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On the Fundamental Solution of a Linearized Homogeneous Coagulation Equation

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Abstract: We describe the fundamental solution of the equation that is obtained by linearization of the coagulation equation with kernel $K(x, y) = (xy)^{\lambda/2}$, around the steady state $f(x) = x^{-(3+\lambda)/2}$ with $\lambda \in (1, 2)$. Detailed estimates on its asymptotics are obtained. Some consequences are deduced for the flux properties of the particles distributions described by such models.

1. Introduction

Under rather general conditions on the kernel $K(x, y)$, a symmetric homogeneous function in x and y, the Cauchy problem for the Smoluchowski coagulation equation:

$$\frac{\partial f}{\partial t} = Q[f], \tag{1.1}$$

$$Q[f]= \frac{1}{2}\int_0^x K(x-y, y)\, f(x-y)\, f(y)\, dy - \int_0^\infty K(x, y)\, f(x)\, f(y)\, dy, \tag{1.2}$$

$$f(0, x) = f_{in}(x), \tag{1.3}$$

has a global solution $f(t, x)$ for all initial data $f_{in}(x) \geq 0$ such that $\int_0^\infty (1+x)\, f_{in}(x)dx < \infty$. Moreover, this solution satisfies the same estimate for all $t > 0$.

Equations (1.1), (1.2) describe the aggregation process of particles of mass x and y with rate $K(x, y)$, assuming that the distribution of particles are uncorrelated at all times. In this context the quantity:

$$\int_0^\infty x\, f(t, x)\, dx \tag{1.4}$$

represents the total mass of particles in the system.

On the other hand, it is known that, when the kernel is of the form $K(x, y) = x^\alpha y^\beta + x^\beta y^\beta$ with $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = \lambda > 1$, the solutions to the Cauchy problem for the Smoluchowski equation undergo the so-called gelation phenomenon. This means that there exists a positive time $T_g < \infty$ such that, for all $t < T_g$,

$$\int_0^\infty x\, f(t,x)\, dx = \int_0^\infty x\, f_{in}(x)\, dx \tag{1.5}$$

and for all $t > T_g$,

$$\int_0^\infty x\, f(t,x)\, dx < \int_0^\infty x\, f_{in}(x)\, dx. \tag{1.6}$$

It is also known that when $\lambda \leq 1$ gelation does not occur and mass is conserved for all time (cf. [7,19]). Two interesting open questions are related with this phenomenon. One is to describe the solution f as $t \to T_g$. We consider here the second one, which is to understand the behaviour of the solution f after the gelling time T_g.

Although no general result is known, several partial results indicate that before the gelling time, at least for a large family of initial data, the solutions to the coagulation equation decay exponentially fast as $x \to +\infty$. The Smoluchowski equation (1.1) has a discrete counterpart for which an explicit exact gelling solution was constructed in [14] for $K(k, j) = k\,j$. Such a solution decays exponentially fast before the gelling time, and as a power law after that time. The exponential decay, before the gelling time, was later shown in [8] for the continuous equation (1.1), $\lambda = 2$ and several initial data. Moreover, it has also been formally shown in [4,8] that, for several initial data and $\lambda \in (1, 2]$, the solution of (1.1) decays, after gelling, like $x^{-(3+\lambda)/2}$ as $x \to +\infty$ (see [13] for more detailed references). On the other hand, it was proved in [9] that $x^{-(3+\lambda)/2}$ is the only possible power law decay for the solutions of (1.1) after gelation. Our main purpose is to prove that for the coagulation kernel

$$K(x,y) = (xy)^{\lambda/2}, \quad \lambda \in (1,2) \tag{1.7}$$

and any initial data f_{in}, regular near the origin and such that:

$$f_{in}(x) \sim x^{-(3+\lambda)/2} \quad \text{as } x \to +\infty, \tag{1.8}$$

the problem (1.1)–(1.3) has a solution f satisfying

$$f(t,x) \sim a(t)\, x^{-(3+\lambda)/2} \;\text{ as }\; x \to +\infty. \tag{1.9}$$

Moreover, this solution satisfies

$$\frac{d}{dt}\int_0^\infty x\, f(x,t)dx = -2\pi a^2(t) \quad \text{for all } t > 0, \tag{1.10}$$

which was formally shown in [6] for the discrete equation. By (1.10) the total mass of the solution f is decreasing. This loss of mass is a characteristic feature of the solutions of (1.1),(1.2) after the gelation time. The choice of exponents $\lambda < 2$ is natural, because $\lambda \leq 2$ excludes instantaneous gelation or non existence of solutions ([3,6,18]) and $\lambda = 2$ is one of the "explicit" cases which has been treated using the Laplace transform (cf. [8]).

In order to prove the existence of classical solutions of (1.1)-(1.3) after gelation we will use the same approach as in [10,11]. The starting point of this approach is to linearize around an initial data f_{in} satisfying $f_{in} \approx x^{-(3+\lambda)/2}$ for x large and to derive detailed estimates on the solutions of the resulting linear equation.

$$\frac{\partial g}{\partial t} = \mathcal{L}(g, f_{in}). \tag{1.11}$$

To this end we will need some rather delicate estimates on the asymptotics of the solutions as x tends to infinity. Moreover, even to prove solvability of the linearized problem (1.11) is nontrivial. We will obtain it treating this problem as a perturbation of the problem obtained replacing f_{in} by its asymptotics as x tends to infinity:

$$\frac{\partial g}{\partial t} = L(g). \tag{1.12}$$

In order to carry on this program we need to derive detailed estimates about the solutions of (1.12). This will be the main goal of this paper.

The linearized equation around the weak solution $x^{-(3+\lambda)/2}$ may be introduced more directly as follows. Consider a solution $f(t,x)$ of the coagulation equation with an initial data f_{in} satisfying (1.8). If one is interested in the behaviour of $f(t,x)$ for x large it is natural to scale the variables as follows: $x = R\,\overline{x}$, $y = R\,\overline{y}$, $t = R^{-(\lambda-1)/2}\overline{t}$ and $f(t,x) = R^{-(3+\lambda)/2} F_R(\overline{t}, \overline{x})$. In these new variables, Eq. (1.1) reads $(F_R)_{\overline{t}} = Q[F_R]$ and the initial data F_{in} satisfies now: $F_R(0,\overline{x}) = R^{(3+\lambda)/2} f_{in}(R\,\overline{x}) \sim (\overline{x})^{-(3+\lambda)/2}$ as $R \to +\infty$. The limit of the function F_R as $R \to +\infty$, if it exists, would then solve the same Eq. (1.1) with initial data $\overline{x}^{\,-(3+\lambda)/2}$. Therefore the linear problem (1.12) appears naturally as the linearisation of the coagulation Eq. (1.1) in the region $x >> 1$. Notice, however that in the region where $\overline{x}$ is small the function F_R is bounded and the approximation by means of the power law $\overline{x}^{\ -(3+\lambda)/2}$ cannot be valid. The analysis of that region would lead naturally to the study of a boundary layer whose description requires the analysis of the operator $\mathcal{L}$. This will be made in a forthcoming work.

On the other hand, the linearized equation (1.12) has some interest by itself. It is indeed a simple model to describe a set of particles at equilibrium, whose density distribution is given by $x^{-(3+\lambda)/2}$, and where a small set of particles is introduced, whose distribution $\varphi(x)$ is considered as a small perturbation. The particles so introduced start to collide both between themselves and with the particles in the background. The equilibrium density distribution $x^{-(3+\lambda)/2}$ is then perturbed. The distribution density function of the resulting set of particles may then be seen at any time t as the equilibrium distribution $x^{-(3+\lambda)/2}$ and a remaining perturbation $\varphi(t,x)$. The linear equation (1.12) only takes into account the collisions of the "particles in the perturbation" with the background and describes how the distribution of these particles evolves in time. It neglects the collisions between particles in the perturbation. This could be a reasonable approximation as long as the perturbation $\varphi(t,x)$ remains small. Notice that the number of clusters in the background as well as the number of particles (the total mass) are infinite (since neither $x^{-(3+\lambda)/2}$ nor $x^{1-(3+\lambda)/2}$ are integrable in $(0,+\infty)$), but the number of clusters and particles in the initial perturbation are finite. Our results show the following:

- There is instantaneously an infinite number of "perturbed clusters", although their mass is finite.
- As $t \to +\infty$, the number of perturbed particles (the mass in the perturbation) tends to zero, but the number of perturbed clusters remains infinite.

– The total flux of particles is perturbed at t finite but tends to the flux corresponding to the original equilibrium distribution as $t \to +\infty$.

Our results are obtained using classical Fourier analysis and the Wiener Hopf method, in a similar way as we did for the linearized Uehling Uhlenbeck operator in [10] although with an important difference. This is the regularising effect of the operator L, absent in the operator studied in [10], and coming from the fact that L is similar to the half derivative operator. The fundamental solution of (1.12) has then very different properties than that obtained in [10].

In Sect. 2 we shortly describe the conventions used for the names of the different constants in the paper. In Sect. 3 we state our main results and transform the integro differential equation (1.12) to a Carleman equation in the complex plane. In Sect. 4 we state the fundamental properties of the auxiliary function Φ appearing in the Carleman equation. This equation is solved in Sects. 5 and 6 using the classical Cauchy integral, which gives an explicit solution. We prove in Sect. 7 that the fundamental solution obtained in this paper is unique in a suitable functional class. In Sects. 8 and 9, the precise asymptotics of the solutions are obtained. Section 10 describes how to solve the initial value problem associated to the linearised coagulation equation. Some properties of the fluxes of particles described by the solutions are considered in Sect. 11. We have finally Sects. 12, 13 and 14 where some necessary technical results are collected.

2. A Guide about the Names of the Constants used in this Paper

In this paper we will use different letters to denote the numerical constants used in the arguments. In order to make easier the reading we will describe the role that constants with different names have in the arguments. Unless specifically stated similar names will be used for these constants in independent arguments.

We will denote as C a positive constant whose value can change from line to line that depends only on λ and in the variables mentioned in each specific lemma.

We denote as ε a positive constant that can be made arbitrarily small. This constant will be always an exponent appearing in estimate containing also a multiplicative constant depending on ε.

We will denote as δ a positive constant, perhaps small and depending only on λ. Occasionally we will denote also as a similar positive constants appearing in exponential functions.

We will use $\varepsilon_0, \ \varepsilon_1, \dots, \ \delta_1, \ \delta_2, \dots$ to denote small positive constants used in technical arguments. The names $\varepsilon_k' s$ will be reserved for variables which are sent to zero or infinity respectively at the end of the argument, and the names $\delta_k' s$ for constants that must remain strictly positive until the end of the argument.

We will denote as $\beta_k \in \mathbb{R}$ the names used to characterize some horizontal lines in the complex plane used in contour integration arguments involving the variables $\eta, \ y, \ \xi$. The values $\beta_0, \ \beta_1, \ \beta_2$ are used to define the functions $\widehat{G}$, $\mathcal{V}$ and G in (5.1), (6.6), (9.1) respectively. They take values in the ranges defined by (5.2), (6.13), (9.2). Due to the fact that the three mentioned functions play a central role in the arguments, the corresponding constants $\beta_0, \ \beta_1, \ \beta_2$ will be used repeatedly. On the other hand we will use the names $\beta_3, \ \beta_4, \dots, \beta_{10}$ to denote horizontal lines in the variables $y, \ \xi$ that will arise in contour deformation arguments. These constants will be used only once.

Finally we will denote as $\gamma_1, \ \gamma_2$ real numbers characterizing horizontal lines in the variable $Y = y - \xi$.

Other letters used for the constants are B that characterizes the domains $D(\xi, B)$, L that measures the width of some strips of the complex plane and M that denotes a large constant at two places.

3. The Linearized Equation

We start this section writing the precise expression of the linearized equation (1.12).

Proposition 3.1. *The linearized equation of (1.1)-(1.2) with* $K(x, y) = (xy)^{\lambda/2}$ *around the solution* $x^{-(3+\lambda)/2}$ *is*

$$\frac{\partial g}{\partial t} = L(g), \tag{3.1}$$

$$\begin{aligned} L(g) = &\int_0^{x/2} \left((x-y)^{-3/2} - x^{-3/2}\right) y^{\lambda/2} g(y)dy \\ &+ \int_0^{x/2} \left((x-y)^{\lambda/2} g(x-y) - x^{\lambda/2} g(x)\right) y^{-3/2} dy \\ &- x^{-3/2} \int_{x/2}^{\infty} y^{\lambda/2} g(y)dy - 2\sqrt{2} x^{(\lambda-1)/2} g(x). \end{aligned} \tag{3.2}$$

Proof. By the symmetry of the kernel K:

$$\frac{1}{2}\int_0^{x/2} K(y, x-y) f(y) f(x-y)dy = \frac{1}{2}\int_{x/2}^{x} K(y, x-y) f(y) f(x-y)dy. \tag{3.3}$$

Then we may write Eq. (1.1)–(1.2) as follows:

$$\begin{aligned} \frac{\partial f}{\partial t} = &\int_0^{x/2} \left[(x-y)^{\lambda/2} f(x-y) - x^{\lambda/2} f(x)\right] y^{\lambda/2} f(y)dy \\ &- \int_{x/2}^{\infty} K(x, y) f(x) f(y)dy. \end{aligned} \tag{3.4}$$

If we linearize around the $x^{-(3+\lambda)/2}$, define $f = x^{-(3+\lambda)/2} + g$ and neglect quadratic terms on g we obtain (3.1), (3.2).

Remark 3.2. The second term in the right-hand side of (3.2) can be seen as some kind of half derivative operator applied to function $x^{\lambda/2} g(x)$. This will appear again in the Fourier analysis that will be made later on the linearized equation.

Remark 3.3. In order for the first integral in the right hand side of (3.2) to be defined we need $y^{1+\lambda/2} g(y)$ to be integrable at the origin. For the second integral we need some kind of regularity of $g(x)$ with respect to x, for example $y^{\lambda/2} g(y)$ γ-Hölder continuous with $\gamma > 1/2$. Finally, for the last one we need $y^{\lambda/2} g(y)$ to be integrable as $y \to \infty$. Assuming power like behaviours we then need bounds on g of the form:

$$g(y) \le C y^{-\lambda/2 - r} \quad \text{as} \quad y \to +\infty, \tag{3.5}$$

$$g(y) \le C y^{-\lambda/2 - \rho} \quad \text{as} \quad y \to 0, \tag{3.6}$$

for some $r > 1$ and $\rho < 2$. We will solve the problem in functional spaces where (3.5) and (3.6) are satisfied in some averaged sense because we solve (3.1), (3.2) with initial value a Dirac mass.

