Abstract. It is shown that any elliptic or parabolic operator in nondivergence form with measurable coefficients has a global fundamental solution verifying certain pointwise bounds.

1. Introduction

Given operators in divergence form

$$Eu = \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju) \quad \text{and} \quad Pu = \sum_{i,j=1}^{n} D_i(a_{ij}(X)D_ju) - D_tu$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $X = (x,t)$, and under the assumptions that the coefficient matrices of both operators are measurable functions verifying for some positive constant $\nu \in (0,1]$

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2, \quad \nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(X)\xi_i\xi_j \leq \nu^{-1}|\xi|^2$$

for all $x, \xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$, it is well known that both operators have a global Green’s function or fundamental solution defined in the whole space. In particular, for both the elliptic operator $E$ and the parabolic operator $P$
and all \( x \in \mathbb{R}^n \) and \( X \in \mathbb{R}^{n+1} \), there exist functions \( g(x,y) \) and \( G(X,Y) \), \( Y = (y, s) \), verifying

\[
\varphi(x) = -\int_{\mathbb{R}^n} g(x,y) E\varphi(y) \, dy
\]

(1.2)

\[
\psi(X) = -\int_{-\infty}^{t} \int_{\mathbb{R}^n} G(X,Y) P\psi(Y) \, dY
\]

(1.3)

for all \( \varphi \in C_0^\infty(\mathbb{R}^n) \) and \( \psi \in C_0^\infty(\mathbb{R}^{n+1}) \).

The basic facts about the fundamental solution for selfadjoint elliptic operators in divergence form were proved by W. Littman, G. Stampachia and H. Weinberger in [32], where they showed that when the dimension \( n \geq 3 \), there is a constant \( N = N(n, \nu) \) verifying

\[
N^{-1}|x-y|^{2-n} \leq g(x,y) \leq N|x-y|^{2-n}
\]

(1.4)

for all \( x, y \in \mathbb{R}^n \). In the two-dimensional setting, the existence and bounds of the global fundamental solution were obtained by C.E. Kenig and Weiming Ni in [27], where it is shown that the fundamental solution necessarily changes sign and for some constant \( N \) as above

\[
|g(x,y)| \leq N (1 + |\log |x-y||)
\]

In [11], S. Chanillo and Li Yanyan derived that \( g(x,y) \) is a function of bounded mean oscillation in the plane whose BMO norm can be estimated in terms of \( \nu \). Finally, the study of the fundamental solution for nonselfadjoint elliptic operators in divergence form was carried out by M. Gr"uter and K.O. Widman [26], where it is shown that (1.4) also holds in this setting.

The first author establishing Gaussian bounds for the fundamental solution \( G(X,Y) \) of a parabolic operator in divergence form with measurable coefficients was D.G. Aronson [2]. He proved that for some constant \( N \) as above,

\[
N^{-1}(t-s)^{-\frac{n}{2}} e^{-\frac{N|x-y|^2}{t-s}} \leq G(X,Y) \leq N(t-s)^{-\frac{n}{2}} e^{-\frac{N|x-y|^2}{t-s}}
\]

(1.5)

when \( X, Y \in \mathbb{R}^{n+1} \) and \( t > s \).

The proofs of these results relied on the fundamental works of Nash, Moser and De Giorgi on the H"older continuity and Harnack’s inequality for solutions of elliptic and parabolic equations in divergence form [12],[33],[34],[37], and some of them, for instance, Aronson’s proof of the Gaussian lower bound, depend strongly on Harnack’s inequality. On the other hand, it is known
that the bounds (1.4) and (1.5) for the fundamental solution imply the corresponding Harnack’s inequality for nonnegative solutions, and thus it was interesting to obtain these estimates independently of Harnack’s inequality. In this direction, we should mention the works [21], [22] by E.B. Fabes and D.W. Stroock.

The purpose of this work is to extend the above results to the setting of elliptic and parabolic equations in nondivergence form with measurable coefficients, and to obtain the corresponding “analog” of the pointwise bounds. The arguments we use rely on a Harnack inequality, but a Harnack inequality for normalized adjoint solutions (See the definition below). We believe that the concept of normalized adjoint solution in the elliptic case was first introduced by P. Bauman in [5], where the corresponding Harnack inequality (Theorem 2.3) was proved. Later, these estimates were of great help in works like [9], [14], [17] and [13]. The normalized adjoint solutions in the parabolic case were first used in [16], here E.B. Fabes, N. Garofalo and S. Salsa proved a Harnack inequality for normalized adjoint solutions but only applying to normalized adjoint solutions vanishing on the lateral boundary of a cylinder. In this work we prove a general Harnack inequality for them and use it to obtain estimates for the fundamental solution.

In the elliptic setting we consider an operator

\[ Eu = \sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u \]

whose coefficient matrix is symmetric, defined in \( \mathbb{R}^n \), and satisfying (1.1) for some \( \nu \in (0,1] \). Also, in what follows \( N \) will denote a constant depending on \( n \geq 2 \) and \( \nu, B_r \) a ball of radius \( r \) centered at the origin and \( B_r(x) \) a ball of radius \( r \) centered at \( x \in \mathbb{R}^n \).

The following definitions are given for operators \( E \) with measurable coefficients, but the functions involved should be regarded as smooth (except for the natural singularities) when the coefficients of \( E \) are smooth.

**Definitions.** A function \( g(x,.) \in L^1_{loc}(\mathbb{R}^n) \) is called a fundamental solution for \( E \), with pole at \( x \in \mathbb{R}^n \) if (1.2) holds for all \( \varphi \in C^\infty_0(\mathbb{R}^n) \). A function \( w \in L^1_{loc}(\Omega) \) is called an adjoint solution for \( E \) in an open set \( \Omega \), if for every \( \varphi \in C^\infty_0(\Omega) \)

\[ \int_{\Omega} w(y)E\varphi(y) \, dy = 0 \, . \]

Let \( W \) be a fixed nonnegative and nontrivial adjoint solution for \( E \), a function \( \tilde{w} \) defined in \( \Omega \) is a normalized adjoint solution for \( E \) relative to \( W \) in \( \Omega \), if \( \tilde{w}W \in L^1_{loc}(\Omega) \) is an adjoint solution for \( E \) in \( \Omega \). A nonnegative function \( w \) in \( \Omega \) is a Muckenhoupt weight in the reverse Hölder class \( \mathcal{B}_{\frac{1}{1-r}}(\Omega) \) if there is
a constant \( N \) such that

\[
(1.7) \quad \left( \int_{B_r(x)} w^{\frac{n}{n-1}} \, dy \right)^{\frac{n-1}{n}} \leq N \int_{B_r(x)} w \, dy
\]

for all balls with \( B_{2r}(x) \subset \Omega \). When \( w \) is defined in \( \mathbb{R}^n \) and (1.7) holds for all the balls \( B_r(x) \) we say \( w \in \mathcal{B}_w^{\frac{n}{n-1}} \).

In the statements of theorems 1.1, 1.2 and 1.3 and until we say the contrary we assume that the coefficients of the operators involved are smooth in the whole space except possibly at infinity.

**Theorem 1.1.** There exists a unique nonnegative adjoint solution \( W \) defined in \( \mathbb{R}^n \) such that

\[
(1.8) \quad W(B_1) = \int_{B_1} W \, dy = |B_1|.
\]

Moreover, \( W \in \mathcal{B}_w^{\frac{n}{n-1}} \) with a constant \( N = N(n, \nu) \) and \( E \) has a global fundamental solution \( g \) with pole at the origin verifying

\[
|g(y)| \leq N \left( 1 + \int_1^{|y|} \frac{s}{W(B_s)} \, ds \right) W(y).
\]

Moreover,

i. \( E \) has a nonnegative fundamental solution if and only if \( \int_1^\infty \frac{s}{W(B_s)} \, ds < +\infty \), and in this case there is a fundamental solution \( g \) such that for all \( y \in \mathbb{R}^n \)

\[
N^{-1} \int_{|y|}^\infty \frac{s}{W(B_s)} \, ds W(y) \leq g(y) \leq N \int_{|y|}^\infty \frac{s}{W(B_s)} \, ds W(y).
\]

ii. \( E \) has a nonpositive fundamental solution if and only if \( \int_0^1 \frac{s}{W(B_s)} \, ds < +\infty \), and in this case there is a fundamental solution \( g \) such that for all \( y \in \mathbb{R}^n \)

\[
N^{-1} \int_0^{|y|} \frac{s}{W(B_s)} \, ds W(y) \leq -g(y) \leq N \int_0^{|y|} \frac{s}{W(B_s)} \, ds W(y).
\]

Finally, if \( \int_0^1 \frac{s}{W(B_s)} \, ds \) and \( \int_1^\infty \frac{s}{W(B_s)} \, ds \) are both infinite, any fundamental solution necessarily changes sign.

