = \emptyset \quad (\ast)$$

for some $k \in (0, 1]$. Then $(X, \tau_d)$ is monotonically normal.
**Quasi-metrizable spaces**

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**Examples**

- If \(d\) is a metric, then condition \((\ast)\) is satisfied with \(k = \frac{1}{2}\).
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- If $d$ is a metric, then condition $(\star)$ is satisfied with $k = \frac{1}{2}$.

- If $d$ is a the Sorgenfrey quasi-metric on $\mathbb{R}$
  
  
  $$d(x, y) = \min\{y - x, 1\} \text{ if } x \leq y \text{ and } d^*(x, y) = 1 \text{ otherwise},$$

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- If $d$ is a metric, then condition $(*)$ is satisfied with $k = \frac{1}{2}$.

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- The Michael line.
**Quasi-metrizable spaces**

**Examples (T₁)**

**Corollary**

Let \((X, d)\) be a \(T₁\) quasi-metric space satisfying:

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x \neq y \implies B_d(x, k \cdot d(x, y)) \cap B_d(y, k \cdot d(y, x)) = \emptyset \quad (\star)
\]

for some \(k \in (0, 1]\). Then \((X, τ_d)\) is monotonically normal.

**Examples**

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- If \(d\) is a the Sorgenfrey quasi-metric on \(\mathbb{R}\)
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- ...
Finally, we can also provide a sufficient condition for a quasi-metric space to be monotonically normal:

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\text{Let } (X, d) \text{ be a quasi-metric space satisfying: }
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B_d(x', d(x', y')) \cap B_d(y', d(y', x')) = \emptyset \quad \forall x' \leq x, y' \leq y.
\]

Then \((X, \tau_d)\) is monotonically normal.

Note that if \(d\) is indeed a metric, the condition above is obviously satisfied. In fact, this is precisely the Hausdorff condition. In this case, the previous proposition is once again nothing but the well-known fact that metrizable spaces are monotonically normal.
Finally, we can also provide a sufficient condition for a quasi-metric space to be monotonically normal:

**Theorem**

*Let \((X, d)\) be a quasi-metric space satisfying:*

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B_d\left(x', \frac{d(x', y)}{2}\right) \cap B_d\left(y', \frac{d(y', x)}{2}\right) = \emptyset \quad \forall x' \leq x, y' \leq y. \quad (\star)
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Finally, we can also provide a sufficient condition for a quasi-metric space to be monotonically normal:

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Quasi-metrizable spaces

Examples (non $T_1$)

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- The reals with the right-order topology (Kolmogorov line).
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- The complexity (quasi-metric) space $(C, d_C)$.
- ...