Definition 3.4.

$$\mathcal{E} = \left\{g \in M^+(\mathbb{R}^+);\ ||g||_{\mathcal{E}} < +\infty\right\},$$
$$||g||_{\mathcal{E}} = \sup_{R\le 1}\left(R^{3/2}\frac{1}{R}\int_{[R/2,R]} dg\right) + \sup_{R\ge 1}\left(R^{(3+\lambda)/2}\frac{1}{R}\int_{[R/2,R]} dg\right),$$

where $M^+(\mathbb{R}^+)$ is the set of nonnegative Radon measures in $\mathbb{R}^+$. For all $T > 0$ we define:

$$\mathcal{H}_T = \left\{g \in L^\infty(0,T);\ \mathcal{E}\right\},$$
$$||g||_{\mathcal{H}_T} = \sup_{0\le t\le T} ||g(t)||_{\mathcal{E}}.$$

We state now the main results of this paper.

Theorem 3.5. *For all $x_0 > 0$, there exists a unique solution of (3.1), (3.2) $g \in \mathcal{H}_T$ for any $T > 0$. satisfying*

$$\lim_{t\to 0}\int_0^\infty g(t,x,x_0)\,\varphi(x)\,dx = \varphi(x_0) \tag{3.7}$$

for all $\varphi \in C_c(\mathbb{R}^+)$. Moreover $g \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ and has the self similar form

$$g(t,x,x_0) = \frac{1}{x_0} g\left(t x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right). \tag{3.8}$$

Remark 3.6. The self similar form (3.8) just follows from the rescaling properties of the problem (3.1), (3.2), (3.7). Notice that due to (3.8) it is enough to restrict our analysis to $x_0 = 1$.

Remark 3.7. It may be relevant to notice the strong regularising effect of Eq. (3.1) (3.2) compared to other kinetic models like for instance the Uehling-Uhlenbeck equation considered in [10].

Theorem 3.8. *Let g be the solution of (3.1), (3.2) that has been obtained in Theorem 3.5. Then, there exists positive constants δ and ϵ_1, only depending on λ, such that for any $0 < \varepsilon < \varepsilon_1$ the following statements hold: For all $t \ge 1$:*

$$g(t,x,1) = t^{\frac{2}{\lambda-1}}\varphi_1(\sigma) + \varphi_2(t,\sigma), \tag{3.9}$$

where σ is the self similar variable:

$$\sigma = t^{\frac{2}{\lambda-1}}\,x, \tag{3.10}$$

and the functions φ_1 and φ_2 satisfy the following estimates:

$$\varphi_1(\sigma) = \begin{cases} a_1\,\sigma^{-\frac{3}{2}} + \mathcal{O}_\varepsilon\left(\sigma^{-\frac{4-\lambda}{2}+\varepsilon}\right) & \textit{for}\ \ 0 \le \sigma < 1 \\ a_2\,\sigma^{-\frac{3+\lambda}{2}} + \mathcal{O}_\varepsilon\left(\sigma^{-(1+\lambda-\varepsilon)}\right) & \textit{for}\ \ \sigma > 1, \end{cases} \tag{3.11}$$

where a_1 and a_2 are two explicit constants,

$$\varphi_2(t,\sigma) = \begin{cases} b_1(t)\,\sigma^{-\frac{3}{2}} + \mathcal{O}\left(t^{\frac{2}{\lambda-1}-\delta}\,\sigma^{-\frac{3}{2}+\delta}\right) & \textit{for}\ \ 0 \le \sigma < 1 \\ b_2(t)\,\sigma^{-\frac{3+\lambda}{2}} + \mathcal{O}\left(t^{\frac{2}{\lambda-1}-\delta}\,\sigma^{-\frac{3+\lambda}{2}-\delta}\right) & \textit{for}\ \ \sigma > 1, \end{cases} \tag{3.12}$$

where b_1 *and* b_2 *are two continuous functions such that* $|b_1(t)| + |b_2(t)| \leq C\, t^{\frac{2}{\lambda-1}-\delta}$. *For all* $0 < t < 1$:

$$g(t,x,1) = \begin{cases} t\,x^{-\frac{3}{2}} + b_3(t)\,x^{-\frac{3}{2}} + \mathcal{O}\left(t\,x^{-\frac{3}{2}+\delta}\right) & \textit{for } 0 \leq x \leq \frac{1}{2}, \\ a_3\,t\,x^{-\frac{3+\lambda}{2}} + b_4(t)\,x^{-\frac{3+\lambda}{2}} + \mathcal{O}\left(t\,x^{-\frac{3+\lambda}{2}-\delta}\right) & \textit{for } x \geq \frac{3}{2}, \\ \mathcal{O}_\varepsilon\left(\frac{t^{1-2\varepsilon}}{|x-1|^{\frac{3}{2}-\varepsilon}}\right) & \textit{for } t^2 < |x-1| < \frac{1}{2}, \end{cases} \tag{3.13}$$

where a_3 *is an explicit numerical constant and* b_3 *and* b_4 *are continuous functions such that* $|b_3(t)| + |b_4(t)| \leq Ct^{1+\delta}$. *Finally*:

$$\lim_{t\to 0} t^2 g(t, 1+t^2\chi, 1) = \Psi(\chi) \quad \textit{uniformly on compact subsets of } \mathbb{R}, \tag{3.14}$$

where the function Ψ *is given by*:

$$\Psi(\chi) = \begin{cases} \frac{\pi}{\chi^{3/2}}\,e^{-\frac{\pi}{\chi}}, & \textit{for all } \chi \geq 0, \\ 0 & \textit{for all } \chi < 0. \end{cases} \tag{3.15}$$

Remark 3.9. We use the standard notation $f = \mathcal{O}(g)$ to denote that $|f| \leq Cg$ in the region under consideration with C depending only on λ. The notation $f = \mathcal{O}_\varepsilon(g)$ has a similar meaning but with C depending also on ε.

Remark 3.10. For $t \geq 1$, (3.11) (3.12) imply:

$$g(t,x,1) = \begin{cases} a_1\,t^{-\frac{1}{\lambda-1}}\,x^{-\frac{3}{2}} + \mathcal{O}\left(t^{-\frac{1-2\delta}{\lambda-1}}\,x^{-\frac{3}{2}+\delta} + t^{-\frac{1}{\lambda-1}-\delta}\,x^{-\frac{3}{2}}\right), \\ \qquad\qquad \text{for } 0 < x < t^{-\frac{2}{\lambda-1}}, \\ a_2\,t^{-\frac{\lambda+1}{\lambda-1}}\,x^{-\frac{3+\lambda}{2}} + \mathcal{O}\left(t^{-\frac{\lambda+1+2\delta}{\lambda-1}}\,x^{-\frac{3+\lambda}{2}-\delta} + t^{-\frac{\lambda+1}{\lambda-1}-\delta}\,x^{-\frac{3+\lambda}{2}}\right) \\ \qquad\qquad \text{for } x > t^{-\frac{2}{\lambda-1}}. \end{cases} \tag{3.16}$$

Our strategy to solve the problem (3.1), (3.2), (3.7) is to use Fourier analysis. The resulting problem is explicitly solvable by means of the Wiener Hopf method [2]. Using the representation formula for the solution, we then prove Theorem 3.5 and Theorem 3.8 by deriving suitable a priori estimates. Related arguments have been used in [10].

3.1. Fourier variables. We reformulate the original problem using Fourier variables. To this end we define $x = e^X$, $X \in \mathbb{R}$, as well as the Fourier transform

$$\widehat{G}(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iX\xi}\, G(t,X)\,dX, \quad G(t,X) = g(t,e^X). \tag{3.17}$$

Then, the problem (3.1), (3.2), (3.7) reads in terms of the new variables:

$$\frac{\partial \widehat{G}}{\partial t}(t,\xi) = \widehat{G}\left(t, \xi + \frac{\lambda-1}{2}i\right) \Phi\left(\xi + \frac{\lambda-1}{2}i\right), \tag{3.18}$$

$$\widehat{G}(0,\xi) = \frac{1}{\sqrt{2\pi}}, \tag{3.19}$$

where the function Φ is given by:

$$\Phi(\xi) = -\frac{2\sqrt{\pi}\ \Gamma(i\xi + 1 + \frac{\lambda}{2})}{\Gamma(i\xi + \frac{\lambda+1}{2})}. \tag{3.20}$$

Indeed, this is a consequence of the fact that $\widehat{G}(\xi, t) = \frac{1}{\sqrt{2\pi}}\mathcal{M}(g)(-i\,\xi)$, where $\mathcal{M}(g)$ is the Mellin transform defined in Sect. 12 (cf. (12.1)). Notice that $\Phi(\xi) = P(i\xi + 1 + \lambda/2)$, where P is as in (12.5). The fact that the function $g(t, \cdot, 1) \in \mathcal{E}$ implies that:

$$\widehat{G}(t, \cdot) \ \text{ is analytic in } \ \mathcal{S} = \{\xi \in \mathbb{C};\ \ Im(\xi) \in (3/2, (3+\lambda)/2)\}. \tag{3.21}$$

Problems of the form (3.18), (3.19) and (3.21) are a particular case of so called Carleman's equations (cf. [2]).

4. The Auxiliary Function

We now summarise some properties of the function Φ in (3.20).

Proposition 4.1. *The function Φ is a meromorphic function in $\mathbb{C}$ with simple poles at:*

$$\xi_p(n) = i\left(1 + \frac{\lambda}{2} + n\right), \quad n = 0, 1, \cdots, \tag{4.1}$$

and whose zeros are:

$$\xi_z(n) = i\left(n + \frac{1+\lambda}{2}\right), \quad n = 0, 1, \cdots. \tag{4.2}$$

For all $M > 0$ fixed:

$$\Phi(\xi) = -\sqrt{2\pi}(1 + iQ)\sqrt{Q\,\xi} + \frac{\sqrt{2\pi}(1 + iQ)i}{\xi}\sqrt{Q\,\xi}\left(\frac{1}{8} + \frac{\lambda}{4}\right) + \mathcal{O}\left(\frac{1}{|\xi|^{3/2}}\right)$$

as $Re(\xi) \to \infty$, uniformly on $Im(\xi) \in (-M, M)$, and where the function Q is defined as:

$$Q \equiv Q(\xi) = sgn\,(Re(\xi)) \tag{4.3}$$

with $sgn(0) = 0$.

Proof. The properties of poles and zeros are a consequence of Proposition 12.2 (Fig. 1). The asymptotic behaviour of $\Phi(\xi)$ as $|Re(\xi)| \to +\infty$ follows from Proposition 12.3 and Taylor's expansion. □

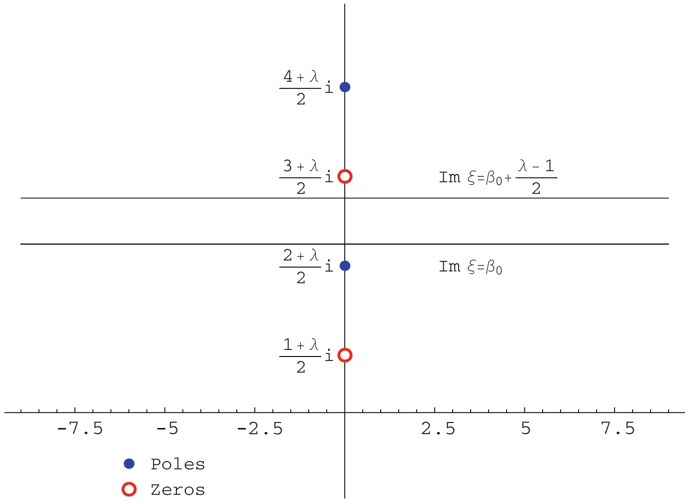

Fig. 1. Some relevant zeros and poles of the function Φ

5. Solving (3.18)-(3.19)

Our goal is to solve the problem (3.18)-(3.19) with $\widehat{G}$ analytic in the strip $\mathcal{S}$. Since we are interested in deriving a solution $G(t, X)$ in the sense of distributions, we want to obtain boundedness of $\widehat{G}$ as $|Re(\xi)| \to \infty$. We will actually obtain, for the particular solution $\widehat{G}(t, \cdot)$ constructed here, exponential decay, something that means that $G(t, \cdot) \in C^{\infty}$ for $t > 0$.