Before dealing with all the generality in the parabolic case we explain what happens when the coefficients are time independent. Again, we are assuming that the coefficients are smooth.
Theorem 1.2. Let $P$ denote the parabolic operator $Pu = Eu - Dt u$. Then, $P$ has a fundamental solution $G(X,Y) = \Gamma(x,t-s,y)$ where for all $x,y \in \mathbb{R}^n$ and $t > 0$

$$N^{-1} \max \left\{ \frac{1}{W(B_{\sqrt{t}}(x))}, \frac{1}{W(B_{\sqrt{t}}(y))} \right\} e^{-\frac{N|x-y|^2}{t}} W(y) \leq \Gamma(x,t,y)$$

$$\Gamma(x,t,y) \leq N \min \left\{ \frac{1}{W(B_{\sqrt{t}}(x))}, \frac{1}{W(B_{\sqrt{t}}(y))} \right\} e^{-\frac{N|x-y|^2}{t}} W(y).$$

Let $P$ denote a parabolic operator

(1.9) $Pu = \sum_{i,j=1}^{n} a_{ij}(X)D_{ij} u - Dt u$

with symmetric coefficients matrix satisfying (1.1). For $Z = (z,\tau) \in \mathbb{R}^{n+1}$ define $Q_r(Z) = B_r(z) \times (\tau - r^2, \tau + r^2)$, and for function $W$ defined on $\mathbb{R}^{n+1}$, $F \subset \mathbb{R}^{n+1}$, $E \subset \mathbb{R}^n$ set

$W(F) = \int_F W \, dY$ , $W(E,\tau) = \int_E W(y,\tau) \, dy$

Definitions. A nonnegative function $G(X,.) \in L_{loc}^1(\mathbb{R}^{n+1})$ is called a fundamental solution for $P$ with pole at $X$ if for all $\alpha < \beta$

$$\int_\alpha^\beta \int_{\mathbb{R}^n} G(X,Y) \, dY < +\infty$$

and (1.3) holds for all $\psi \in C_0^\infty(\mathbb{R}^{n+1})$. A function $w \in L_{loc}^1(\Omega)$ is called and adjoint solution for $P$ in an open set $\Omega \subset \mathbb{R}^{n+1}$ if for all $\psi \in C_0^\infty(\Omega)$

$$\int_\Omega w(Y)P\psi(Y) \, dY = 0.$$

A nonnegative function $w$ is called a parabolic Muckenhoupt weight in the reverse Hölder class $B_{n+1}^{n+1}(\Omega)$ if for all $Q_{2r}(Z) \subset \Omega$

(1.10) $\left( \int_{Q_r(Z)} w^{n+1} \, dY \right)^{\frac{1}{n+1}} \leq N \int_{Q_r(Z)} w \, dY,$

and when $w$ is defined in $\mathbb{R}^{n+1}$ and (1.10) holds for all $Q_r(Z)$, we say $w \in B_{n+1}^{n+1}$. 
Theorem 1.3. Let $P$ be as above, then there exists a unique nonnegative adjoint solution $W$ defined on $\mathbb{R}^{n+1}$ verifying $W(B_1 \times (0,1)) = |B_1|$ and
\[(1.11)\] 
$W(B_{2r}(z), \tau + \theta r^2) \leq N W(B_r(z), \tau)$, $W(B_{2r}(z), \tau) \leq N W(B_r(z), \tau + \theta r^2)$
for all $Z \in \mathbb{R}^{n+1}$, $r > 0$ and $0 \leq \theta \leq 1$. Moreover, $W \in B_{\mathbb{R}^{n+1}}$ with constant $N = N(n, \nu)$ and $P$ has a global fundamental solution $G$ verifying
\[N^{-1} \max \{ \frac{1}{W(B_{\sqrt{t}}(x), s)} , \frac{1}{W(B_{\sqrt{t}}(y), s)} \} \ e^{-\frac{|x-y|^2}{N t}} \leq G(X, Y) \leq N \min \{ \frac{1}{W(B_{\sqrt{t}}(x), s)} , \frac{1}{W(B_{\sqrt{t}}(y), s)} \} \ e^{-\frac{|x-y|^2}{N t}} \]
for all $X, Y \in \mathbb{R}^{n+1}$ with $t > s$. Also, $G(X, \cdot) \in B_{\mathbb{R}^{n+1}}(\Omega(X, K, \epsilon))$ for all $K > 0$ and $0 < \epsilon < 1$ with constant $N = N(n, \nu, K, \epsilon)$, where
\[\Omega(X, K, \epsilon) = \{ Y \in \mathbb{R}^{n+1} : |x-y| \leq K \sqrt{t} - s \, , \, s \leq t \} . \]

When the operators $E$ and $P$ have measurable coefficients, the usual compactness arguments and the above estimates show that the claims in theorems 1.1 and 1.2 still hold, though (1.11) should be understood as if the associated nonnegative adjoint solution $W$ for $P$ had a restriction \[18\] to each time $\tau \in \mathbb{R}$ as a measure $W(dy, \tau)$ verifying (1.11) and
\[(1.12) \quad \int_\tau^{+\infty} \int_{\mathbb{R}^{n}} W(Y) P\psi(Y) \, dY = \int_{\mathbb{R}^{n}} \psi(y, \tau) W(dy, \tau) \]
for all $\psi \in C^\infty_0 (\mathbb{R}^{n+1})$.

Recall that an equivalent formulation of what has been called the “weak uniqueness property” for elliptic equations is the following: The weak uniqueness property holds for the operator $E$ on a smooth domain $\Omega \subset \mathbb{R}^n$ when for all $f \in L^\infty(\Omega)$, there is a unique generalized or viscosity solution (in the sense of [9], [10], [29], [36] and [39]) for the problem $Eu = -f$ in $\Omega$, $u = 0$ on $\partial \Omega$.

A simple corollary of the above results and along the lines of the results in [29] is the following.

**Theorem 1.4.** The weak uniqueness property holds for $E$ on all bounded smooth domains if and only if there is a unique nonnegative adjoint solution $W$ for $E$ on $\mathbb{R}^n$ verifying (1.8).

On the other hand N.S. Nadirashvili [36], has constructed an elliptic operator $E$ for which the weak uniqueness fails in the unit ball, and as a consequence this operator or any of its possible extensions to $\mathbb{R}^n$ has at least two different nonnegative and global adjoint solutions verifying (1.8).

Finally, if $W$ is one of the global adjoint solutions associated to an operator $E$ the following holds
Theorem 1.5. If the coefficients of $E$ are continuous at infinity, then for all $\epsilon > 0$ there is $N_\epsilon$ such that for $r \geq 1$

$$N_\epsilon^{-1} r^{n-\epsilon} \leq W(B_r) \leq N_\epsilon r^{n+\epsilon}.$$ 

If the coefficients are continuous at zero, then for all $\epsilon > 0$ there is $N_\epsilon$ such that for $r \leq 1$

$$N_\epsilon^{-1} r^{n+\epsilon} \leq W(B_r) \leq N_\epsilon r^{n-\epsilon}.$$ 

Therefore, if $n \geq 3$ and the coefficients are continuous at infinity there is a nonnegative fundamental solution, and can not exist a nonpositive fundamental solution if the coefficients are continuous at zero. On the other hand, when $n = 2$ and even if the coefficients are both continuous at zero and at infinity the information in the above theorem does not imply that a fundamental solution must change sign, and in fact we show examples where any of the possible cases occur. When $n \geq 3$ it is possible to give examples of operators were again all possibilities take place, but of course these operators fail to have nice coefficients at some of the points.

In section 2 we prove theorems 1.1, 1.4, 1.5 and give examples of elliptic operators showing the different behaviors of the fundamental solution. In section 3 we prove the results related to parabolic operators.

2. The elliptic case

Recall that a Green’s function for $E$ on a smooth domain $\Omega$ with pole at $x \in \Omega$, is a nonnegative function $g_{\Omega}(x,.) \in L^1(\Omega)$ verifying

$$\varphi(x) = - \int_{\Omega} g_{\Omega}(x,y) E \varphi(y) \, dy$$

for all $\varphi \in C^2(\overline{\Omega})$ with $\varphi = 0$ on $\partial \Omega$.

Then, under the assumption that the coefficients of $E$ are smooth on $\mathbb{R}^n$ except possibly at infinity the following results are well known.

Theorem 2.1. For any bounded smooth domain $\Omega$ and all $x \in \Omega$ there is a unique Green’s function for $E$ and $\Omega$ with pole at $x$. Moreover, $g_{\Omega}(x,.) \in B_{\frac{n}{n-1}}(\Omega)$ with a constant $N$ and

$$\left( \int_{\Omega} g_{\Omega}(x,y) \frac{n}{n-1} \, dy \right)^{\frac{n-1}{n}} \leq N d(\Omega)$$

where $d(\Omega)$ denotes the diameter of $\Omega$. 
Theorem 2.2. (Muckenhoupt properties of adjoint solutions) Let \( w \) be a nonnegative adjoint solution for \( E \) in \( \Omega \). Then, \( w \in B^{n-1}_{\infty}(\Omega) \) with a constant \( N = N(n, \nu) \).