The following transformation allows to reduce (3.18), (3.19) to an equation for a function depending only on one variable ξ:

$$\widehat{G}(t, \xi) = -\frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{Im(y)=\beta_0} \frac{\mathcal{V}(\xi)}{\mathcal{V}(y)} t^{\frac{2i(\xi-y)}{\lambda-1}} \Gamma\left(-\frac{2i(\xi-y)}{\lambda-1}\right) dy \tag{5.1}$$

for some

$$\beta_0 \in (3/2, 2). \tag{5.2}$$

5.1. A heuristic derivation of (5.1). A heuristic explanation for the formula (5.1) can be given using the Laplace transform. Suppose that we define the Laplace transform of $\widehat{G}(t, \xi)$ in t as:

$$\widetilde{G}(z, \xi) = \int_0^{\infty} \widehat{G}(t, \xi) e^{-zt} dt.$$

Then, (3.18), (3.19) becomes:

$$z\widetilde{G}(z, \xi) = \widetilde{G}\left(t, \xi + \frac{\lambda-1}{2} i\right) \Phi\left(\xi + \frac{\lambda-1}{2} i\right) + \frac{1}{\sqrt{2\pi}}. \tag{5.3}$$

The solution of this equation can be formally reduced to (5.11) by means of the transformation:

$$\widetilde{G}(z,\xi) = \exp\left(-\frac{2i}{\lambda-1}\log(-z)\,\xi\right)\mathcal{V}(\xi)\,H(z,\xi)\,. \tag{5.4}$$

The reason for using $\log(-z)$ instead of $\log(z)$ is that $\widetilde{G}(\cdot,\xi)$ can be expected to be analytic for $Re(z) > a$ for some $a \in \mathbb{R}$ and in this form the function $\log(-z)$ can be expected to be analytic in this region assuming that $\arg(z) \in (-2\pi, 0)$. The transformation (5.4) brings (5.3) to

$$H(z,\xi) - H\left(z,\xi + \frac{\lambda-1}{2}i\right) = \frac{1}{\sqrt{2\pi}}\frac{e^{\frac{2i}{\lambda-1}\log(-z)\xi}}{z\mathcal{V}(\xi)}. \tag{5.5}$$

Equation (5.5) can be transformed into a Riemann-Hilbert problem by means of the following conformal mapping:

$$H(z,\xi) = h(z,\zeta)\,,\quad \zeta = e^{\frac{4\pi}{\lambda-1}(\xi-\beta_0 i)} \tag{5.6}$$

where, for the sake of simplicity we will write, with some slight abuse of notation

$$\mathcal{V}(\xi) = \mathcal{V}(\zeta)\,.$$

Then (5.5) becomes:

$$h(z,\zeta+i0) - h(z,\zeta-i0) = \frac{e^{\frac{4\pi}{\lambda-1}\beta_0\alpha(z)}}{\sqrt{2\pi}}\frac{\zeta^{\alpha(z)}}{z\mathcal{V}(\zeta)},\quad \zeta\in\mathbb{R}^+ \tag{5.7}$$

with h analytic in $\mathbb{C}\backslash\mathbb{R}^+$ and:

$$\alpha(z) = \frac{1}{2\pi i}\arg(-z)\,.$$

It is well known that the solution of Riemann-Hilbert problems can be obtained using Wiener Hopf methods (cf. [16,17]). However, in this particular case, assuming that $\frac{\zeta^{\alpha(z)}}{\mathcal{V}(\zeta)}$ satisfies suitable boundedness estimates for small and large ζ, we can solve (5.7) just using Cauchy's formula to obtain:

$$h(z,\zeta) = \frac{1}{2\pi i}\frac{1}{\sqrt{2\pi}}\frac{e^{\frac{4\pi}{\lambda-1}\beta_0\alpha(z)}}{z}\int_0^\infty \frac{s^{\alpha(z)}}{\mathcal{V}(s)}\frac{ds}{(s-\zeta)},$$

and, using (5.6):

$$H(z,\xi) = \frac{1}{2\pi i}\frac{1}{\sqrt{2\pi}}\frac{1}{z}\int_{-\infty}^{\infty}\frac{e^{\frac{4\pi\alpha(z)}{\lambda-1}y}}{\mathcal{V}(y)}\frac{dy}{1-e^{\frac{4\pi}{\lambda-1}(\xi-y)}}. \tag{5.8}$$

It then follows from (5.4) that:

$$\widetilde{G}(z,\xi) = \frac{1}{2\pi i}\frac{1}{\sqrt{2\pi}}\frac{1}{z}\mathcal{V}(\xi)\int_{-\infty}^{\infty}\frac{e^{\frac{4\pi\alpha(z)}{\lambda-1}(y-\xi)}}{\mathcal{V}(y)}\frac{dy}{1-e^{\frac{4\pi}{\lambda-1}(\xi-y)}}, \tag{5.9}$$

and inverting the Laplace transform we finally obtain (5.1).

5.2. Proving that (5.1) solves (3.18) (3.19). We have:

Lemma 5.1. *Suppose that $\mathcal{V}(\eta)$ is analytic in the strip $\mathcal{S}$ and that $\mathcal{V}$ satisfies*

$$\int_{Im(y)=\beta_0} \left| \frac{1}{\mathcal{V}(y)} \right| e^{-\frac{\pi}{(\lambda-1)}|y|} \sqrt{|y|+1}\, |dy| < \infty \tag{5.10}$$

for any $\beta_0 \in \left(\frac{3}{2}, 2\right)$, as well as:

$$\mathcal{V}(\eta) = -\mathcal{V}\left(\eta + \frac{\lambda-1}{2} i\right) \Phi\left(\eta + \frac{\lambda-1}{2} i\right) \tag{5.11}$$

for $Im(\eta) \in \left(\frac{3}{2}, 2\right)$. Define $\widehat{G}(t,\xi)$ by means of (5.1) for $Im(\xi) > \beta_0$. Then $\widehat{G}$ can be extended analytically to $\mathcal{S}$ and it solves (3.18), (3.19) for $Im(\xi) \in \left(\frac{3}{2}, 2\right)$.

Remark 5.2. In formula (5.11) and all the remainder of the paper we denote as $|dy|$ the total variation of the signed measure dy.

Proof. It just follows by direct computation. Indeed, notice that Stirling's formula, that is uniformly valid for $\Gamma(z)$, $\arg(z) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ with $\varepsilon_0 > 0$ (cf. [1]) implies:

$$\left| \Gamma\left(-\frac{2i(\xi - y)}{\lambda - 1} \right) \right| \le C_R \frac{e^{-\frac{\pi}{(\lambda-1)}|y|}}{\sqrt{|y|+1}}$$

for $|\xi| \le R$, $Im(y) = \beta_0$. Therefore, the integral on the right-hand side of (5.1) converges for any $\xi \in \mathcal{S} \cap \left\{ \xi : Im(\xi) \in \left(\beta_0, \frac{3+\lambda}{2}\right) \right\}$ due to (5.10) and the function $\widehat{G}(t,\xi)$ satisfies:

$$\left| \widehat{G}(t,\xi) \right| \le C_R \int_{Im(y)=\beta_0} \left| \frac{1}{\mathcal{V}(y)} \right| e^{-\frac{\pi}{(\lambda-1)}|y|} \frac{|dy|}{\sqrt{|y|+1}} < \infty, \quad |\xi| \le R.$$

Taking β_0 arbitrarily close to $\frac{3}{2}$ we obtain analyticity of $\widehat{G}$ in $\mathcal{S}$.

Moreover, the derivative with respect to t of $\widehat{G}$ in (5.1) can be computed by means of:

$$\frac{\partial \widehat{G}}{\partial t}(t,\xi) = \frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{Im(y)=\beta_0} \frac{\mathcal{V}(\xi)}{\mathcal{V}(y)} t^{\left[\frac{2i(\xi-y)}{\lambda-1} - 1\right]} \Gamma\left(-\frac{2i(\xi-y)}{\lambda-1} + 1 \right) dy, \tag{5.12}$$

where we have used $z\Gamma(z) = \Gamma(z+1)$. On the other hand, using (5.1) we obtain:

$$\widehat{G}\left(t, \xi + \frac{(\lambda-1)}{2} i\right) = -\frac{\sqrt{2}}{\sqrt{\pi} i (\lambda-1)} \int_{Im(y)=\beta_0} \frac{\mathcal{V}\left(\xi + \frac{(\lambda-1)}{2} i\right)}{\mathcal{V}(y)} t^{\frac{2i(\xi-y)}{\lambda-1} - 1} \times \Gamma\left(-\frac{2i(\xi-y)}{\lambda-1} + 1 \right) dy,$$

and using (5.11) and (5.12), (3.18) follows.

It only remains to check (3.19). To this end we use contour deformation and the residue theorem to transform (5.1) into:

$$\widehat{G}(t,\xi)=\frac{1}{\sqrt{2\pi}}-\widehat{G}_r(t,\xi),$$
$$\widehat{G}_r(t,\xi)=\frac{\sqrt{2}}{\sqrt{\pi}i(\lambda-1)}\int_{Im(y)=Im(\xi)+\delta_1}\frac{\mathcal{V}(\xi)}{\mathcal{V}(y)}t^{\frac{2i(\xi-y)}{\lambda-1}}\Gamma\left(-\frac{2i(\xi-y)}{\lambda-1}\right)dy,$$

with $\delta_1>0$ small. Using (5.10) it follows that:

$$\left|\widehat{G}_r(t,\xi)\right|\le C_R\, t^{\frac{2\delta_1}{\lambda-1}},\quad \xi\in\mathcal{S}\cap\{|\xi|\le R\},$$

and therefore $\widehat{G}_r$ converges to zero as $t\to 0$ uniformly in bounded sets of ξ, whence (3.19) follows. □

6. On the solutions of (5.11)

Equation (5.11) admits infinitely many solutions. Indeed, given any solution $\mathcal{V}_{part}(\xi)$ we can obtain any other one by means of:

$$\mathcal{V}(\xi)=\mathcal{V}_{part}(\xi)\,p(\xi),$$

where

$$p(\xi)=p\left(\xi+\frac{\lambda-1}{2}i\right). \tag{6.1}$$

Notice that (6.1) has infinitely many solutions, some of them being $e^{4\pi\ell\xi/(\lambda-1)}$, $\ell\in\mathbb{N}$, and linear combinations of them. Given such a non uniqueness a natural and essential question is then how to choose one of them. We may state several sufficient conditions that would ensure that $\widehat{G}$ is the Fourier transform of a tempered distribution. First we want the function $\widehat{G}$ to be defined. This is guaranteed by the condition (5.10) above. However, this condition is not sufficient to prove that $\widehat{G}(t,\xi)$ is globally bounded with respect to ξ. The difficulty comes from the fact that, if the behaviours of $\mathcal{V}(\xi)$ are too disparate as $Re(\xi)$ tends to plus or minus infinity, the quotient $\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}$ may be strongly increasing in some regions of the integral in (5.1). A sufficient condition to avoid this difficulty is to have:

$$|\mathcal{V}(\xi)|\approx e^{B_\pm|\xi|},\quad |B_\pm|\le\frac{\pi}{(\lambda-1)} \tag{6.2}$$

as $Re(\xi)\to\pm\infty$. The decay rate of the Gamma function in (5.1) may then control the possible growth of the quotient $\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}$ uniformly on ξ.

Another requirement that we need for the function $\mathcal{V}$ comes from the requirement that $\widehat{G}$ must be analytic in the strip $\mathcal{S}$. This is ensured by imposing also that $\mathcal{V}$ is also analytic in $\mathcal{S}$.

We will then construct a function $\mathcal{V}$ analytic in that strip, satisfying Eq. (5.11) for $Im(\xi)\in(3/2,2)$, and conditions (5.10) and (6.2).

A particular solution of (5.11) can be easily obtained using Cauchy's formula. To this end we take the logarithm of both sides of (5.11) to obtain:

$$\log\left(\mathcal{V}\left(\xi\right)\right)=\log\left(\mathcal{V}\left(\xi+\frac{\lambda-1}{2}i\right)\right)+\log\left(-\Phi\left(\xi+\frac{\lambda-1}{2}i\right)\right) \tag{6.3}$$

or equivalently

$$\log\left(\mathcal{V}\left(\xi-\frac{\lambda-1}{2}i\right)\right)=\log\left(\mathcal{V}\left(\xi\right)\right)+\log\left(-\Phi\left(\xi\right)\right). \tag{6.4}$$

Let us take any β_1 such that $\Phi\left(\xi\right)$ has no zeros nor poles along the line $Im\left(\xi\right)=\beta_1$. We define:

$$\psi\left(\zeta\right)=\log\left(\mathcal{V}\left(\xi\right)\right),\quad \zeta=e^{\frac{4\pi}{\lambda-1}\left(\xi-\beta_1 i\right)},\quad Q\left(\zeta\right)=\log\left(-\Phi\left(\xi\right)\right).$$

Equation (6.4) then becomes

$$\psi\left(\zeta+i0\right)=\psi\left(\zeta-i0\right)+Q\left(\zeta-i0\right),\quad \zeta\in\mathbb{R}^+ \tag{6.5}$$

with ψ analytic in $\mathbb{C}\backslash\mathbb{R}^+$. Taking into account that $\left|Q\left(\zeta\right)\right|\leq C\left(1+\left|\log\left(\zeta\right)\right|\right)$ we can obtain a particular solution of (6.5) as:

$$\psi\left(\zeta\right)=\frac{1}{2\pi i}\int_{\mathbb{R}^+}Q\left(s\right)\left[\frac{1}{s-\zeta}-\frac{1}{s+1}\right]ds,$$

where the term $1/(s+1)$ has been added to the classical Cauchy integral in order to ensure the convergence of the integral. Then, returning to the variable ξ we obtain:

$$\mathcal{V}_{part,\beta_1}\left(\xi\right)=\exp\left(\frac{2}{\left(\lambda-1\right)i}\int_{Im(\eta)=\beta_1}\log\left(-\Phi\left(\eta\right)\right)\left[\frac{1}{1-e^{\frac{4\pi}{\lambda-1}\left(\xi-\eta\right)}}-\frac{1}{1+e^{-\frac{4\pi}{\lambda-1}\eta}}\right]d\eta\right), \tag{6.6}$$

where $Im\left(\xi\right)\in\left(\beta_1-\frac{\lambda-1}{2},\beta_1\right)$.

Formula (6.6) provides a particular solution of (5.11). On the other hand, we can obtain an infinite family of solutions of (6.1) given by:

$$p\left(\xi\right)=e^{\frac{4\pi}{\lambda-1}\ell\xi},\quad \ell\in\mathbb{Z}. \tag{6.7}$$

Let us define a family of solutions of (5.11):

$$\mathcal{V}_\ell\left(\xi\right)=e^{\frac{4\pi\ell}{\lambda-1}\xi}\mathcal{V}_{part,\beta_1}\left(\xi\right). \tag{6.8}$$

Actually, using Fourier series, it can be seen that any solution of (6.1) can be written as an infinite linear combination of the functions $\mathcal{V}_\ell\left(\xi\right)$.

The formula (6.6) does not define uniquely the function $\mathcal{V}_{part,\beta_1}$ unless we prescribe the value of β_1 and the argument of the function $\ln(-\Phi(\eta))$. The different possible choices of this argument just differ by a factor $2\pi\ell i$ and therefore the resulting functions $\mathcal{V}_{part,\beta_1}$ would differ by a multiplicative factor (6.7). Proposition 4.1 implies that $\arg(-\Phi(\eta))\to\pi/4+2\pi\ell i$ as $Re(\eta)\to+\infty$. In order to avoid exponential factors in some of the forthcoming formulas, we determine uniquely the function $\ln(-\Phi(\eta))$ by choosing:

$$\lim_{Re(\eta)\to+\infty} \arg(-\Phi(\eta)) = \frac{\pi}{4}. \tag{6.9}$$

Notice that in the formula (6.6) there exists an infinite possibility of choices of the constant β_1. These functions may be extended analytically moving ξ and simultaneously the contour of integration in such a way that the condition $-(\lambda-1)/2 < Im(\xi-\eta) < 0$ always holds. The only true obstruction to extend analytically the functions $\mathcal{V}_{part,\beta_1}$ arises from crossing with the contour deformation the zeros or poles of the function Φ. Suppose that ξ_{sing} is a zero or a pole of Φ and β_1, $\widetilde{\beta}_1$ are such that:

$$\xi_{sing} - \frac{1}{2} < \beta_1 < \xi_{sing} < \widetilde{\beta}_1 < \xi_{sing} + \frac{1}{2}.$$

Then

$$\frac{\mathcal{V}_{part,\beta_1}(\xi)}{\mathcal{V}_{part,\widetilde{\beta}_1}(\xi)} = -\left(\frac{e^{\frac{4\pi}{\lambda-1}(\xi_{sing}-\xi)} - 1}{1 + e^{\frac{4\pi}{\lambda-1}\xi_{sing}}}\right)^{-n}, \tag{6.10}$$

where

$$n = \begin{cases} 1 & \text{if } \xi_{sing} \text{ is a zero} \\ -1 & \text{if } \xi_{sing} \text{ is a pole.} \end{cases} \tag{6.11}$$

Combining (6.10) with (6.6) we can then extend any function $\mathcal{V}_{part,\beta_1}$ to the whole complex plane as a meromorphic function. As it could be expected the different functions $\mathcal{V}_{part,\beta_1}$ can be related to each other by means of linear combinations of functions of the form given in (6.8).

In order to obtain the function $\mathcal{V}(\xi)$ with the properties requested above, it is sufficient to take

$$\mathcal{V}(\xi) = \mathcal{V}_{part,\beta_1}(\xi), \quad Im(\xi) \in \left(\beta_1 - \frac{\lambda-1}{2}, \beta_1\right), \tag{6.12}$$

with

$$\beta_1 \in \left(\frac{2+\lambda}{2}, \frac{3+\lambda}{2}\right). \tag{6.13}$$

Moving the contour of integration if needed, inside the strip $Im(\xi) \in (2+\lambda)/2, (3+\lambda)/2$ we obtain that the function $\mathcal{V}(\xi)$ has no zeros nor poles in the whole strip $\mathcal{S}$. It only remains to check that this function satisfies the two conditions (5.10) and (6.2). It follows from Proposition 13.2 in Sect. 13 that:

$$C_\varepsilon e^{-\frac{1}{2}\left(\frac{\pi}{\lambda-1}+\varepsilon\right)|\xi|} \le |\mathcal{V}(\xi)| \le C_\varepsilon e^{-\frac{1}{2}(\frac{\pi}{\lambda-1}-\varepsilon)|\xi|} \text{ for } \xi \in \mathcal{S} \tag{6.14}$$

for $\delta > 0$ arbitrarily small and $C_\delta > 0$ a constant depending on δ. This behaviour implies both (5.10) and (6.2).

Finally, we can extend $\mathcal{V}$ meromorphically using (5.11). The positions of the poles and zeros of $\mathcal{V}$ can then be obtained using (3.20) as well as the properties of the Gamma function.

Summarizing, we have shown:

Proposition 6.1. *The function $\mathcal{V}(\xi)$ defined by means of (6.6), (6.12) with β_1 as in (6.13) can be extended analytically to the strip $\mathcal{S}$ and meromorphically to the whole complex plane. It satisfies Eq. (5.11) as well as the estimates (6.14). Morever, $\mathcal{V}(\xi) \neq 0$ in all the strip $\mathcal{S}$ and we have the following representation formula:*

$$\frac{\mathcal{V}(\xi)}{\mathcal{V}(y)} = \exp\left[\frac{2}{(\lambda-1)i}\int_{Im\,\eta=\beta_1} \ln\left(-\Phi(\eta)\right) \times \left(\frac{1}{1-e^{\frac{4\pi}{\lambda-1}(\xi-\eta)}} - \frac{1}{1-e^{\frac{4\pi}{\lambda-1}(y-\eta)}}\right) d\eta\right] \tag{6.15}$$

for ξ and y such that $\beta_1-(\lambda-1)/2 < Im(\xi) < \beta_1$ and $\beta_1-(\lambda-1)/2 < Im(y) < \beta_1$. The poles of the function $\mathcal{V}$ are:

$$\left(\frac{1+\lambda}{2} + n + k\frac{\lambda-1}{2}\right)i, \quad n=1,2,\ldots, \quad k=0,1,\ldots, \tag{6.16}$$

$$\left(1+\frac{\lambda}{2} - k\frac{\lambda-1}{2}\right)i, \quad k=1,2,\ldots \tag{6.17}$$

The zeros of the function $\mathcal{V}$ are

$$\left(1+\frac{\lambda}{2} + n + k\frac{\lambda-1}{2}\right)i, \quad n=1,2,\ldots, \quad k=0,1,\ldots, \tag{6.18}$$

$$\left(\frac{1+\lambda}{2} - k\frac{\lambda-1}{2}\right)i, \quad k=1,2,\ldots. \tag{6.19}$$

Corollary 6.2. *The function $\widehat{G}(t,\xi)$ defined by means of (5.1) with $\mathcal{V}$ defined in Proposition 6.1, solves (3.18), (3.19).*

7. Uniqueness of Solutions

In this section we prove the uniqueness of the solution g in $\mathcal{H}_T$ stated in Theorem 3.5. Suppose that there exist two solutions g_1, g_2 in the space $\mathcal{H}_T$ of the problem (3.1), (3.2), (3.7). Then the difference $g = g_1 - g_2 \in \mathcal{H}_T$, solves (3.1), (3.2) and satisfies $g(t,\cdot) \rightharpoonup 0$ as $t \to 0^+$. We then define $\widehat{G}(t,\xi)$ as in (3.17). Notice that since $g \in \mathcal{H}_T$ we have that $\widehat{G}(t,\cdot)$ is analytic and bounded in $\mathcal{S}$ for $0 \le t \le T$ and it satisfies (3.18) in $(0,T)\times\mathcal{S}$. Moreover due to (3.7) we have $\lim_{t\to 0^+}\widehat{G}(t,\xi) = 0$ uniformly in compact sets of ξ. Let $\sigma(t)$ be a C^∞ cut-off function satisfying $\sigma(t)=1$ for $0 \le t \le T/2$, $\sigma(t)=0$ if $t \ge T$. Define $\overline{G}(t,\xi) = \widehat{G}(t,\xi)\,\sigma(t)$, then

$$\frac{\partial\overline{G}}{\partial t}(t,\xi) = \overline{G}\left(t, \xi + \frac{\lambda-1}{2}i\right)\Phi\left(\xi+\frac{\lambda-1}{2}i\right) + r(t,\xi), \tag{7.1}$$

where the function r is bounded in $(0,T)\times\mathcal{S}$ and $r(t,\cdot)\equiv 0$ for $0\le t\le T/2$. Taking the Laplace transform on t of (7.1) we obtain:

$$z\widetilde{G}(z,\xi) = \widetilde{G}\left(z,\xi+\frac{\lambda-1}{2}i\right)\Phi\left(\xi+\frac{\lambda-1}{2}i\right) + \widetilde{r}(z,\xi), \tag{7.2}$$

$$Re(z) > 0, \quad \xi \in \mathcal{S}, \tag{7.3}$$

where, for some positive constant C:

$$|\widetilde{r}(z,\xi)| \leq C\, e^{-\frac{T}{2}\,Re(z)} \quad \text{for all } \ \xi \in \mathcal{S},\ \ Re(z) > 0. \tag{7.4}$$

Due to the linearity of (7.2), (7.3) we can split $\widetilde{G}$ as $\widetilde{G} = \widetilde{G}_{part} + \widetilde{G}_{hom}$, where $\widetilde{G}_{part}$ may be obtained as in the heuristic argument in Subsect. 5.1 (cf. (5.9)):

$$\widetilde{G}_{part}\,(z,\xi) = \frac{1}{2\pi i}\frac{\mathcal{V}\,(\xi)}{z}\int_{-\infty}^{\infty}\frac{e^{\frac{4\pi\alpha(z)}{\lambda-1}(y-\xi)}}{\mathcal{V}\,(y)}\,\frac{\tilde{r}(z,\xi)\,dy}{1-e^{\frac{4\pi}{\lambda-1}(\xi-y)}}, \tag{7.5}$$

and where $\widetilde{G}_{hom}$ solves

$$z\,\widetilde{G}_{hom}\,(z,\xi) = \widetilde{G}_{hom}\left(z,\xi+\frac{\lambda-1}{2}i\right)\Phi\left(\xi+\frac{\lambda-1}{2}i\right), \tag{7.6}$$

$$Re(z) > 0,\ \ \xi \in \mathcal{S}. \tag{7.7}$$

We assume in (7.5) that $\mathcal{V}(\xi)$ is defined as in (6.6), (6.12) (cf. also Proposition 6.1). Then due to our choice of $\log(-z)$ we have $\alpha(z) \in (-3/4, -1/4)$ for $Re(z) > 0$. Therefore using also (6.14) and (7.4) it follows that the integral defining $\widetilde{G}_{part}$ in (7.5) is convergent for any $\xi \in \mathcal{S}$ and $\widetilde{G}_{part}(\cdot,\xi)$ is analytic in $Re(z) > 0$ for any $\xi \in \mathcal{S}$. Moreover,

$$|\widetilde{G}_{part}(z,\xi)| \leq C\, e^{-\frac{T}{2}\,Re(z)}, \quad \text{for all } \ \xi \in \mathcal{S},\ \ Re(z) > 0. \tag{7.8}$$

To study the function $\widetilde{G}_{hom}$ we define (cf. (5.4))

$$H(z,\xi) = \frac{\widetilde{G}_{hom}(z,\xi)\, e^{\frac{2i}{\lambda-1}\log(-z)\,\xi}}{\mathcal{V}(\xi)}, \tag{7.9}$$

and we then define $h(z,\xi)$ by means of (5.6). Then $h(z,\cdot)$ is analytic on $\mathbb{C}\backslash\mathbb{R}^{+}$ and

$$h\,(z,\zeta+i0) = h\,(z,\zeta-i0)\,,\quad \zeta \in \mathbb{R}^{+}, \tag{7.10}$$

whence $h(z,\cdot)$ is analytic in $\mathbb{C}\backslash\{0\}$. The boundedness of $\widetilde{G}$, (6.14), (7.8), (7.9) and (5.6),

$$|h(z,\zeta)| \leq C_{\varepsilon}(z)|\zeta|^{\frac{1}{2}-\alpha(z)}\max\{|\zeta|^{-\varepsilon}, |\zeta|^{\varepsilon}\}, \tag{7.11}$$

where ε can be chosen arbitrarily small. Liouville's theorem then implies $h(z,\zeta) = 0$ whence $\widetilde{G}_{hom} = 0$ and $\widetilde{G} = \widetilde{G}_{part}$. Laplace's inversion formula then yields

$$\overline{G}(t,\xi) = \frac{1}{2\pi\, i}\int_{b-i\,\infty}^{b+i\,\infty}\widetilde{G}_{part}(z,\xi)\, e^{z\,t}\,dz \tag{7.12}$$

for any $b > 0$. Therefore, (7.8) implies $\overline{G}(t,\xi) = \widehat{G}(t,\xi) = 0$ for all $0 \leq t < T/2$ and all $\xi \in \mathcal{S}$, whence $g(t,x) = 0$ for $0 \leq t \leq T/2$ and $x \in \mathbb{R}^{+}$ and the result follows. □

8. Decay Estimates for the Function $\widehat{G}(t, \xi)$

For any $L > 0$ we define

$$\mathcal{T}_L := \mathcal{S} \cup \{\xi \in \mathbb{C};\ |Im(\xi)| \le L,\ \ |\xi| \ge 2L\},$$

where $\mathcal{S}$ is defined in (3.21). The main result of this section is the following.

Proposition 8.1. *For any $L > 0$ there exists positive constants C and a, depending only on L, such that the function $\widehat{G}$ defined in (5.1) satisfies:*

$$|\widehat{G}(t, \xi)| \le C e^{-a\sqrt{|\xi|}t}, \tag{8.1}$$

$$\left(1 + |\xi|^{1/2}\right) \left| \frac{\partial}{\partial \xi} \widehat{G}(t, \xi) \right| + \left(1 + |\xi|^{3/2}\right) \left| \frac{\partial^2}{\partial \xi^2} \widehat{G}(t, \xi) \right| \le C t\, e^{-a\sqrt{|\xi|}t} \tag{8.2}$$

for all $t \in (0, 1)$, $\xi \in \mathcal{T}_L$.

We will prove this proposition by means of a general lemma that provides estimates of a class of integrals using contour deformation and the Laplace method. This technical result is one of the key ingredients of this paper. In order to formulate it we need some definitions.

$$\Theta(\sigma, Y) := \frac{1}{1 - e^{-\frac{4\pi}{\lambda-1}\sigma}} - \frac{1}{1 - e^{\frac{4\pi}{\lambda-1}(-\sigma+Y)}}, \tag{8.3}$$

$$\begin{aligned} \Psi(\xi, Y, t) := {} & \frac{2}{(\lambda - 1)\, i} \int_{Im\,\eta=\beta_1} \ln\left(-\Phi(\eta)\right) \Theta(\eta - \xi, Y) d\eta \\ & - \frac{2iY}{\lambda - 1} \ln(t) - \frac{2iY}{\lambda - 1} + \left(\frac{2iY}{\lambda - 1} - \frac{1}{2}\right) \ln\left(\frac{2iY}{\lambda - 1}\right), \end{aligned} \tag{8.4}$$

$$A(z) := \frac{\Gamma(z)}{\sqrt{2\pi} e^{-z} z^{z-1/2}}. \tag{8.5}$$

Stirling's formula yields $\lim A(z) = 1$ uniformly as $|z| \to \infty$ and $arg(z) \in (-\pi + \varepsilon, \pi - \varepsilon)$ for any $\varepsilon > 0$ small.