Theorem 2.3. (Harnack Inequality for normalized adjoint solutions) Let \( W \) be a nonnegative and nonzero adjoint solution for \( E \) on \( \Omega \) and \( \tilde{w} \) a nonnegative normalized adjoint solution for \( E \) relative to \( W \) on \( B_r(x) \subset \Omega \). Then,

\[
\sup_{B_r(x)} \tilde{w} \leq N \inf_{B_r(x)} \tilde{w}.
\]

Theorem 2.4. (Strong maximum principle for normalized adjoint solutions) Let \( W \) be as above and \( \tilde{w} \) a normalized adjoint solution for \( E \), relative to \( W \) in \( \Omega \). Then, \( \tilde{w} \) can not attain a maximum or minimum in the interior of \( \Omega \) unless it is constant.

Theorem 2.5. (Hölder continuity of normalized adjoint solutions) Let \( W \) be as above and \( \tilde{w} \) be a normalized adjoint solution in a ball \( B_r(x) \subset \Omega \). Then, there exists \( \alpha = \alpha(n, \nu) \in (0, 1] \) such that for all \( y \in B_r(x) \)

\[
|\tilde{w}(y) - \tilde{w}(x)| \leq N \left( \frac{|y - x|}{r} \right)^\alpha \sup_{B_r(x)} |\tilde{w}|
\]

Theorem 2.6. (Bounds for normalized adjoint solutions) Let \( \tilde{w} \) be a normalized adjoint solution relative to \( W \) in a ball \( B_r(x) \subset \Omega \). Then

\[
\sup_{B_{r/2}(x)} |\tilde{w}| \leq N \int_{B_r(x)} |\tilde{w}| Wdy
\]

where the average is taken with respect to \( Wdx \).

The first claim in theorem 2.1 follows from the fundamental works by A.D. Aleksandrov [1] and C. Pucci [38], the second claim and theorem 2.2 were proved by E.B. Fabes and D.W. Stroock [22] and theorem 2.3 by P. Bauman [5, Theorem 4.4]. Theorems 2.4 and 2.5 are standard consequences of Harnack’s inequality and the fact that constants are normalized adjoint solutions. A proof of theorem 2.6 can be found in [13, Theorem 2.3].

Remark 2.1. Before we proceed we make the following remarks which will help the reader to understand in what sense are true the results in theorem 1.1 for operators with measurable coefficients:

i. The weak uniqueness property holds for \( E \) in \( \Omega \) if and only if there is a unique Green’s function for \( E \) in \( \Omega [29] \).

ii. If \( w \in L^1_{loc}(\Omega) \) is a weak nonnegative adjoint solution for \( E \) in \( \Omega \), defining \( w^\varepsilon = w * \theta_\varepsilon + \varepsilon \), where \( \theta \) is a smooth regularization of the identity
supported in the unit ball, one easily verifies that \( w^\varepsilon \) is a nonnegative adjoint solution in \( \Omega^\varepsilon = \{ x \in \Omega : d(x, \partial \Omega) > \varepsilon \} \), for the operator \( E^\varepsilon \) with coefficients given by \( a^\varepsilon_{ij} = \left( (a_{ij} w)^\ast \theta + \varepsilon \delta_{ij} \right)/w^\varepsilon \). Then, letting \( \varepsilon \to 0 \) it follows from theorem 2.2 that \( w \in B_{n-1}(\Omega) \) with constant \( N = N(n, \nu) \).

iii. Reviewing the proof of theorem 2.3, if \( W \) and \( \tilde{w} \) are as in theorem 2.3 with \( B_{5r} \subset \Omega \), the integration by parts argument used in [5, Theorem 4.4] gives

\[
N^{-1} g_1(2re_n, y) \int_{B_{4r}} W^\varepsilon \, dx \leq W^\varepsilon(y) \leq N g_1(2re_n, y) \int_{B_{4r}} W^\varepsilon \, dx
\]

\[
N^{-1} g_2(2re_n, y) \int_{B_{4r}} (\tilde{w}W)^\varepsilon \, dx \leq (\tilde{w}W)^\varepsilon(y) \leq N g_2(2re_n, y) \int_{B_{4r}} (\tilde{w}W)^\varepsilon \, dx
\]

for all \( y \in B_{r} \), and where \( g_1, g_2 \) are respectively the Green’s functions for the smooth operators associated as before to \( W \) and \( \tilde{w}W \) in \( B_{5r} \). Passing to the limit as \( \varepsilon \to 0 \) gives

\[
N^{-1} g_1(2re_n, y) \int_{B_{4r}} W \, dx \leq W(y) \leq N g_1(2re_n, y) \int_{B_{4r}} W \, dx
\]

\[
N^{-1} g_2(2re_n, y) \int_{B_{4r}} \tilde{w}W \, dx \leq \tilde{w}W(y) \leq N g_2(2re_n, y) \int_{B_{4r}} \tilde{w}W \, dx
\]

for almost every \( y \in B_{r} \), and where both \( g_1, g_2 \) are Green’s functions for \( E \) in \( B_{4r} \) with pole at \( 2re_n \). Hence, if the weak uniqueness holds for \( E \) on subsets of \( \Omega \), \( g_1 = g_2 \), and the above inequalities imply that theorem 2.3 still remain true for weak normalized adjoint solutions. Of course, the same happens with theorem 2.4.

Proof of theorem 1.1. Letting \( g_R \) denote the Green’s function for \( E \) on \( B_R \) with pole at the origin, define

\[
W_R(y) = \frac{g_2\alpha(y) - g_R(y)}{\alpha(R)} \quad \text{where} \quad \alpha(R) = \int_{B_1} (g_{2R} - g_R) \, dy
\]

The maximum principle and theorem 2.2 imply that \( W_R \) is a nonnegative adjoint solution in \( B_R \), \( W_R \in B_{n-1}(B_R) \) with constants depending on \( n \) and \( \nu \). Also, \( W_R(B_1) = |B_1| \), and the Muckenhoupt property of \( W_R \) [35] implies there exists \( \theta = \theta(n, \nu) > 0 \), possibly large, such that for \( 0 < s \leq t \leq R/2 \)

\[
\left( \frac{|B_s|}{|B_t|} \right)^\theta \leq N \frac{W_R(B_s)}{W_R(B_t)}
\]


and in particular $W_R(B_r) \leq N|B_r|^\theta$ for $1 \leq r \leq R/2$. The last bound and compactness imply the existence of a sequence $R_k$ tending to infinity such that $W_{R_k}$ converges weakly in $L^\infty_{loc}(\mathbb{R}^n)$ to a global nonnegative adjoint solution $W$ defined in $\mathbb{R}^n$ verifying $W \in B_{n-1}$ with constants depending on $n$ and $\nu$, $W(B_1) = |B_1|$.

If $W_1$, $W_2$ are two global nonnegative adjoint solutions verifying (1.8), $W_1/(W_1 + W_2)$ is a bounded normalized adjoint solution relative to $W_1 + W_2$ in $\mathbb{R}^n$, and the identity $W_1 = W_2$ follows from the Liouville-type theorem implied by theorem 2.5 and the normalization condition (1.8).

Recall that the normalized Green’s function for a bounded domain $\Omega$ is defined as $\tilde{g}\Omega(x,y) = g\Omega(x,y)/W(y)$, $\tilde{g}\Omega \in C(\Omega \times \Omega \setminus \{(x,y): x = y\})$, $\tilde{g}\Omega(x,y) = 0$ for $y \in \partial\Omega$ and $x \in \Omega$. The normalized Green’s function for $B_R$ with pole at the origin will be denoted $\tilde{g}_R$. The following is proved by P. Bauman in [7, Theorem 2.3]

**Theorem 2.7.** There is a constant $N = N(n,\nu)$ such that for $x,y \in B_{R/2}(z)$ with $|x-y| \leq R/2$

$$N^{-1} \int_{|x-y|}^R \frac{s}{W(B_s(x))} \, ds \leq \tilde{g}_{B_{R}(z)}(x,y) \leq N \int_{|x-y|}^R \frac{s}{W(B_s(x))} \, ds .$$

The second claim in theorem 1.1 follows by compactness and the following estimate.