For any $\xi \in \mathbb{C}$, and $B > 0$ we set:

$$\begin{aligned} D(\xi, B) := \Bigg\{ & Z \in \mathbb{C};\ \ Im(Z) < 0,\ |Im(Z)| \le B \left| Re(Z) + \frac{Q}{8}\sqrt{|\xi|} \right|, \\ & sgn(Re(Z)) = sgn(Re(\xi)) \Bigg\} \end{aligned} \tag{8.6}$$

(cf. Fig. 6 in Sect. 14), with Q as in (4.3). We then have:

Lemma 8.2. *Suppose that $L > 0$, $B > 2\sqrt{\pi}$, $\gamma_1 > 0$. Assume that for every $\xi \in \mathcal{T}_L$ the function m is such that $m(\xi, \cdot)$ is analytic in*

$$D(\xi, B) \cup \left\{ Z \in \mathbb{C};\ Im(Z) \in \left[-\gamma_1, \gamma_1 + \frac{\lambda - 1}{2}\right]\right\}.$$

Let us consider the function:

$$W(t, \xi) = \int_{Im(Y)=-\gamma_1} m(\xi, Y) e^{\Psi(\xi, Y, t)}\, dY. \tag{8.7}$$

Then there exists $\xi_0 > 0$ sufficiently large and $C > 0$, both depending on L, B and γ_1, such that:

– *If*

$$|m(\xi, Y)| \leq 1 \tag{8.8}$$

then

$$|W(t,\xi)| \leq C\, e^{-a|\xi|^{1/2}\, t} \tag{8.9}$$

for all $t \in [0, 1]$ *and all* ξ *such that* $|Re(\xi)| \geq \xi_0$ *and* $\xi \in \mathcal{T}_L$.

– *If*

$$|m(\xi, Y)| \leq (1 + |Y|) \;\; \textit{and} \;\; m(\xi, 0) = 0 \tag{8.10}$$

then,

$$|W(t,\xi)| \leq C\, t\, |\xi|^{1/2}\, e^{-a|\xi|^{1/2}\, t} \tag{8.11}$$

for all $t \in [0, 1]$ *and all* ξ *such that* $|Re(\xi)| \geq \xi_0$ *and* $\xi \in \mathcal{T}_L$.

Proof of Lemma 8.2. We introduce the new variable Z as: $Y = \sqrt{|\xi|}\, Z$. Then the function Ψ becomes:

$$\begin{aligned}\Psi(\xi, Y, t) = \widetilde{\Psi}(\xi, Z, t) &= \frac{2}{(\lambda-1)\, i} \int_{Im\, \eta = \beta_0 + \frac{\lambda-1}{2} - \varepsilon} \ln\left(-\Phi(\eta)\right) \Theta(\eta - \xi, \sqrt{|\xi|} Z) d\eta \\ &\quad - \sqrt{|\xi|} \left(\frac{2iZ}{\lambda-1} \ln(t) + \frac{2iZ}{\lambda-1} - \left(\frac{2iZ}{\lambda-1} - \frac{1}{2\sqrt{|\xi|}} \right) \ln\left(\frac{2iZ}{\lambda-1} \right) \right. \\ &\quad \left. - \left(\frac{2iZ}{\lambda-1} - \frac{1}{2\sqrt{|\xi|}} \right) \ln |\xi|^{1/2} \right).\end{aligned} \tag{8.12}$$

The function $\widetilde{\Psi}(\xi, Z, t)$ can be extended analytically as a function of Z to the set $D(\xi, B) \cup B_{\sqrt{|\xi|}/8}(0)$ (cf. Lemma 14.1). Moreover we prove in Lemma 14.3 the existence of a critical point Z_c of the function $\widetilde{\Psi}(\xi, Z, t)$ with the asymptotics (14.11).

Using the analyticity properties of the functions $\widetilde{\Psi}(\xi, Z, t)$ and $m(\xi, Y)$ we can obtain the new representation formula for W using contour deformation:

$$W(t,\xi) = \sqrt{|\xi|} \int_{\mathcal{C}_t} e^{\widetilde{\Psi}(\xi, Z, t)} m(\xi, \sqrt{|\xi|} Z) dZ, \tag{8.13}$$

where $\mathcal{C}_t$ is defined as (see Fig. 2):

$$\begin{aligned}\mathcal{C}_t &= \left\{ Z_1 + \mathbb{R}^- \right\} \cup \gamma_{3,t}(M) \cup \gamma_{1,t}(M) \cup \gamma_{2,t}(M) \cup \left\{ Z_2 + \mathbb{R}^+ \right\}, \\ Z_1 &= Re(Z_c) - M\, t + i\, \gamma_1 t, \quad Z_2 = Re(Z_c) + M\, t + i\, \gamma_1 t\end{aligned}$$

with $\gamma_{\ell,t}(M)$, $\ell = 1, 2, 3$ and M are as in (14.19)-(14.21).
Let $R > 0$ be as in Lemmas (14.8) and (14.9). We now consider separately the two different cases $|\xi|^2\, t \geq R$ and $|\xi|^2\, t \leq R$ bounded.

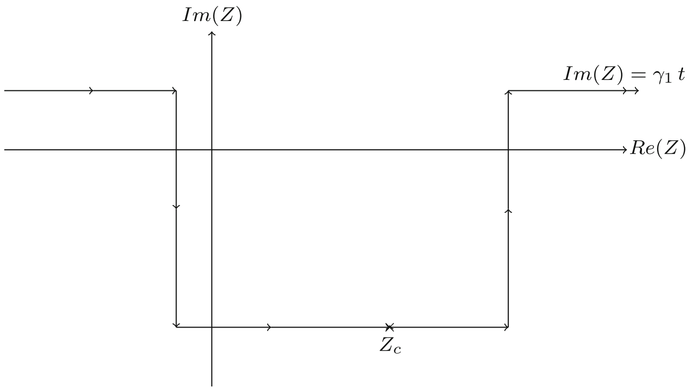

Fig. 2. The curve $\mathcal{C}_t$

The estimate of $W(t,\xi)$ for $|\xi|^2 t \geq R$. We write the function $W(t,\xi)$ as follows:

$$W(t,\xi) = \mathcal{I}_1 + \mathcal{I}_2, \tag{8.14}$$

$$\mathcal{I}_1 = \sqrt{|\xi|} \int_{\mathcal{C}_t \cap \{Z;\, |Z - Z_c| \leq \varepsilon t\}} e^{\widetilde{\Psi}(\xi,Z,t)} m(\xi, \sqrt{|\xi|} Z) dZ, \tag{8.15}$$

$$\mathcal{I}_2 = \sqrt{|\xi|} \int_{\mathcal{C}_t \cap \{Z;\, |Z - Z_c| \geq \varepsilon t\}} e^{\widetilde{\Psi}(\xi,Z,t)} m(\xi, \sqrt{|\xi|} Z) dZ \tag{8.16}$$

for some positive constant ε to be fixed.

We estimate the integral $\mathcal{I}_1$ using Lemmas 14.1–14.5 and Taylor's expansion we obtain:

$$\widetilde{\Psi}(\xi, Z, t) = \widetilde{\Psi}(\xi, Z_c, t) - \frac{1}{2} \frac{\sqrt{|\xi|}}{\sqrt{2\pi}\, t\, (1 + i\, Q)} (1 + \delta(\xi, Z, t)) |Z - Z_c|^2, \tag{8.17}$$

where $|\delta(\xi, Z, t)|$ can be made arbitrarily small if $|\xi| t^2 \geq R$ and ε sufficiently small. Therefore:

$$\mathcal{I}_1 = \sqrt{|\xi|} e^{\widetilde{\Psi}(\xi,Z_c,t)} \int_{\mathcal{C}_t \cap \{Z;\, |Z - Z_c| \leq \varepsilon t\}} e^{-\frac{1}{2} \frac{\sqrt{|\xi|}}{\sqrt{2\pi}\, t\, (1+i\, Q)} (1+\delta(\xi,Z,t)) |Z - Z_c|^2} m(\xi, \sqrt{|\xi|} Z) dZ$$

which gives, in the case (8.8):

$$|\mathcal{I}_1(t,\xi)| \leq C\, e^{-a\sqrt{|\xi|}\, t} \quad \text{if } |\xi|\, t^2 \geq R, \ \ 0 < t < 1, \tag{8.18}$$

and in the case (8.10):

$$|\mathcal{I}_1(t,\xi)| \leq C\, t \sqrt{|\xi|}\, e^{-a\sqrt{|\xi|}\, t} \quad \text{if } |\xi|\, t^2 \geq R, \ \ 0 < t < 1. \tag{8.19}$$

We must now estimate the $\mathcal{I}_2$ given by (8.16). To this end we split the integral as follows:

$$\mathcal{I}_2 = \mathcal{I}_{2,1} + \mathcal{I}_{2,2} + \mathcal{I}_{2,3},$$
$$\mathcal{I}_{2,1} = \sqrt{|\xi|}\int_{\mathcal{C}_t\cap\{Z;\ |Z-Z_c|\geq\varepsilon t\}\cap\gamma_t(M)} e^{\widetilde{\Psi}(\xi,Z,t)} m(\xi,\sqrt{|\xi|}Z)dZ,$$
$$\mathcal{I}_{2,2} = \sqrt{|\xi|}\int_{\mathcal{C}_t\setminus\gamma_t(M)\cap\{Z;\ |Z|\leq\varepsilon_1\sqrt{|\xi|}\}} e^{\widetilde{\Psi}(\xi,Z,t)} m(\xi,\sqrt{|\xi|}Z)dZ,$$
$$\mathcal{I}_{2,3} = \sqrt{|\xi|}\int_{\mathcal{C}_t\setminus\gamma_t(M)\cap\{Z;\ |Z|\geq\varepsilon_1\sqrt{|\xi|}\}} e^{\widetilde{\Psi}(\xi,Z,t)} m(\xi,\sqrt{|\xi|}Z)dZ,$$

where $\gamma_t(M)$ is the portion of the curve $\mathcal{C}_t$ along which $Im(Z) < \gamma_1 t$ and ε_1 is as in Lemma 14.9. Using Lemma 14.8 it follows that, in the case (8.8):

$$\begin{aligned}\left|\mathcal{I}_{2,1}\right| &\leq Ce^{\widetilde{\Psi}(\xi,Z_c,t)}e^{-\delta_0\sqrt{|\xi|}\,t}\sqrt{|\xi|}\int_{\mathcal{C}_t\cap\{Z;\ |Z-Z_c|\geq\varepsilon t\}\cap\gamma_t(M)} |m(\xi,\sqrt{|\xi|}Z)|dZ\\ &\leq Ce^{\widetilde{\Psi}(\xi,Z_c,t)}e^{-\delta_0\sqrt{|\xi|}\,t}\sqrt{|\xi|}\,t \leq Ce^{\widetilde{\Psi}(\xi,Z_c,t)}e^{-\frac{\delta_0\sqrt{|\xi|}}{2}\,t},\end{aligned} \tag{8.20}$$

and in the case (8.10):

$$\left|\mathcal{I}_{2,1}\right| \leq Ce^{\widetilde{\Psi}(\xi,Z_c,t)}e^{-\delta_0\sqrt{|\xi|}\,t}(\sqrt{|\xi|}\,t)^2 \leq C(\sqrt{|\xi|}\,t)\,e^{\widetilde{\Psi}(\xi,Z_c,t)}e^{-\frac{\delta_0\sqrt{|\xi|}}{2}\,t}. \tag{8.21}$$

The estimate of $\mathcal{I}_{2,2}$ follows using Lemma 14.9. In the case (8.8) we obtain:

$$\left|\mathcal{I}_{2,2}\right| \leq C\sqrt{|\xi|}\int_{|Z|\geq M\,t} e^{-a\sqrt{|\xi|}|Z|}\,dZ = Ce^{-a\,M\sqrt{|\xi|}t}, \tag{8.22}$$

and in case (8.10) we have:

$$\left|\mathcal{I}_{2,2}\right| \leq C\sqrt{|\xi|}\int_{|Z|\geq M\,t} e^{-a\sqrt{|\xi|}|Z|}\sqrt{|\xi|}\,Z\,dZ \leq C(\sqrt{|\xi|}\,t)e^{-a\,M\sqrt{|\xi|}t}, \tag{8.23}$$

since $|\xi|^2\,t \geq R$.

The third integral $\mathcal{I}_{2,3}$ is estimated using Proposition 13.2. To this end we use the variable Y and the identity:

$$e^{\Psi(\xi,Z,t)} = t^{-\frac{2iY}{\lambda-1}}\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}\frac{\Gamma\left(\frac{2iY}{\lambda-1}\right)}{A\left(\frac{2iY}{\lambda-1}\right)},$$

where A is defined by formula (8.5). Then

$$|\mathcal{I}_{2,3}| \leq \int_{Im(Y)=\gamma_1,|Y|\geq\varepsilon_1|\xi|}\left|\frac{m(\xi,Y)}{A\left(\frac{2iY}{\lambda-1}\right)}\right|\left|\frac{\mathcal{V}(\xi)}{\mathcal{V}(Y+\xi)}\right|\left|t^{-\frac{2iY}{\lambda-1}}\right|\left|\Gamma\left(\frac{2iY}{\lambda-1}\right)\right|dY. \tag{8.24}$$

Using that $\gamma_1 > 0$, $0 \leq t \leq 1$ and Stirling's formula, it follows that, in both cases (8.8) and (8.10):

$$|\mathcal{I}_{2,3}| \leq \int_{Im(Y)=\gamma_1,|Y|\geq\varepsilon_1|\xi|}(1+|Y|)\left|\frac{\mathcal{V}(\xi)}{\mathcal{V}(Y+\xi)}\right|e^{-\frac{\pi}{\lambda-1}|Y|}\,dY.$$

Proposition 13.2 gives the following bounds:

$$|\mathcal{I}_{2,3}| \leq C \int_{Im(Y)=\gamma_1, |Y|\geq \varepsilon_1 |\xi|} (1+|Y|)\, e^{\varepsilon\,\pi\,|\xi|}\, e^{-\frac{\pi}{2(\lambda-1)}|Y|} dY$$
$$\leq Ce^{-\frac{\pi}{2(\lambda-1)}\varepsilon_1 |\xi|} e^{\varepsilon\,\pi\,|\xi|}. \tag{8.25}$$

This last term can be estimated by the right hand side of (8.9) choosing ε sufficiently small. Combining this with (8.20) and (8.22) we obtain (8.9) for $|\xi| t^2 \geq R$. In the case (8.10), we have for some $\delta_1 > 0$,

$$|\mathcal{I}_{2,3}| \leq Ce^{-\delta_1 |\xi|} \leq Ce^{-\frac{\delta_1}{2}|\xi|} e^{-\frac{\delta_1}{2t^2}} \leq Ce^{-\frac{\delta_1}{2}\sqrt{|\xi|}\,t} e^{-\frac{\delta_1}{2t^2}}$$

which is estimated by the right hand side of (8.11). Combining this with (8.21) and (8.23) we obtain (8.11) for $|\xi| t^2 \geq R$.