**Lemma 2.1.** There is a constant $N = N(n,\nu)$ such that for $|y| \leq R/8$

$$|g_R(y) - \left( \int_{B_y} g_R \, dx \right) W(y) | \leq N \left( 1 + \left| \int_1^{|y|} \frac{s}{W(B_s)} \, ds \right| \right) W(y) .$$

*Proof Lemma 2.1.* Recall that the Muckenhoupt property of $W$ [35] implies there is $N$ such that

$$W(B_{2r}(x)) \leq N W(B_r(x)) \quad \text{for all} \quad x \in \mathbb{R}^n, \quad r > 0 \quad (2.1)$$

The maximum principle (theorem 2.4) implies $\tilde{g}_R - \tilde{g}_r \leq \max_{\partial B_r} \tilde{g}_R$ on $B_r$ for $r < R$, and in particular

$$0 \leq \max_{\partial B_r} \tilde{g}_R - (\tilde{g}_R - \tilde{g}_{r/2}) + (\tilde{g}_r - \tilde{g}_{r/2}) \quad \text{on} \quad B_{r/2} .$$

From (2.1) and theorem 2.7, $\tilde{g}_r - \tilde{g}_{r/2} \leq N \frac{r^2}{W(B_r)}$ on $B_{r/2}$. Hence, on this ball

$$\tilde{g}_R - \tilde{g}_{r/2} \leq N \frac{r^2}{W(B_r)} + \max_{\partial B_r} \tilde{g}_R ,$$
and taking the maximum of the function in the left hand side of the last inequality over \( B_{r/2} \)

\[
(2.2) \quad \max_{\partial B_{r/2}} \tilde{g}_R - \max_{\partial B_r} \tilde{g}_R \leq N \frac{r^2}{W(B_r)} \quad \text{for} \quad r < R.
\]

Again, by the maximum principle, \( \tilde{g}_R \leq \max_{\partial B_{r/2}} \tilde{g}_R \) on \( B_R \setminus B_{r/2} \), and from (2.2)

\[
N \frac{r^2}{W(B_r)} + \max_{\partial B_r} \tilde{g}_R - \tilde{g}_R \geq 0 \quad \text{on} \quad B_R \setminus B_{r/2}.
\]

Applying the Harnack inequality to the above nonnegative normalized adjoint solution in \( B_{2r} \setminus B_{r/2} \) gives

\[
\sup_{\partial B_r} \left( N \frac{r^2}{W(B_r)} + \max_{\partial B_r} \tilde{g}_R - \tilde{g}_R \right) \leq N \inf_{\partial B_r} \left( N \frac{r^2}{W(B_r)} + \max_{\partial B_r} \tilde{g}_R - \tilde{g}_R \right)
\]

when \( r \leq R/2 \). Thus,

\[
(2.3) \quad \max_{\partial B_r} \tilde{g}_R - \min_{\partial B_r} \tilde{g}_R \leq N \frac{r^2}{W(B_r)} \quad \text{for} \quad r \leq R/2.
\]

Taking averages with respect to the measure \( Wdy \) we have

\[
\int_{B_r} |\tilde{g}_R - \min_{\partial B_r} \tilde{g}_R| Wdy = \int_{B_r} (\tilde{g}_R - \tilde{g}_r - \min_{\partial B_r} \tilde{g}_R) Wdy + \int_{B_r} \tilde{g}_r Wdy
\]

and from the maximum principle for normalized adjoint solutions, (2.3) and theorem 2.1

\[
\int_{B_r} |\tilde{g}_R - \min_{\partial B_r} \tilde{g}_R| Wdy \leq N \frac{r^2}{W(B_r)} \quad \text{for} \quad r \leq R/2.
\]

Using standard arguments for functions of bounded mean oscillation and (2.1) the last inequality imply that

\[
(2.4) \quad \int_{B_r} |\tilde{g}_R - \int_{B_{2r}} \tilde{g}_R Wdx| Wdy \leq N \frac{r^2}{W(B_r)}
\]

\[
\left| \int_{B_1} \tilde{g}_R Wdy - \int_{B_{2k}} \tilde{g}_R Wdy \right| \leq N \int_{k}^{2k} \frac{s}{W(B_s)} ds
\]

when \( r \leq R/2 \) and \( k \in \mathbb{Z} \) with \( 2^k \leq R/8 \). Then, if \( 2^{k-1} \leq |y| \leq 2^k \leq R/8 \), theorem 2.6, (2.1) and the first inequality in (2.4) imply

\[
|\tilde{g}_R(y)| - \int_{B_{2k+1}} \tilde{g}_R Wdx \leq N \int_{B_{2k+1}} |\tilde{g}_R| - \int_{B_{2k+1}} \tilde{g}_R Wdx | Wdy \leq N \frac{2^k}{W(B_{2k})},
\]
and from the second inequality in (2.4) and (2.1)

$$|\tilde{g}_R(y) - \int_{B_1} \tilde{g}_R \, W \, dx| \leq N \left( \frac{2^{2k}}{W(B_{2^k})} + \frac{r^{2k}}{W(B_2)} \int_1^{2^k} s \, ds \right)$$

proving the lemma.

The existence of a nonnegative fundamental solution when \( \int_{B_R}^\infty \frac{s}{W(B_s)} \, ds \) is finite is obvious by compactness and theorem 2.7. Also, if there is a

solution in \( B \), by that \( \limsup_{y \to 0} \tilde{g}_R(y) \leq g/\partial B \), and again the estimate in theorem 2.7 shows that the integral is bounded by \( N \max_{\partial B} \tilde{g} \).

If \( E \) has a nonpositive fundamental solution \( g \), \( \tilde{g}_1 - \tilde{g} \) is a normalized adjoint solution in \( B_1 \) with boundary values \( -\tilde{g} \), and then \( \tilde{g}_1 - \tilde{g} \leq -\min_{\partial B} \tilde{g} \) on \( B_1 \). In particular \( \tilde{g}_1(y) \leq -\min_{\partial B} \tilde{g} \) for all \( y \in B_1 \), and the finiteness of the integral follows from theorem 2.7 with \( R = 1 \).

On the other hand, if \( \int_0^1 \frac{s}{W(B_s)} \, ds \) is finite and \( R > 0 \), theorem 2.7 implies that \( \limsup_{y \to 0} \tilde{g}_R(y) = \tilde{g}_R(0) < +\infty \) and

$$N^{-1} \int_0^R \frac{s}{W(B_s)} \, ds \leq \tilde{g}_R(0) \leq N \int_0^R \frac{s}{W(B_s)} \, ds .$$

The maximum principle implies that in fact there exists \( \lim_{y \to 0} \tilde{g}_R(y) = \tilde{g}_R(0) \) and \( \tilde{g}_R(0) - \tilde{g}_R \geq 0 \) in \( B_R \). For \( r \leq R/2 \), the function \( \tilde{g}_R(0) - \tilde{g}_R - \tilde{g}_R(0) + \tilde{g}_R \) is a normalized adjoint solution in \( B_r \), vanishing at zero, hence there is \( x_r \in \partial B_r \) such that \( \tilde{g}_R(0) - \tilde{g}_R(x_r) - \tilde{g}_R(0) + \tilde{g}_R(x_r) = \tilde{g}_R(0) - \tilde{g}_R(x_r) - \tilde{g}_R(0) = 0 \). Thus, \( \tilde{g}_R(0) - \tilde{g}_R(x_r) = \tilde{g}_R(0) \), and from Harnack’s inequality \( \max_{\partial B_r} (\tilde{g}_R(0) - \tilde{g}_R) \leq N \min_{\partial B_r} (\tilde{g}_R(0) - \tilde{g}_R) \). Therefore, from (2.5)

$$N^{-1} \int_0^{\|y\|} \frac{s}{W(B_s)} \, ds \leq \tilde{g}_R(0) - \tilde{g}_R(y) \leq N \int_0^{\|y\|} \frac{s}{W(B_s)} \, ds$$

for all \( y \in B_{R/2} \),

and the existence and bounds of a nonpositive fundamental solution follow from the above estimate, compactness and as a limit when \( R \to +\infty \) of \( g_R - \tilde{g}_R(0) \).

**Proof of theorem 1.5.** Assuming that the coefficients matrix \( a(x) = (a_{ij}(x)) \) is continuous at infinity, we may assume \( a(\infty) = I \), where \( I \) is the identity matrix, and given \( \epsilon > 0 \) there is \( r_\epsilon > 0 \) such that \( |a(y) - I| \leq \epsilon \) for \( \|y\| \geq r_\epsilon \). Also (2.1) implies there is \( \alpha = \alpha(n, \nu) \in (0, 1) \) such that \( W(B_{r_\epsilon}) \leq N(r_\epsilon/r)^{\alpha} W(B_r) \) for \( r > r_\epsilon \). Then, integration by parts and the identity

$$E (r^2 - \|y\|^2) = -2 \text{trace}(a)$$

give

$$r \int_{B_r} \text{trace}(a) \, W \, dy = \int_{\partial B_r} a(y)y \cdot y \, W \, dy .$$
and setting $f(r) = W(B_r)$, the above identity and previous estimates give
$$(n - N\epsilon^n) f(r) \leq r f'(r) \leq (n + N\epsilon^n) f(r) \quad \text{for } r \geq r_\epsilon/\epsilon$$
which implies the first claim. The case of continuity at zero follows in a similar way.