This ends the proof of the estimate of $\mathcal{I}_2$ and then of Proposition 8.1 in the domain where $|\xi|\, t^2 \geq R$, $0 < t < 1$.

The estimate of $W(t,\xi)$ *for* $|\xi|^2\, t \leq R$.. We suppose first that condition (8.8) holds. We split the integral in (8.33) in two pieces:

$$W(t,\xi) = J_1 + J_2, \tag{8.26}$$
$$J_1(t,\xi) = \int_{Im\,Y=-\gamma_1\,t,\ |Y|\leq\varepsilon_0|\xi|} \frac{m(\xi,Y)}{A\left(\frac{2\,i\,Y}{\lambda-1}\right)} \frac{\mathcal{V}(\xi)}{\mathcal{V}(Y+\xi)} t^{-\frac{2iY}{\lambda-1}}\, \Gamma\left(\frac{2iY}{\lambda-1}\right) dY,$$
$$J_2(t,\xi) = \frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)} \int_{Im\,Y=-\gamma_1\,t,\ |Y|\geq\varepsilon_0|\xi|} \frac{m(\xi,Y)}{A\left(\frac{2\,i\,Y}{\lambda-1}\right)} \frac{\mathcal{V}(\xi)}{\mathcal{V}(Y+\xi)} t^{-\frac{2iY}{\lambda-1}}$$
$$\times \Gamma\left(\frac{2iY}{\lambda-1}\right) dY.$$

The integral J_2 is estimated with the same argument used to bound the integral $\mathcal{I}_{2,3}$ in (8.24),

$$|J_2| \leq Ce^{-\delta_1|\xi|}. \tag{8.27}$$

We rewrite J_1 as:

$$J_1(t,\xi) = \sqrt{|\xi|}\,t \int_{Im(\zeta)=-\frac{\gamma_1}{\sqrt{|\xi|}},\, |\zeta|\leq\varepsilon_0\frac{\sqrt{|\xi|}}{t}} e^{\widetilde{\Psi}(\xi,\zeta\,t,t)} m(\xi, \sqrt{|\xi|}\,\zeta\,t) d\zeta,$$

where

$$\widetilde{\Psi}(\xi,\zeta t,t) = -\sqrt{|\xi|}\,t\frac{2i\zeta}{\lambda-1}\left[1-\ln\left(\frac{2i\zeta}{\lambda-1}\right)+\ln\left(2\sqrt{\pi}e^{i\,Q\frac{\pi}{4}}\right)\right]$$
$$-\frac{1}{2}\ln\left(t\,|\xi|^{1/2}\right) - \frac{1}{2}\ln\left(\frac{2\,i\,\zeta}{\lambda-1}\right) + h(\xi,\zeta t,t). \tag{8.28}$$

Notice that Lemma 14.2 implies:

$$|h(\xi,\zeta t,t)| \leq C\left(|\zeta|^2\, t^2 + \frac{1}{|\xi|}\right)$$

for $t\,|\zeta| \leq \varepsilon_0 \sqrt{|\xi|}$, $\zeta t \in D(\xi, B)$, $|\xi| \geq \xi_0$. A detailed computation of $Re(\widetilde{\Psi}(\xi, \zeta t, t))$ yields

$$Re(\widetilde{\Psi}(\xi, \zeta t, t)) \leq \sqrt{|\xi|}\, t \left(C - \frac{\pi}{2(\lambda - 1)} |\zeta| \right) - \frac{1}{2} \ln \left(t\, |\xi|^{1/2} \right) + C\varepsilon_0 \sqrt{|\xi|}\, t\, |\zeta| + C$$
$$\leq \sqrt{|\xi|}\, t \left(C - \frac{\pi}{4(\lambda - 1)} |\zeta| \right) - \frac{1}{2} \ln \left(t\, |\xi|^{1/2} \right) + C$$

for $\zeta \in B_{\varepsilon_0 \sqrt{|\xi|}/t}(0) \cap \left\{ \zeta \in \mathbb{C};\ Im(\xi) = -\gamma_1/\sqrt{|\xi|} \right\}$, $|\xi| > \xi_0$, $\sqrt{|\xi|}\, t \leq R$ and for some positive constant C independent of ξ and ζ. Using again that $t\sqrt{|\xi|} \leq R$ it follows that, in the case (8.8):

$$|J_1(t, \xi)| \leq C \left(\sqrt{|\xi|}\, t \right)^{1/2} \int_{Im(\zeta) = -\frac{\gamma_1}{\sqrt{|\xi|}}, |\zeta| \leq \varepsilon_0 \frac{\sqrt{|\xi|}}{t}} e^{-\frac{\pi}{\lambda - 1} \sqrt{|\xi|}\, t |\zeta|} \frac{|d\zeta|}{\sqrt{|\zeta|}},$$

and computing the integral:

$$|J_1(t, \xi)| \leq C.$$

Combining this estimate with (8.27), estimate (8.9) of the lemma follows.

In the case (8.10) we deform the contour of integration in formula (8.7) using the analyticity properties of the function $m(\xi, Y)$ as well as the fact that $m(\xi, 0) = 0$ to obtain:

$$\begin{aligned} W(t, \xi) &= -\frac{\pi\,(\lambda - 1)}{A(-1)} m \left(\xi, \frac{\lambda - 1}{2} i \right) \frac{\mathcal{V}(\xi)}{\mathcal{V}\left(\frac{\lambda - 1}{2} i + \xi \right)} t \\ &\quad + \int_{Im(Y) = \gamma_1 + \frac{\lambda - 1}{2}} m(\xi, Y) e^{\Psi(\xi, Y, t)}\, dY \\ &= \frac{\pi(\lambda - 1)}{A(-1)} m \left(\xi, \frac{\lambda - 1}{2} i \right) \Phi \left(\frac{\lambda - 1}{2} i + \xi \right) t \\ &\quad + \int_{Im(Y) = \gamma_1 + \frac{\lambda - 1}{2}} m(\xi, Y) e^{\Psi(\xi, Y, t)}\, dY = K_1 + K_2, \end{aligned} \tag{8.29}$$

using (5.11). By (8.10), $m(\xi, \frac{\lambda - 1}{2} i)$ is uniformly bounded. Using then Proposition 4.1 we obtain that

$$|K_1| \leq C(1 + |\xi|^{1/2}) t. \tag{8.30}$$

We are then left with the integral K_2 which is formally very similar to (8.7) although the integral contour is different. We split this integral in two terms J_1 and J_2 like in (8.26). The term J_2 is bounded as (8.27) by a similar argument as before with one small difference which is that the term $m(\xi, Y)$ is now bounded as $(1 + |Y|)$. The term J_1 can be written as:

$$J_1(t, \xi) = \int_{Im\, Y = \gamma_1 + \frac{\lambda - 1}{2},\ |Y| \leq \varepsilon_0 |\xi|} \frac{m(\xi, Y)}{A\left(\frac{2iY}{\lambda - 1} \right)} \frac{\mathcal{V}(\xi)}{\mathcal{V}(Y + \xi)} t^{-\frac{2iY}{\lambda - 1}} \Gamma \left(\frac{2iY}{\lambda - 1} \right) dY.$$

Using Lemma 14.1, we may estimate $\left|\frac{\mathcal{V}(\xi)}{\mathcal{V}(Y+\xi)}\right|$ by $Ce^{\left(\frac{\pi}{2(\lambda-1)}+\delta_0\right)|Y|}$ in the domain of integration. On the other hand, since $Im\ Y = \gamma_1 + \frac{\lambda-1}{2}$ we may estimate $\left|t^{-\frac{2iY}{\lambda-1}}\right|$ by $t^{1+\frac{2\gamma_1}{\lambda-1}}$. Therefore, using the decay of the Gamma function:

$$|J_1(t,\xi)| \leq Ct^{1+\frac{2\gamma_1}{\lambda-1}} \int_{Im(Y)=\gamma_1+\frac{\lambda-1}{2}} e^{\left(\frac{\pi}{2(\lambda-1)}+\delta_0\right)|Y|} e^{-\frac{\pi}{\lambda-1}|Y|}|dY| \tag{8.31}$$

$$\leq Ct^{1+\frac{2\gamma_1}{\lambda-1}}. \tag{8.32}$$

Combining (8.30) and (8.31) we deduce the estimate (8.11) when $|\xi|t^2 \leq R$. This concludes the proof of Lemma 8.2. □

Proof of Proposition 8.1. Using the change of variables: $y - \xi = Y$ in (5.1) we obtain

$$\widehat{G}(t,\xi) = -\frac{\sqrt{2}}{\sqrt{\pi}i(\lambda-1)} \int_{Im\,Y=-\gamma_1} \frac{\mathcal{V}(\xi)}{\mathcal{V}(Y+\xi)} t^{-\frac{2iY}{\lambda-1}} \Gamma\left(\frac{2iY}{\lambda-1}\right) dY, \tag{8.33}$$

where γ_1 is a positive constant sufficiently small. We rewrite the function $\widehat{G}(t,\xi)$ as follows.

$$\widehat{G}(t,\xi) = \frac{\sqrt{2}}{\sqrt{\pi}i(\lambda-1)} \int_{Im(Y)=-\gamma_1} e^{\Psi(\xi,Y,t)} A\left(\frac{2iY}{\lambda-1}\right) dY, \tag{8.34}$$

where the functions Ψ and θ are given by (8.4) and (8.3).

In order to obtain estimates (8.1) and (8.2) for bounded values of ξ we use contour deformation. In particular, crossing the pole at $Y = 0$ in integral (8.34) and using the residue theorem we obtain:

$$\widehat{G}(t,\xi) = \frac{1}{\sqrt{2\pi}} + \widehat{G}_1(t,\xi), \tag{8.35}$$

$$\widehat{G}_1(t,\xi) = -\frac{\sqrt{2}}{\sqrt{\pi}i(\lambda-1)} \int_{Im\,Y=\gamma_1} \frac{\mathcal{V}(\xi)}{\mathcal{V}(Y+\xi)} t^{-\frac{2iY}{\lambda-1}} \Gamma\left(\frac{2iY}{\lambda-1}\right) dY. \tag{8.36}$$

Using Proposition 13.2 it follows that

$$|\widehat{G}_1(t,\xi)| \leq Ct^{\frac{2\gamma_1}{\lambda-1}}$$

uniformly for ξ in bounded sets. This yields estimate (8.1) for ξ in bounded sets. If we differentiate in (8.36) with respect to ξ we obtain:

$$\frac{\partial^\ell}{\partial\xi^\ell}\widehat{G}(t,\xi) = -\frac{\sqrt{2}}{\sqrt{\pi}i(\lambda-1)} \int_{Im\,Y=\gamma_1} m_\ell(\xi,Y) t^{-\frac{2iY}{\lambda-1}} \Gamma\left(\frac{2iY}{\lambda-1}\right) dY, \tag{8.37}$$

$$m_\ell(\xi,Y) = \frac{\partial^\ell}{\partial\xi^\ell}\left(\frac{\mathcal{V}(\xi)}{\mathcal{V}(Y+\xi)}\right). \tag{8.38}$$

Using the analyticity properties of the functions $m_\ell(\xi,Y)$ we deform the integration contour in the integral (8.37). The first singularity that is met is the pole of function $\Gamma\left(\frac{2iY}{\lambda-1}\right)$ located at $Y = (\lambda-1)i/2$. (This point is below the first zero of the function

$\mathcal{V}(\xi+Y)$ which is located at $Y=(2+(\lambda/2)-\xi)\,i$ and $2+(\lambda/2)-\xi>(\lambda-1)/2$.) We then deduce

$$\frac{\partial^\ell}{\partial\xi^\ell}\widehat{G}(t,\xi)=\frac{2\sqrt{2\pi}}{\sqrt{\pi}i\,(\lambda-1)}t\,m_\ell(\xi,\frac{\lambda-1}{2}i) \tag{8.39}$$

$$-\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im\,Y=\gamma_1+\frac{\lambda-1}{2}}m_\ell(\xi,Y)t^{-\frac{2iY}{\lambda-1}}\Gamma\left(\frac{2iY}{\lambda-1}\right)dY, \tag{8.40}$$

using the analyticity properties of the functions $m_\ell(\xi,Y)$ we deduce estimate (8.2) for bounded values of ξ.

In order to complete the proof of Proposition 8.1 we apply the method of the stationary phase. To this end we differentiate the expression in (8.34) with respect to ξ, and obtain:

$$\frac{\partial^\ell}{\partial\xi^\ell}\widehat{G}(t,\xi)=\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im\,Y=-\gamma_1}\left[\frac{\partial^\ell\Psi}{\partial\xi^\ell}(\xi,Y,t)\,A\left(\frac{2iY}{\lambda-1}\right)\right]e^{\Psi(\xi,Y,t)}dY \tag{8.41}$$

for $\ell=1,2$ and use Lemma 8.2 with three choices for $m(t,\xi)$:

$$m(\xi,Y)=\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}A\left(\frac{2\,i\,Y}{\lambda-1}\right), \tag{8.42}$$

$$m(\xi,Y)=\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}(|\xi|+1)^\ell\frac{\partial^\ell\psi}{\partial\xi^\ell}(\xi,Y,t)\,A\left(\frac{2\,i\,Y}{\lambda-1}\right) \tag{8.43}$$

for $\ell=1,2$. (Notice that $\frac{\partial\psi}{\partial\xi}(\xi,Y,t)$ and $\frac{\partial^2\psi}{\partial\xi^2}(\xi,Y,t)$ are independent of t.)

The function m in (8.42) satisfies condition (8.8). On the other hand, using Lemma 14.10, it follows that for the choices (8.43), $\ell=1,2$: $|m(\xi,Y)|\le C|Y|$. Moreover, by the definition of the function Ψ (cf. (8.4))

$$\frac{\partial\psi}{\partial\xi}(\xi,Y)=\frac{\partial}{\partial\xi}\left(\ln\left(\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}\right)\right),$$

and therefore $m(\xi,0)=0$ with the choice (8.43) and $\ell=1$. This follows by a similar argument for the choice (8.43), $\ell=2$. Applying then Lemma 8.2 the estimates (8.1) and (8.2) follow for $|\xi|$ sufficiently large and this concludes the proof of Proposition 8.1. □

We consider now the case $t>1$.