**Some examples.**
For $\varphi \in C((0, +\infty))$ with $1 + \varphi > 0$ set
$$E_\varphi = \sum_{i,j=1}^{n} a_{ij}(x)D_{ij} + a_{ij} = \delta_{ij} + \varphi(|x|)|x_i x_j|^{1/2}.$$ These operators [24] have continuous coefficients in $\mathbb{R}^n \setminus \{0\}$ and the weak uniqueness holds in all domains ([9], [10], [39]). When $W(x) = W(|x|)$, $\tilde{w} = \tilde{w}(|x|)$ a calculation shows that $D_j(a_{ij}W) = 0$ for $i = 1, \ldots, n$ and $D_j(a_{ij}\tilde{w}W) = 0$ on $\mathbb{R}^n \setminus \{0\}$ provided that
$$W'(1 + \varphi) + W(\varphi + \frac{\alpha + 1}{r} \varphi) = 0, \quad (1 + \varphi)\tilde{w}' + \frac{\alpha + 1}{r} \tilde{w}' = 0$$
Also, if $\lim_{\epsilon \to 0} W(\epsilon)\epsilon^n = 0$, $W$ verifies in the distribution sense, $D_j(a_{ij}W) = 0$ for $i = 1, \ldots, n$ on $\mathbb{R}^n$. For a constant $\beta > -1$ and $n \geq 2$ the corresponding global nonnegative adjoint solution for $\varphi \equiv \beta$ is given by $c(n, \beta)|x|^{-\frac{(n + 1)\beta}{2}}$. In these cases the integral over $(1, +\infty)$ of $s/W(B_s)$ is finite when $\beta < n - 2$, the integral over $(0, 1)$ is finite when $\beta > n - 2$ and both integrals are infinite when $\beta = n - 2$, and fundamental solutions with pole at zero are given respectively by
$$\frac{1}{\omega_n(n - 2\beta)} |x|^{2-n} \cdot \frac{1}{\omega_n(n - 2\beta)} |x|^{2-n} \cdot -\frac{1}{\omega_n(n - 1)} |x|^{2-n} \log |x|$$
where $\omega_n$ is the surface area of the unit ball in $\mathbb{R}^n$.

For $\alpha \in \mathbb{R}$ the function $\log |x|^\alpha$ satisfies $D_j(a_{ij}W) = 0$ for $i = 1, \ldots, n$ when $\varphi$ solves $(r^{n-1} \log r)^{\alpha} = -\alpha r^{n-2} |\log r|^\alpha / \log r$. If $n = 2$, $\alpha, \beta > 0$, setting
$$g_\alpha = \frac{\alpha}{r|\log r|^\alpha} \int_0^r |\log t|^{\alpha-1} dt$$
$$f_\beta = \frac{\beta}{r(|\log r|)^\beta} \int_0^r (|\log t|)^{\beta-1} dt + \frac{A}{r(|\log r|)^\beta}$$
it is possible to choose $\varepsilon$ small, $R$ large and $A$ positive such that $1 + g_\alpha$, $1 + f_\beta$ are respectively positive and bounded for $r \leq \varepsilon$ and $r \geq R$, and $g_\alpha(\varepsilon) = f_\beta(R) = \gamma$. Defining
$$\varphi = \begin{cases} g_\alpha & r \leq \varepsilon \\ \gamma & \varepsilon \leq r \leq R \\ f_\beta & r \geq R \end{cases}$$
$$W(x) = \begin{cases} |\log |x||^{\alpha} & r \leq \varepsilon \\ \alpha B |x|^{-\gamma} & \varepsilon \leq r \leq R \\ C |\log |x||^\beta & r \geq R \end{cases}$$
where \( B, C > 0 \) are chosen so that \( W \) is continuous on \( \mathbb{R}^2 \setminus \{0\} \), it turns out that a positive multiple of \( W \) is the global nonnegative adjoint solution associated to \( E_\varphi \) which is an operator with continuous coefficients in the plane and at infinity, and choosing different values of \( \alpha, \beta \) one can make at will the integrals of \( s/W(E_s) \) over the intervals \((0, 1)\) and \((1, \infty)\) be either finite or infinite.

**Proof of theorem 1.4.** If the weak uniqueness holds the Green’s function for any bounded smooth domain is unique, and the remark after theorem 2.6 imply that the Harnack inequality does hold for nonnegative weak-normalized adjoint solutions and as a consequence, a Liouville-type theorem holds for a global and bounded normalized adjoint solution for \( E \) obtained as the ratio of two nonnegative and nonzero weak-adjoint solutions. Then, the uniqueness for \( W \) follows as in the case of smooth coefficients.

To prove the reciprocal we need to recall the following results which hold for operators \( E \) with smooth coefficients and where normalized adjoint solutions are defined with respect to the global adjoint solution \( W \) associated to \( E \) in theorem 1.1.

**Theorem 2.8.** Let \( \Omega \) be a Lipschitz domain in \( \mathbb{R}^n \), \( \varphi \in C(\partial \Omega) \). Then, there is a unique normalized adjoint solution \( \tilde{w}_\varphi \) in \( \Omega \) verifying \( \tilde{w}_\varphi \in C(\overline{\Omega}) \), \( \tilde{w}_\varphi = \varphi \) on \( \partial \Omega \). The moduli of continuity of \( \tilde{w}_\varphi \) in \( \Omega \) can be controlled in terms of \( n \), \( \nu \), the Lipschitz character of \( \Omega \) and the moduli of continuity of \( \varphi \). Moreover, for \( y \in \Omega \) there is a probability measure \( d\tilde{\omega}^y \) on \( \partial \Omega \), called the normalized harmonic measure, such that

\[
\tilde{w}_\varphi(y) = \int_{\partial \Omega} \varphi \, d\tilde{\omega}^y \quad \text{for all} \quad \varphi \in C(\partial \Omega)
\]

**Theorem 2.9.** Let \( \tilde{w}_1, \tilde{w}_2 \) be nonnegative normalized adjoint solutions in \( B_R \setminus B_R/4(z) \) vanishing continuously on \( \partial B_R(z) \). Then, if \( y_R = z + \frac{R}{2} \, e_n \)

\[
\frac{\tilde{w}_1(y)}{\tilde{w}_2(y)} \leq N \, \frac{\tilde{w}_1(y_R)}{\tilde{w}_2(y_R)}
\]

for all \( y \in B_R(z) \setminus B_R/2(z) \).

Theorem 2.8 is nowhere explicitly stated but it is an standard consequence of Harnack’s inequality and the results by E.B. Fabes, N. Garofalo, S. Marín Malave y S. Salsa in [17, Theorem I.1.6], theorem 2.9 follows from [17, Theorem I.3.7].

Now, if \( E \) has a unique global and nonnegative adjoint solution \( W \) verifying (1.8), given any sequence of operators \( E_k \) with smooth coefficients verifying (1.1) and whose coefficient matrices converge pointwise to the coefficients
of $E$, necessarily the corresponding sequence of global adjoint solutions $W_k$ converges weakly in $L^{\infty}_{loc}(\mathbb{R}^n)$ to $W$. This, theorem 2.7 and (2.1) imply there is a constant $N$ such that if $g_1, g_2$ are two Green’s functions for $E$ on $B_{4r}(z)$ generated by compactness and regularizations of the coefficients, then $g_1(x, y) \leq N g_2(x, y)$ for $x, y \in B_{2r}(z)$. This and the third remark after theorem 2.6 imply that the Harnack inequality in theorem 2.3 holds for weak-nonnegative normalized adjoint solutions relative to $W$. These clearly implies that the normalized harmonic measure associated to $E$ implies that the normalized harmonic measure associated to $W$ is unique, for if $\tilde{w} \in C(\Omega)$ is a normalized adjoint solution in $\Omega$ relative to $W$ vanishing on $\partial \Omega$, $\tilde{w}$ must attain its maximum at some point $z \in \Omega$, and $\tilde{w}(z) - \tilde{w}$ is a weak-nonnegative normalized adjoint solution for $E$ in $\Omega$, and by Harnack’s inequality $\tilde{w} = \tilde{w}(z) = 0$ in $\Omega$.

On the other hand, it is shown by N.V. Krylov in [29, Lemma 3.1] that if $q$ is a weak-Green’s function with pole at zero for $E$ in $B_R$, there is a sequence $E_k$ of operators with smooth coefficients verifying (1.1) and converging pointwise almost everywhere to the coefficients of $E$ such that the corresponding sequence of Green’s functions with pole at zero converges weakly in $L^{\infty}_{loc}(B_R)$ to $g$. It is also proved in the last work that suffices to know that the weak uniqueness holds for $E$ on $B_R$ at zero in order to conclude that the weak uniqueness holds in $B_R$.

Now, if $g_1, g_2$ are two Green’s functions for $E$ in $B_R$ the above remarks and theorem 2.7 imply that for some $N$, $g_1(y) \leq N g_2(y)$ for $y \in B_{R/2}$. Also, since normalized harmonic measure relative to $W$ is unique in $B_R \setminus B_{R/4}$, compactness and theorem 2.8 give

$$\tilde{g}_j(y) = \int_{\partial B_{R/4}} \tilde{g}_j \ d\tilde{\omega}^y,$$

for $y \in B_R \setminus B_{R/4}$, $j = 1, 2$, and where $d\tilde{\omega}^y$ is the normalized harmonic measure for $E$ on $B_R \setminus B_{R/4}$, and this and theorem 2.9 and 2.7 imply the existence of a constant $N$ such that $g_1(y) \leq N g_2(y)$ for all $y \in B_R$. If $N > 1$, setting $A = (N - 1)^{-1}$, $g_1 + A(g_1 - g_2)$ is a weak-nonnegative Green’s function for $E$ in $B_R$ with pole at zero, thus $g_2 \leq N (g_1 + A(g_1 - g_2))$ in $B_R$ and iterating $g_1 + A(g_1 - g_2) + A[(g_1 - g_2) + A(g_1 - g_2)]$ is also a weak-Green’s function for $E$ on $B_R$ with pole at zero. Continuing in this manner, we find there is $\{A_k\}$, $A_k \to +\infty$ such that $g_1 + A_k(g_1 - g_2) \leq N g_1$ and this is only possible if $g_1 \leq g_2$. Similarly $g_2 \leq g_1$. Finally, from [29, Theorem 2.1] the weak uniqueness holds for $E$ if and only if it does hold for all balls, and this is what was just shown.

3. The Parabolic case

Let $P$ denote the parabolic operator in (1.9) verifying (1.1). The letters
$X$, $Y$, $Z$ will denote points in $\mathbb{R}^{n+1}$ with $X = (x,t)$, $Y = (y,s)$, and $Z = (z,\tau)$. For $R$, $r > 0$ set $Q_R = B_R \times \mathbb{R}$, $Q_r(Z) = B_r(z) \times (\tau - r^2, \tau + r^2)$ and $C_r(Z) = B_r(z) \times (\tau - r^2, \tau)$. Also, in the next theorems and proofs and until we say the opposite we assume that the coefficients of $P$ are smooth in $\mathbb{R}^{n+1}$. Then, under these hypothesis the following results are well known.

**Theorem 3.1.** (Harnack Inequality) Let $u$ be a nonnegative solution to $Pu = 0$ on $Q_2r(Z)$. Then,

$$\sup_{B_r(z) \times (\tau - 3r^2, \tau - r^2)} u \leq N \inf_{B_r(z) \times (\tau, \tau + 4r^2)} u$$

**Theorem 3.2.** (Interior elliptic-type Harnack inequality) Suppose $u$ is a nonnegative solution to $Pu = 0$ in $B_2 \times (\tau, \tau + T]$ vanishing continuously on $\partial B_2 \times (\tau, \tau + T)$, and $0 < \delta \leq \frac{1}{2} \min\{1, \sqrt{T}\}$. Then

$$\sup_{B_{2 - \delta} \times (\tau + \delta^2, \tau + T]} u \leq N \inf_{B_{2 - \delta} \times (\tau + \delta^2, \tau + T]} u$$

where the constant $N = N(n, \nu, \delta, T)$.

**Theorem 3.3.** (Bounds for the normal derivative) Let $Z$ lie in the lateral boundary of $Q_1$, $0 < r \leq 1/2$ and $u$ be a nonnegative solution to $Pu = 0$ in $Q_2r(Z) \cap Q_1$ vanishing continuously on $Q_2r(Z) \cap (\partial B_1 \times \mathbb{R})$. Then, for $X \in Q_r(Z) \cap Q_1$

$$\frac{u((1 - r)z, \tau - 2r^2)(4 - |x|^2)}{Nr} \leq u(X) \leq \frac{Nu((1 - r)z, \tau + 2r^2)(4 - |x|^2)}{r}$$

and

$$\frac{\partial u}{\partial \nu}(X) = -\sum_{i,j=1}^{n} a_{ij}(X)D_i u(X)x_j$$

is the conormal derivative of $u$ at $X$ in $Q_r(Z) \cap (\partial B_1 \times \mathbb{R})$.

Recall that a Green’s function for $P$ on $Q_R$ with pole at $X \in Q_R$ is a nonnegative function $g_R(X,) \in L^1(B_R \times (\alpha, \beta))$ for all $\alpha < \beta$ and verifying

$$\psi(X) = -\int_{\tau}^{t} \int_{B_R} g_R(X,Y) \, P \psi \, dY + \int_{B_R} g_R(X,Y,\tau) \psi(y, \tau) \, dy$$

for all $\tau < t$ and $\psi \in C^{2,1}(\overline{Q_R})$ vanishing on $\partial B_R \times \mathbb{R}$. 


Theorem 3.4. Given $R > 0$ there is a unique Green’s function $g_R(X,Y)$ for $P$ on $Q_R$ with pole at $X \in Q_R$. Moreover, $g_R(X,Y) = 0$ for $t < s$ and

\[
\left( \int_{Q_R} g_R(X,Y) \frac{dY}{R} \right)^{\frac{1}{n+1}} \leq NR^{\frac{1}{n+1}},
\]

and $P$ has a unique global fundamental solution $G(X,Y)$ in $\mathbb{R}^{n+1}$ with $G(X,Y) = 0$ for $t < s$ and verifying

\[
\left( \int_{\alpha}^{\beta} \int_{B_R(z)} G(X,Y) \frac{dY}{R} \right)^{\frac{n}{n+1}} \leq N(\beta - \alpha)^{\frac{n}{2(n+1)}}.
\]

Theorem 3.5. (Doubling property of caloric measure) There is a constant $N$ verifying

\[
\int_{B_{2r}(z)} G(X,y,\tau) \, dy \leq N \int_{B_r(z)} G(X,y,\tau) \, dy
\]

for all $r > 0$, $X, Z \in \mathbb{R}^{n+1}$ with $|x - z| \leq \sqrt{t - \tau}$.

Theorem 3.6. There is a constant $N$ verifying

\[
N \int_{B_{2r}(z)} G(X,y,\tau) \, dy \geq 1
\]

for all $Z \in \mathbb{R}^{n+1}$, $r > 0$ and $X \in B_r(z) \times [\tau, \tau + r^2]$.

Theorem 3.1 was proved by N.V. Krylov and M.V. Safonov [30], and theorems 3.2 and 3.3 by N. Garofalo [23]. The estimate (3.1) is called by some authors the Krylov-Bakelman-Aleksandrov-Tso inequality ([28], [42]) and the estimate (3.2) is due to X. Cabré [8, Theorem 1.15 and Remark 1.14]. It is not clear that there is any place where the bound (3.3) is explicitly stated, but it follows from (3.2). Finally, theorems 3.5 and 3.6 are due to M.V. Safonov and Yu Yuan [41, Theorem 1.1], [41, Lemma 4.1].

Before proving theorem 1.3 we need the following lemmas.

Lemma 3.1. For all $r > 0$, $X, Z \in \mathbb{R}^{n+1}$ with $t \geq \tau$

\[
\int_{B_r(z)} G(X,y,\tau) \, dy \leq \left( 1 + \frac{t - \tau}{r^2} \right)^{-\frac{n}{2}} \exp \left( \frac{\nu}{4} \left( 1 - \frac{|x - z|^2}{t - \tau + r^2} \right) \right).
\]

Proof. The function $u(X)$ on the left hand side in the above inequality is the solution to $Pu = 0$ on $\mathbb{R}^n \times (\tau, +\infty)$ with data given by $\chi_{B_r(z)}$. A simple calculation shows that the function $v(X)$ on the right hand side, verifies $Pv \leq 0$ on $\mathbb{R}^n \times (\tau, +\infty)$ and $v \geq u$ on the parabolic boundary, and the lemma follows from the maximum principle.
Lemma 3.2. Given $0 \leq \theta \leq 1$ and $r > 0$, the inequalities

\begin{align}
\int_{B_{2r}(z)} G(X, y, \tau + \theta r^2) \, dy &\leq N \int_{B_r(z)} G(X, y, \tau) \, dy \quad \text{(3.5)} \\
\int_{B_{2r}(z)} G(X, y, \tau) \, dy &\leq N \int_{B_r(z)} G(X, y, \tau + \theta r^2) \, dy \quad \text{(3.6)}
\end{align}

hold respectively when $|x - z| \leq \sqrt{t - \tau}$ and $t \geq \tau + \theta r^2$, and when $|x - z| \leq \sqrt{t - \tau - \theta r^2}$.