Lemma 8.3. *The function $\widehat{G}$ defined in (5.1) satisfies that for any $\varepsilon_0>0$ arbitrarily small, there exist two positive constants κ_1 and a such that*

$$|\widehat{G}(t,\xi)|\le\kappa_1\,t^{-\frac{1}{\lambda-1}+\varepsilon_0}e^{-a\sqrt{|\xi|}} \tag{8.44}$$

for all ξ such that $Im(\xi)\in(3/2,(3+\lambda)/2)$ and all $t>1$.

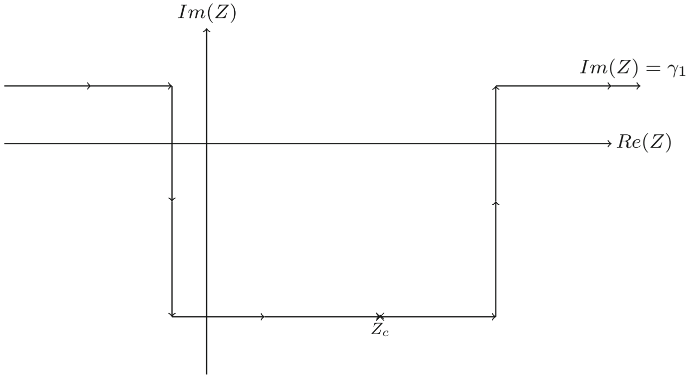

Fig. 3. The curve $\mathcal{D}$

Proof. We deform the integration contour in the expression (8.33) to the line: $Im(Y) = (\lambda - 1)/2 - \varepsilon_0$ with ε_0 arbitrarily small in order to avoid the zero of the function $\mathcal{V}(\xi + Y)$ at $\xi + Y = 1$. We then use the change of variables $Y = \sqrt{|\xi|}\, Z$, and deform the contour of integration in the Z variable to the curve $\mathcal{D}$ in Fig. 3 to obtain:

$$\widehat{G}(t, Z) = \frac{\sqrt{2}\sqrt{|\xi|}}{\sqrt{\pi} i\, (\lambda - 1)} \int_{\mathcal{D}} t^{-\frac{2i}{\lambda-1}\sqrt{|\xi|}\, Z} e^{\widetilde{\Psi}(\xi, Z, 1)} A\left(\frac{2i\sqrt{|\xi|}Z}{\lambda - 1}\right) dZ. \tag{8.45}$$

Then,

$$|\widehat{G}(t, Z)| \leq C\sqrt{|\xi|}\, t^{-\frac{1}{\lambda-1}+\varepsilon_0} \int_{\mathcal{D}} \left| e^{\widetilde{\Psi}(\xi, Z, 1)} A\left(\frac{2i\sqrt{|\xi|}Z}{\lambda - 1}\right)\right| dZ.$$

We argue now in the same way as in the previous case with $t = 1$ splitting the integral in the same pieces to obtain (8.44). □

9. Estimates on the Function $G(t, X)$

We may now take the inverse Fourier transform of $\widehat{G}(t, \xi)$ to obtain the function:

$$G(t, X) = \frac{1}{\sqrt{2\pi}} \int_{Im(\xi)=\beta_2} e^{iX\xi}\, \widehat{G}(t, \xi) d\xi \tag{9.1}$$

with

$$\beta_2 \in \left(\frac{3}{2}, \frac{3 + \lambda}{2}\right). \tag{9.2}$$

Proposition 9.1. *For all $t > 0$ the function $G(t, \cdot)$ defined by (9.1) belongs to $C^\infty(\mathbb{R})$ and for any fixed $R > 0$ and $\varepsilon_0 > 0$, it satisfies:*

$$\left|\frac{\partial^\ell G}{\partial X^\ell}(t, X)\right| \leq \frac{C_{\ell,R}}{t^{2(1+\ell)}}, \quad \text{for } 0 < t \leq 1, \ |X| \leq R, \ \ell = 0, 1, 2, \cdots, \tag{9.3}$$

$$\left|\frac{\partial^\ell G}{\partial X^\ell}(t, X)\right| \leq \frac{C_{\ell,\varepsilon_0,R}}{t^{\frac{1}{\lambda-1}-\varepsilon_0}}, \quad \text{for } t \geq 1, \ |X| \leq R, \ \ell = 0, 1, 2, \cdots, \tag{9.4}$$

for suitable positive constants $C_{\ell,R}$ and $C_{\ell,\varepsilon_0,R}$.

Proof. This proposition is just a consequence of (9.1) as well as from (8.1) for $t \in (0, 1)$ and (8.44) for $t > 1$. □

The estimates (9.3) and (9.4) provide the regularity result for g in Theorem 3.5, but they do not give any detailed description of the function G in the different regions of X and t. We derive such results in the remainder of this section.

9.1. Behaviour as $t \to 0$, $0 \le |X| \le 1$: *Heuristics.* We begin by showing that the function G behaves like a mollifier of the Dirac measure when $t \to 0$ and for X small. The asymptotic profile of the solution $G(t, X)$ can be heuristically guessed as follows. If we assume that the function g, solution of (3.1) with $g(0, x) \sim \delta(x-1)$ is small, then Eq. (3.1) may be approximated, for t and $x - 1$ small, as

$$\frac{\partial g}{\partial t} = \int_0^{1/2} \frac{g(x-y) - g(x)}{y^{3/2}}\, dy, \qquad g(x, 0, 1) = \delta(x-1). \tag{9.5}$$

The equation in (9.5) describes the probability distribution for the size of a particle, initially equal to one, and increasing its size by an amount y at the rate $1/y^{3/2}$. If we write the function g as

$$g(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{g}(t, x)\, e^{i\,k\,x}\, dk,$$

applying the Fourier transform we obtain:

$$\frac{\partial \widehat{g}}{\partial t}(t, k) = m(k)\, \widehat{g}(t, k), \qquad \widehat{g}(k, 0) = \frac{e^{-i\,k}}{\sqrt{2\pi}},$$
$$m(k) = \int_0^{1/2} \frac{e^{-i\,k\,y} - 1}{y^{3/2}}\, dy.$$

The multiplier $m(k)$ can be computed explicitly but its exact formula is not needed to compute the asymptotics of the function $g(t, x)$. The only relevant information that we really need is

$$m(k) \approx -2\sqrt{\pi\, k\, i}, \quad \text{for } |k| \to +\infty.$$

Then,

$$\widehat{g}(t, k) \approx \frac{e^{-i\,k}}{\sqrt{2\pi}}\, e^{-2\sqrt{\pi\,k\,i}\,t}, \quad \text{for } |k| \to +\infty,$$

and inverting the Fourier transform we obtain

$$g(t, x) \approx \frac{2}{t^2}\, \Psi\left(\frac{x-1}{t^2}\right) \quad \text{as } t \to 0,$$

with

$$\Psi(\chi) = \begin{cases} \frac{\pi}{\chi^{3/2}}\, e^{-\frac{\pi}{\chi}} & \text{for } \chi \ge 0, \\ 0 & \text{for } \chi < 0. \end{cases} \tag{9.6}$$

Finally, since $x = e^X$, $x - 1 = e^X - 1 \sim X$ when $X \sim 0$ we obtain

$$G(t, X) \approx \frac{2}{t^2} \Psi\left(\frac{X}{t^2}\right) \quad \text{as} \quad t \to 0. \tag{9.7}$$

The fact that the support of $G(t, X)$ is contained in $\mathbb{R}^+$ is a consequence of the interpretation of (9.5) in terms of coagulation of particles given above.

9.1.1. The region $X = \mathcal{O}(t^2)$ We now prove that the fundamental solution $G(t, X)$ behaves in the self similar form (9.7) in the region where $X = \mathcal{O}(t^2)$ using the explicit representation formula given by (5.1), (6.15) and (9.1). We are interested in the limit $\lim_{t\to 0^+} t^2 G(t, t^2 \chi) = \Psi(\chi)$ in compact sets of χ. To this end we rewrite (9.1) as follows:

$$G(t, \chi) = \frac{1}{\sqrt{2\pi}\, t^2} \int_{Im(\eta)=\beta_2 t^2} e^{i\eta\chi} \widehat{G}\left(t, \frac{\eta}{t^2}\right) d\eta. \tag{9.8}$$

Formula (9.8) suggests that in order to compute $\Psi(\chi)$ we need to obtain $\lim_{t\to 0^+} \widehat{G}\left(t, \frac{\eta}{t^2}\right)$ for η such that $Im(\eta) = \beta_2 t^2$. To this end we use the expression (8.33) that implies:

$$\widehat{G}(t, \eta/t^2) = -\frac{\sqrt{2}}{\sqrt{\pi} i\, (\lambda - 1)} \int_{Im\, Y=-\gamma_1} \frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(Y + \eta/t^2)} t^{-\frac{2iY}{\lambda-1}}\, \Gamma\left(\frac{2iY}{\lambda - 1}\right) dY. \tag{9.9}$$

Proposition 9.2.

$$\lim_{t\to 0^+} \left(t^2 G(t, t^2 \chi)\right) = \psi(\chi)$$

uniformly for χ in compact sets of $\mathbb{R}$ where $\psi(\chi)$ is as in (9.6).

The proof of Proposition 9.2 requires several lemmas.

Lemma 9.3. *For all $\varepsilon_0 > 0$ and $M > 0$ there exists a function $h_{\varepsilon_0,M}(t)$ such that*

$$\lim_{t\to 0^+} h_{\varepsilon_0,M}(t) = 0 \tag{9.10}$$

and

$$\left| \frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(Y + \eta/t^2)} t^{-\frac{2iY}{\lambda-1}} - e^{-\frac{2iY}{\lambda-1} \ln(2\sqrt{\pi}\sqrt{i\eta})} \right| \le h_{\varepsilon_0,M}(t) \tag{9.11}$$

for all Y such that $|Im(Y)| \le 1/4$, $|Re(Y)| \le \frac{1}{t\,|\ln t|}$, for $Im(\eta/t^2)$ in compact subsets of $(3/2, (3+\lambda)/2)$ and $\varepsilon_0 \le |\eta| \le M$. Moreover there is $\delta_0 > 0$ small (depending on ε_0 and M) such that:

$$\left| \frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(Y + \eta/t^2)} t^{-\frac{2iY}{\lambda-1}} \right| \le C\, e^{\frac{3\pi}{4}\frac{|Y|}{\lambda-1}} \tag{9.12}$$

for all Y such that $|Im(Y)| \le 1/4$, $|Re(Y)| \le \frac{\delta_0}{t^2}$, for $Im(\eta/t^2)$ in compact subsets of $(3/2, (3+\lambda)/2)$ and $\varepsilon_0 \le |\eta| \le M$.

Remark 9.4. In Lemmas 9.3 until 9.6 and in Proposition 9.2 we choose the branch of the function square root as follows:

$$\sqrt{z} = |z|^{1/2} e^{i\frac{\theta}{2}} \text{ with } \theta \in (-\pi, \pi].$$

Proof. Lemma 9.3 is a consequence of Lemma 14.1. By formula (6.15) we have:

$$\begin{aligned}\frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(\eta/t^2+Y)} &= \exp\left[\frac{2}{(\lambda-1)\,i}\int_{Im\,\zeta=\beta_1} \ln\left(-\Phi(\zeta)\right)\Theta(\zeta-\zeta/t^2, Y)d\zeta\right]\\ &= \exp\left[F\left(\eta/t^2, t\,Y/\sqrt{|\eta|}\right)\right],\end{aligned}$$

and using (14.3):

$$\left|F\left(\eta/t^2, t\,Y/\sqrt{|\eta|}\right) + \frac{2\,i\,Y}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)\right| \le C\,t^2\frac{|Y|^2}{|\eta|}.$$

Therefore, in the region where $|Re(Y)| \le 1/(t|\ln t|)$ we obtain (9.11). On the other hand, if $|Re(Y)| \le \delta_0/t^2$ we deduce (9.12) using that:

$$\left|Re\left(\frac{2\,i\,Y}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)\right)\right| \le \frac{\pi}{2(\lambda-1)}|Y| + C$$

as $|\eta| \to +\infty$ and for $Im(Y) = -\gamma_1$ with γ_1 as in (9.9). □

Lemma 9.5. *For all positive constants* M, ε_0 *such that* $M > \varepsilon_0$:

$$\lim_{t\to 0^+} \widehat{G}\left(t, \frac{\eta}{t^2}\right) = -\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im(Y)=-\gamma_1} e^{-\frac{2\,i\,Y}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)}\Gamma\left(\frac{2iY}{\lambda-1}\right)dY,$$

uniformly for $Im(\eta/t^2)$ *in compact subsets of* $(3/2, (3+\lambda)/2)$ *and* $\varepsilon_0 \le |\eta| \le M$.

Proof. We split the integral in (9.9) as follows:

$$\begin{aligned}&\widehat{G}\left(t, \frac{\eta}{t^2}\right)\\ &= \frac{-\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im\,Y=-\gamma_1,\,|Re(Y)|\le\frac{1}{t^2|\ln t|}} \frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(Y+\eta/t^2)}t^{-\frac{2iY}{\lambda-1}}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)dY\\ &= \frac{-\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im\,Y=-\gamma_1,\,\frac{1}{t^2|\ln t|}\le|Re(Y)|\le\frac{\delta_0}{t^2}} \frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(Y+\eta/t^2)}t^{-\frac{2iY}{\lambda-1}}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)dY\\ &\quad+\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im\,Y=-\gamma_1,\,|Re(Y)|\ge\frac{\delta_0}{t^2}} \frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(Y+\eta/t^2)}t^{-\frac{2iY}{\lambda-1}}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)dY\\ &= J_1 + J_2 + J_3.\end{aligned}$$

$$\begin{aligned}J_1 &= -\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\\ &\quad\times\int_{Im\,Y=-\gamma_1,\,|Re(Y)|\le\frac{1}{t^2|\ln t|}}\left(\frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(Y+\eta/t^2)}t^{-\frac{2\,i\,Y}{\lambda-1}} - e^{-\frac{2\,i\,Y}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)}\right)\end{aligned}$$

$$\times \Gamma\left(\frac{2iY}{\lambda-1}\right) dY$$
$$+\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im\,Y=-\gamma_1,} e^{-\frac{2iY}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right) dY$$
$$-\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im\,Y=-\gamma_1,\,|Re(Y)|\geq\frac{1}{t^2\,|\ln t|}} e^{-\frac{2iY}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right) dY. \quad (9.13)$$

Using that

$$\left|e^{-\frac{2iY}{\lambda-1}\ln\left(\sqrt{i\eta}\right)}\right| \leq C\,e^{\frac{\pi}{2(\lambda-1)}|Y|} \quad \text{and} \quad \left|\Gamma\left(\frac{2\,i\,Y}{\lambda-1}\right)\right| \leq C\,e^{-\frac{\pi}{(\lambda-1)}|Y|},$$

the last term in (9.13) is estimated as:

$$\left|\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im\,Y=-\gamma_1,\,|Re(Y)|\geq\frac{1}{t^2\,|\ln t|}} e^{-\frac{2iY}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right) dY\right|$$
$$\leq Ce^{-\frac{a}{t^2\,|\ln t|}}$$

for some positive constant a. In the first term of (9.13), using (9.11) in Lemma 9.3 we are led to estimate: $h_{\varepsilon_0,M}(t)\int_{Im\,Y=-\gamma_1} e^{-\frac{\pi}{(\lambda-1)}|Y|}|dY|$ which tends to zero as $t \to 0^+$ by (9.10).