Proof. Given $Z \in \mathbb{R}^{n+1}$, $r > 0$ and $0 \leq \theta \leq 1$ theorem 3.5 gives

\[ \int_{B_{2r}(z)} G(X, y, \tau) \, dy \leq N \int_{B_r(z)} G(X, y, \tau) \, dy \quad \text{when} \quad |x - z| \leq \sqrt{t - \tau} \]

and from theorem 3.6

\[ N \int_{B_{2r}(z)} G(X, y, \tau) \, dy \geq 1 \quad \text{on} \quad B_{2r}(z) \times [\tau, \tau + 4r^2]. \]

Thus, to prove (3.5) suffices to show that for $t \geq \tau + \theta r^2$ and $x \in \mathbb{R}^n$

\[ \int_{B_{2r}(z)} G(X, y, \tau + \theta r^2) \, dy \leq N \int_{B_r(z)} G(X, y, \tau), \]

but this is implied by the previous inequality and the maximum principle.

To prove (3.6), we let $u(X)$ denote the solution to $Pu = 0$ in $\mathbb{R}^n \times (\tau, +\infty)$ with initial data $\chi_{B_{2r}(z)}$. Since for $t > \sigma \geq \tau$ we have

\[ u(X) = \int_{\mathbb{R}^n} G(X, y, \sigma) u(y, \sigma) \, dy, \]

(3.7) \[ \int_{B_{2r}(z)} G(X, y, \tau) \, dy = \int_{\mathbb{R}^n} G(X, y, \tau + \theta r^2) u(y, \tau + \theta r^2) \, dy. \]

Also, an iteration of the inequality in theorem 3.5 and lemma 3.1 give respectively that for $|x - z| \leq \sqrt{t - \tau - \theta r^2}$, $k \geq 1$ and $y \in \mathbb{R}^n$

\[ \int_{B_{2r}(z)} G(X, y, \tau + \theta r^2) \, dy \leq N^{k+1} \int_{B_r(z)} G(X, y, \tau) \, dy \]

\[ u(y, \tau + \theta r^2) \leq Ne^{-\frac{|y-z|^2}{N r^2}} \]

and (3.6) follows from (3.7) and the exponential decay of $u$.

Now, if we set $\theta = 1$ in (3.5), (3.6) and integrate the inequalities with respect to the time variable the following follows.
Lemma 3.3. (Space-time doubling property) There is a constant $N$ such that for $X, Z \in \mathbb{R}^{n+1}$ the inequalities
\[
\int_{B_{2r}(z)} \int_{\tau}^{\tau+4r^2} G(X,Y) \, dY \leq N \int_{B_r(z)} \int_{\tau}^{\tau+4r^2} G(X,Y) \, dY
\]
\[
\int_{B_{2r}(z)} \int_{\tau}^{\tau+4r^2} G(X,Y) \, dY \leq N \int_{B_r(z)} \int_{\tau}^{\tau+4r^2} G(X,Y) \, dY
\]
hold respectively when when $|x - z| \leq \sqrt{t - \tau - 3r^2}$, $t \geq \tau + 4r^2$, and when $|x - z| \leq \sqrt{t - \tau}$.

Lemma 3.4. Let $W$ be a nonnegative adjoint solution for $P$ on $C_{4r}(Z)$, then
\[
\left( \int_{C_r(Z)} W^{\frac{n+1}{n+1}} \, dY \right)^{\frac{n}{n+1}} \leq N \int_{C_{2r}(Z)} W \, dY
\]

Proof. The proof of this result in the elliptic case is done in [22, Theorem 2.1]. There the argument is based in the Pucci-Aleksandrov inequality (Theorem 2.1) and integration by parts. In the parabolic setting, a proof of the above estimate can be obtained with similar arguments but replacing the Pucci-Aleksandrov estimate by its parabolic analog (inequality (3.1)).

Proof of theorem 1.3. To prove the existence of the global nonnegative adjoint solution define for $T > 0$
\[
W_T(Y) = \frac{G(0,T,Y)}{\alpha(T)} \quad \text{where} \quad \alpha(T) = \int_{B_1 \times (0,1)} G(0,T,y,s) \, dY
\]
Then, $W_T(B_1 \times (0,1)) = |B_1|$ and lemmas 3.2, 3.3 and 3.4 give
\[
W_T(B_{2r}(z), \tau + \theta \tau) \leq N W_T(B_{r}(z), \tau)
\]
\[
W_T(B_{2r}(z), \tau) \leq N W_T(B_{r}(z), \tau + \theta \tau)
\]
\[
\left( \int_{Q_r(z)} W_T^{\frac{n+1}{n+1}} \, dY \right)^{\frac{n}{n+1}} \leq N \int_{Q_r(z)} W_T \, dY
\]
for $|z| \leq \sqrt{T/2}$, $\tau \leq T/4$, $0 < r \leq \sqrt{T}/8$ and $0 \leq \theta \leq 1$. Then, proceeding as in section 2 and from the analog properties of parabolic Muckenhoupt weights, compactness and (1.12) one obtains the existence of a nonnegative adjoint solution $W$ defined on $\mathbb{R}^{n+1}$ verifying the properties stated in theorem 1.3.
Definition. A function \( \tilde{w} \) is a normalized adjoint solution for \( P \) relative to \( W \) in an open set \( \Omega \subset \mathbb{R}^{n+1} \) if \( \tilde{w}W \in L_{loc}^1(\Omega) \) is an adjoint solution in \( \Omega \).

**Theorem 3.7.** (Harnack inequality for normalized adjoint solutions) Let \( Z \in \mathbb{R}^{n+1} \) and \( \tilde{w} \) be a nonnegative normalized adjoint relative to \( W \) in \( C_2(Z) \). Then,

\[
\sup_{B_r(z) \times (\tau - 2r^2, \tau - r^2)} \tilde{w} \leq N \inf_{B_r(z) \times (\tau - 4r^2, \tau - 3r^2)} \tilde{w}
\]

**Proof.** By translation and scaling we may assume \( Z = (0, 4), r = 1 \), and that \( \tilde{w} \) is a nonnegative normalized adjoint solution relative to \( W \) in \( C_2(Z) \).

Defining the normalized Green’s function for the cylinder \( Q_2 \) as \( \tilde{g}_2(X,Y) = g_2(X,Y)/W(Y) \) and denoting the surface measure on the lateral boundary of the cylinder as \( d\sigma \), the following representation formula holds for \( \tilde{w} \) and \( Y \in C_2(Z) \)

\[
\tilde{w}(Y) = \int_{B_2} \tilde{g}_2(x,4,Y)\tilde{w}(x,4)W(x,4)dx + \int_{\partial B_2} \int_s^4 \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y)\tilde{w}(X)W(X)d\sigma
\]

In particular, for \( Y \in B \times (2,3), \tilde{w}(\tilde{Y}) \in B \times (0,1) \)

\[
\tilde{w}(\tilde{Y}) \leq \int_{B_2} \tilde{g}_2(x,4,Y)\tilde{w}(x,4)W(x,4)dx + \int_{\partial B_2} \int_2^4 \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y)\tilde{w}(X)W(X)d\sigma
\]

\[
\tilde{w}(\tilde{Y}) \geq \int_{B_2} \tilde{g}_2(x,4,Y)\tilde{w}(x,4)W(x,4)dx + \int_{\partial B_2} \int_2^4 \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y)\tilde{w}(X)W(X)d\sigma
\]

From theorems 3.1, 3.2 and 3.3

\[
\tilde{g}_2(x,4,Y) \leq N \tilde{g}_2(0,5,Y)(4 - |x|^2) \quad \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y) \leq N \tilde{g}_2(0,5,Y)
\]

\[
N \tilde{g}_2(x,4,Y) \geq \tilde{g}_2(0,5,Y)(4 - |x|^2) \quad N \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y) \geq \tilde{g}_2(0,5,Y)
\]

for all \( x \in B_2, X \in \partial B_2 \times [2,4] \), and from these inequalities

\[
\frac{\tilde{w}(\tilde{Y})}{\tilde{w}(\tilde{Y})} \leq N \frac{\tilde{g}_2(0,5,Y)}{\tilde{g}_2(0,5,Y)}.
\]

Since the constant function one is a normalized adjoint solution relative to \( W \), for \( Y \in C_2(Z) \)

\[
1 = \int_{B_2} \tilde{g}_2(x,4,Y)W(x,4)dx + \int_{\partial B_2} \int_s^4 \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y)W(X)d\sigma
\]
and then

\[
1 \geq \int_{B_2} \tilde{g}_2(x,4,Y) W(x,4) \, dx + \int_{\partial B_2} \int_{\tau}^{4} \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y) W(X) \, d\sigma
\]

\[
1 \leq \int_{B_2} \tilde{g}_2(x,4,Y) W(x,4) \, dx + \int_{\partial B_2} \int_{0}^{4} \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y) W(X) \, d\sigma .
\]

Using again theorems 3.1, 3.2 and 3.3 it follows that for \( x \in B_2, \, X \in \partial B_2 \times [\frac{31}{8}, 4] \)

\[
N \tilde{g}_2(x,4,Y) \geq (4 - |x|^2)\tilde{g}_2(0,5,Y) \quad N \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y) \geq \tilde{g}_2(0,5,Y)
\]

and for \( x \in B_2, \, X \in \partial B_2 \times [0, 4] \)

\[
\tilde{g}_2(x,4,Y) \leq N(4 - |x|^2)\tilde{g}_2(0,5,Y) \quad \frac{\partial \tilde{g}_2}{\partial \nu}(X,Y) \leq N\tilde{g}_2(0,5,Y)
\]

and these inequalities imply

\[
\frac{\tilde{g}_2(0,5,Y)}{\tilde{g}_2(0,5,Y)} \leq N \frac{\int_{B_2} (4 - |x|^2) \, W(x,4) \, dx + \int_{\partial B_2} \int_{0}^{4} W(X) \, d\sigma}{\int_{B_2} (4 - |x|^2) \, W(x,4) \, dx + \int_{\partial B_2} \int_{0}^{4} W(X) \, d\sigma} .
\]

Thus, to finish the proof suffices to show that the above fraction is bounded by a constant depending on \( n \) and \( \nu \). Now, integration by parts and the identity \( L(4 - |x|^2) = -2\text{trace } a(X) \), where \( a(X) \) denotes the coefficients matrix of \( P \) give

\[
\int_{B_2} (4 - |x|^2) \, W(x,\tau) \, dx + \int_{\partial B_2} \int_{\tau}^{4} 2\text{trace } a(X) \, W(X) \, dX = \int_{B_2} (4 - |x|^2) \, W(x,4) \, dx + \int_{\partial B_2} \int_{\tau}^{4} a(X)x \cdot x \, W(X) \, d\sigma
\]

for \( \tau \leq 4 \), and from (1.1) we obtain that the above quotient is bounded by

\[
N \frac{\int_{B_2} (4 - |x|^2) \, W(x,0) \, dx + \int_{B_2} \int_{0}^{4} W(X) \, dX}{\int_{B_2} (4 - |x|^2) \, W(x, \frac{31}{8}) \, dx + \int_{B_2} \int_{0}^{4} W(X) \, dX}
\]

and this last ratio is bounded by a universal constant due to the doubling properties of \( W \), in particular (1.11).

A standard consequence of Harnack’s inequality is the following theorem
Theorem 3.8 (Hölder continuity). Let \( \tilde{w} \) be a normalized adjoint solution relative to \( W \) in \( Q_r(Z) \). Then, for some \( \alpha = \alpha(n, \nu) \in (0, 1) \)

\[
|\tilde{w}(Y) - \tilde{w}(Z)| \leq N \left( \frac{|y-z|^2 + |s-t|}{r^2} \right)^{\alpha/2} \sup_{Q_r(Z)} |\tilde{w}|
\]

This result implies a Liouville-type theorem for bounded normalized adjoint solutions defined in \( \mathbb{R}^{n+1} \) and gives again as in the elliptic case the uniqueness of a nonnegative and global adjoint solution \( W \) verifying (1.11) and the normalization condition \( W(B_1 \times (0,1)) = |B_1| \).

Defining \( \tilde{G}(X,Y) = G(X,Y)/W(Y) \), theorem 3.7 with \( Z = (y,t) \) and \( r = \sqrt{t-s} \) implies

\[
\tilde{G}(X,Y) \leq N \tilde{G}(X,z,4s-3t) \quad \text{for } z \in B_{\sqrt{t-s}}(y) , \ s < t .
\]

Multiplying this inequality by \( W(z,4s-3t) \) and integrating the result over \( B_{\sqrt{t-s}}(y) \) with respect to \( dz \) one obtains

\[
W(B_{\sqrt{t-s}}(y),4s-3t) \tilde{G}(X,Y) \leq N \int_{B_{\sqrt{t-s}}(y)} G(X,z,4s-3t) \, dz ,
\]

and from lemma 3.1

(3.8) \[
W(B_{\sqrt{t-s}}(y),4s-3t) \tilde{G}(X,Y) \leq Ne^{-\frac{|x-y|^2}{2(t-s)}} .
\]

Then, the upper bound in theorem 1.3 follows from (3.8) and the inequalities

\[
W(B_{\sqrt{t-s}}(y),s) \leq N W(B_{\sqrt{t-s}}(y),4s-3t)
\]

\[
W(B_{\sqrt{t-s}}(x),s) \leq N \left( 1 + \frac{|x-y|}{\sqrt{t-s}} \right)^N W(B_{\sqrt{t-s}}(y),s)
\]

which are standard consequences of (1.11).

To obtain the lower bound we apply theorem 3.7 with \( Z = (y,t) \), \( r = \sqrt{t-s/2} \) obtaining

\[
N \tilde{G}(X,Y) \geq \tilde{G}(X,z,\frac{t+s}{2}) \quad \text{for } z \in B_{\sqrt{t-s/4}}(y) , \ s < t ,
\]

and multiplying this inequality by \( W(z,\frac{t+s}{2}) \) and integrating the result over \( B_{\sqrt{t-s/4}}(y) \) with respect to \( dz \)

\[
N W(B_{\sqrt{t-s/4}}(y),\frac{t+s}{2}) \tilde{G}(X,Y) \geq \int_{B_{\sqrt{t-s/4}}(y)} G(X,z,\frac{t+s}{2}) \, dz .
\]
Also, from (1.11)
\[ N W(B_{\sqrt{t-s}}(y), s) \geq W(B_{\sqrt{t-s/4}}(y), \frac{t+s}{4}) \]
\[ N \left( 1 + \frac{|x-y|}{\sqrt{t-s}} \right) W(B_{\sqrt{t-s}}(x), s) \geq W(B_{\sqrt{t-s}}(y), s) . \]

The function \( u(\cdot) = \int_{B_{\sqrt{t-s/4}}(y)} G(\cdot, z, \frac{t+s}{4}) \, dz \) is nonnegative and verifies
\[ Pu = 0 \text{ on } \mathbb{R}^n \times (\frac{t+s}{4}, +\infty), \]
and with an standard iteration of theorem 3.1 (See the proof of the lower bound in [21, Theorem 2.7]) we have
\[ u(X) \geq N^{-1} e^{-\frac{|x-y|^2}{t-s}} u(y, \frac{t+s}{4} + \left( \frac{\sqrt{t-s}}{8} \right)^2) \]
and from theorem 3.6, \( Nu(y, \frac{t+s}{4} + \left( \frac{\sqrt{t-s}}{8} \right)^2) \geq 1 \), obtaining
\[ \int_{B_{\sqrt{t-s/4}}(y)} G(X, z, \frac{t+s}{4}) \, dz \geq N^{-1} e^{-\frac{|x-y|^2}{t-s}} , \]
and the lower bound follows from the previous inequalities. Finally, from the bounds we proved there is a constant \( N = N(n, \nu, K, \epsilon) \) such that \( N^{-1} W(Y) \leq G(X, \cdot) \leq N W(Y) \) for \( Y \in \Omega(X, K, \epsilon) \), and this implies the last claim.

**Remark 4.1.** A standard consequence of the Harnack’s inequality for normalized adjoint solutions is that when the coefficients of \( P \) are smooth and \( Q = \Omega \times (-\infty, T) \), where \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^n \), \( \varphi \in C(\partial_p Q) \) and \( \partial_p Q = \Omega \times T \) and \( \partial \Omega \times (-\infty, T) \), there exists a unique normalized adjoint solution \( \tilde{w}_\varphi \) in \( Q \) verifying \( \tilde{w}_\varphi \in C(\overline{Q}) \), \( \tilde{w}_\varphi = \varphi \) on \( \partial_p Q \). Also, the maximum principle holds in this setting and there is \( \alpha_0 = \alpha_0(n, \nu, \Omega) \in (0, 1) \) such that if \( \varphi \in C^{\alpha, \alpha/2}(\partial_p Q) \) and \( \alpha \leq \alpha_0 \), then \( \tilde{w}_\varphi \in C^{\alpha, \alpha/2}(\overline{Q}) \).

From the arguments in [19], [20] and [41], once Harnack’s inequality for normalized adjoint solutions and the bounds in theorem 1.3 for the fundamental solution are established, the results proved in the above works (the so called: Carleson-type inequality, Interior elliptic-type Harnack inequality, Comparison principle, Backward Harnack inequality near the boundary, Hölder continuity for quotients and the Doubling property of parabolic measure) can be carried out again in the context of normalized adjoint solutions, and in particular, once those results are established the analog of theorem 1.4 can be proved again in the parabolic setting with similar arguments.

**ACKNOWLEDGMENTS**

Part of this work was done at the time the author was attending the half-year program in Harmonic Analysis held at M.S.R.I. from July to December
1997. He would like to thank the members of the Institute and the organizers of the program for their hospitality. The author is supported by Universidad del País Vasco grant UPV 127.310 EA-210/96 and by the Basque Government grant P-1997/22.

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25