We now consider the integral J_3. This term is estimated using the estimates of the function $\mathcal{V}$ proved in Proposition 13.2 and Stirling's formula:

$$|J_3| \leq C_\varepsilon t^{-\frac{2\gamma_1}{\lambda-1}} e^{-\frac{1}{t^2}\left(\frac{\pi\,\delta_0}{\lambda-1}-2\varepsilon M\right)} \leq C_\delta t^{-\frac{2\gamma_1}{\lambda-1}} e^{-\frac{a}{t^2}}$$

for some positive constant a choosing δ sufficiently small.

The integral J_2 is estimated using (9.12) in Lemma 9.3:

$$|J_2| \leq C\int_{Im\,Y=-\gamma_1,\,\frac{1}{t^2\,|\ln t|}\leq|Re(Y)|\leq\frac{\delta_0}{t^2}} e^{\frac{3\pi}{4(\lambda-1)}|Y|}e^{-\frac{\pi}{\lambda-1}|Y|}|dY| \leq Ce^{-\frac{a}{t^2\,|\ln t|}}$$

for some positive constant a. □

Lemma 9.6.

$$\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im(Y)=-\gamma_1} e^{-\frac{2iY}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right) dY = -\sqrt{2\pi}\,e^{-2\sqrt{\pi\,\eta\,i}}.$$

Proof. The change of variable $2iY/(\lambda-1) = s$ yields

$$\frac{\sqrt{2}}{\sqrt{\pi}i\,(\lambda-1)}\int_{Im(Y)=-\gamma_1} e^{-\frac{2iY}{\lambda-1}\ln\left(2\sqrt{\pi}\sqrt{i\,\eta}\right)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right) dY$$
$$= -\frac{1}{\sqrt{2\pi}}\int_{Re(s)=2\gamma_1/(\lambda-1)} e^{-s\,\ln\left(2\sqrt{\pi\,\eta\,i}\right)}\,\Gamma(s)\,ds.$$

This integral can then be computed adding the residues of the integrand at the poles $s = -n$ of the Gamma function. □

Proof of Proposition 9.2. We split the integral in (9.8)

$$t^2 G(t,\chi) = I_1 + I_2 + I_3,$$

where

$$I_k = \frac{1}{\sqrt{2\pi}} \int_{\{Im(\eta)=\beta_2 t^2\}\cap D_k} e^{i\eta\chi} \widehat{G}\left(t, \frac{\eta}{t^2}\right) d\eta$$

with $D_1 = B_{\varepsilon_0}(0)$, $D_2 = B_M(0)\backslash B_{\varepsilon_0}(0)$, $D_3 = \mathbb{C}\backslash B_M(0)$, where ε_0 and M are for awhile arbitrary positive constants with $\varepsilon_0 < M$. Using Proposition 8.1:

$$|I_1| + |I_3| \leq C\left(\varepsilon_0 + e^{-aM}\right). \tag{9.14}$$

The integral I_2 is estimated using the previous lemmas:

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{\{Im(\eta)=\beta_2 t^2\}\cap D_2} e^{i\eta\chi} (\widehat{G}\left(t, \frac{\eta}{t^2}\right) - \sqrt{2\pi}\, e^{-2\sqrt{\pi\,\eta\, i}}) d\eta$$
$$+ \int_{\{Im(\eta)=\beta_2 t^2\}\cap D_2} e^{i\eta\chi}\, e^{-2\sqrt{\pi\,\eta\, i}} d\eta = I_{2,1} + I_{2,2}.$$

The first term is bounded as

$$|I_{2,1}| \leq C \int_{\{Im(\eta)=\beta_2 t^2\}\cap D_2} \left|\widehat{G}\left(t, \frac{\eta}{t^2}\right) - \sqrt{2\pi}\, e^{-2\sqrt{\pi\,\eta\, i}}\right| |d\eta|$$

and tends to zero by Lemma 9.5 and Lemma 9.6. And,

$$\lim_{t\to 0^+} I_{2,1} = \int_{\{Im(\eta)=0\}\cap D_2} e^{i\eta\chi}\, e^{-2\sqrt{\pi\,\eta\, i}} d\eta.$$

Arguing as in the derivation of (9.14) we obtain that

$$\left|\int_{\{Im(\eta)=0\}\cap D_2} e^{i\eta\chi}\, e^{-2\sqrt{\pi\,\eta\, i}} d\eta - \int_{Im(\eta)=0} e^{i\,\eta\,\chi} e^{-2\sqrt{\pi}\,\eta\, i}\, d\eta\right| \leq C\left(\varepsilon_0 + e^{-aM}\right).$$

We have then shown that, for t sufficiently small (depending on ε_0 and M):

$$\left|t^2 G(t,x) - \int_{Im(\eta)=0} e^{i\,\eta\,\chi} e^{-2\sqrt{\pi}\,\eta\, i}\, d\eta\right| \leq 2C(\varepsilon_0 + e^{-aM}),$$

which means exactly:

$$\lim_{t\to 0^+} t^2 G(t,\chi) = \int_{Im(\eta)=0} e^{i\eta\chi}\, e^{-2\sqrt{\pi\,\eta\, i}} d\eta. \tag{9.15}$$

The integral in the right hand side of (9.15) may be calculated explicitly using contour deformation. For $\chi < 0$ the contour is sent to the region where $Im(\eta) \to -\infty$ and the integral gives zero. When $\chi > 0$ the deformation is made to the upper half plane in such a way that it avoids the cut along the half line $\eta \in i\,\mathbb{R}^+$ and the integral is reduced to

$$2\int_0^\infty \sin\left(2\sqrt{\pi\,\lambda}\right) e^{-\chi\,\lambda}\, d\lambda = \frac{\pi}{\chi^{3/2}}\, e^{-\frac{\pi}{\chi}}.$$

□

9.1.2. Estimates of $G(t,X)$ for $t^2 \le X \le 1$. We derive a self similar estimate for the function G in this region as $t \to 0$.

Proposition 9.7. *For any $\varepsilon \in (0, 1/2)$, there exists a positive constant C_ε such that*

$$|G(t,X)| \le C \frac{t^{1-2\varepsilon}}{|X|^{\frac{3}{2}-\varepsilon}} \quad \text{for } t^2 \le |X| \le 1.$$

Proof. We integrate by parts twice in formula (9.1) and obtain:

$$G(t,X) = \frac{1}{\sqrt{2\pi}} \frac{1}{X^2} \int_{Im(\xi)=\beta_2} \left(e^{iX\xi} - 1\right) \frac{\partial^2}{\partial \xi^2} \widehat{G}(t,\xi) d\xi. \tag{9.16}$$

Using that $\left|e^{iX\xi} - 1\right| \le C_\varepsilon |X|^{1/2+\varepsilon} |\xi|^{1/2+\varepsilon}$ for $\varepsilon \in [0, 1/2]$, as well as (8.2) we deduce:

$$\begin{aligned} |G(t,X| &\le \frac{C_\varepsilon}{X^2} \int_{Im(\xi)=\beta_2} \frac{t}{(1+|\xi|^{3/2})} |X|^{1/2+\varepsilon} |\xi|^{1/2+\varepsilon} e^{-t\sqrt{|\xi|}} |d\xi| \\ &\le \frac{C_\varepsilon\, t}{|X|^{\frac{3}{2}-\varepsilon}} \int_{Im(\xi)=\beta_2} e^{-t\sqrt{|\xi|}} \frac{|d\xi|}{|\xi|^{1-\varepsilon}}, \end{aligned}$$

and the result follows. □

9.2. Behaviour of $G(t,X)$ for $t \ge 1$. The behaviour of the function G as $t \to +\infty$ has a self similar structure. This is seen by writing the function $G(t,X)$, given in (5.1) and (9.1), in terms of the variable

$$\theta = X + \frac{2}{\lambda - 1} \ln(t), \tag{9.17}$$

$$G(t,X) = \frac{i}{\pi(\lambda-1)} \int_{Im(\xi)=\beta_2} d\xi\, e^{i\xi\theta} \int_{Im(y)=\beta_0} dy\, \frac{\mathcal{V}(\xi)}{\mathcal{V}(y)} t^{-\frac{2iy}{\lambda-1}} \Gamma\left(-\frac{2i}{\lambda-1}(\xi - y)\right) \tag{9.18}$$

with $3/2 < \beta_0 < \beta_2 < (3+\lambda)/2$ (cf. Lemma 5.1).

Moving the integration contour of y downward in the expression of $G(t,X)$, the first singularity to be met in the integrand of (9.18) is $y = i$ which is a zero of $\mathcal{V}(y)$ (cf. Proposition 6.1). This gives

$$G(t,X) = t^{\frac{2}{\lambda-1}} \Psi_1(\theta) + G_1(t,X),$$

where

$$\Psi_1(\theta) = \frac{2}{\lambda-1} \int_{Im(\xi)=\beta_2} d\xi\, e^{i\xi\theta} \frac{\mathcal{V}(\xi)}{\mathcal{V}'(i)} \Gamma\left(-\frac{2i}{\lambda-1}(\xi - i)\right), \tag{9.19}$$

$$G_1(t,X) = \frac{i}{\pi(\lambda-1)} \int_{Im(\xi)=\beta_2} d\xi\, e^{i\xi\theta} \int_{Im(y)=\beta_3} dy\, \frac{\mathcal{V}(\xi)}{\mathcal{V}(y)} t^{-\frac{2iy}{\lambda-1}} \Gamma\left(-\frac{2i}{\lambda-1}(\xi - y)\right), \tag{9.20}$$

and now $\beta_3 \in ((3-\lambda)/2, 1)$. Notice that $\mathcal{V}'(i) \neq 0$. Indeed, differentiating (5.11) we obtain:

$$\mathcal{V}'(i) = 2\pi i \mathcal{V}\left(\frac{\lambda+1}{2} i\right),$$

where we use that $\Phi((\lambda+1)/2) = 0$ and $\Phi'((\lambda+1)/2) = -2\pi i$ (cf. (3.20)). By Proposition 6.1, $\mathcal{V}(\frac{\lambda+1}{2} i) \neq 0$. Then:

$$\Psi_1(\theta) = \frac{1}{\pi(\lambda-1)\, i\, \mathcal{V}\left(\frac{\lambda+1}{2} i\right)} \int_{Im(\xi)=\beta_2} d\xi\, e^{i\xi\theta} \mathcal{V}(\xi) \Gamma\left(\frac{2i}{\lambda-1}(i-\xi)\right). \tag{9.21}$$

We study now the function $\Psi_1(\theta)$ and give its behaviour as $\theta \to \pm\infty$.

Proposition 9.8. *There exists $\epsilon > 0$ such that the following estimates hold:*

$$\Psi_1(\theta) = a_1 e^{-\frac{3}{2}\theta} + \mathcal{O}\left(e^{-\left(\frac{4-\lambda}{2}+\varepsilon\right)\theta}\right) \quad as \ \theta \to -\infty, \tag{9.22}$$

$$\Psi_1(\theta) = a_2 e^{-\frac{3+\lambda}{2}\theta} + \mathcal{O}\left(e^{-(\lambda+1-\varepsilon)\theta}\right) \quad as \ \theta \to +\infty, \tag{9.23}$$

where, (cf. (3.11)),

$$a_1 = \frac{2i}{(\lambda-1)} \frac{\mathcal{V}((1+\frac{\lambda}{2})i)}{\mathcal{V}(\frac{\lambda+1}{2} i)} \quad and \quad a_2 = -\frac{\Gamma\left(\frac{\lambda+1}{\lambda-1}\right)}{2\pi i} \frac{\mathcal{V}(2i)}{\mathcal{V}(\frac{\lambda+1}{2} i)}.$$

Proof of Proposition 9.8. We use again contour deformation. In order to obtain the behaviour as $\theta \to -\infty$ we deform the contour integration in Ψ downward. The first singularity of the integrand that we meet is $\xi = 3i/2$ which is a pole of $\mathcal{V}(\xi)$. Using (5.11) and (3.20) we obtain:

$$\mathcal{R}es\left(\mathcal{V}, \xi = \frac{3i}{2}\right) = -i\, \mathcal{V}\left(\left(1+\frac{\lambda}{2}\right) i\right), \tag{9.24}$$

$$\mathcal{R}es\left(\mathcal{V}, \xi = \frac{(3+\lambda)i}{2}\right) = -\frac{\mathcal{V}(2i)}{4\pi i}. \tag{9.25}$$

Therefore

$$\begin{aligned} \Psi_1(\theta) &= \frac{2i}{(\lambda-1)} \frac{\mathcal{V}((1+\lambda/2)i)}{\mathcal{V}((\lambda+1)/2)i)} e^{-\frac{3}{2}\theta} \\ &\quad + \frac{1}{\pi(\lambda-1)\, i\, \mathcal{V}\left(\frac{\lambda+1}{2} i\right)} \int_{Im(\xi)=\beta_4} d\xi\, e^{i\xi\theta} \mathcal{V}(\xi) \Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right), \end{aligned} \tag{9.26}$$

$$\begin{aligned} \Psi_1(\theta) &= -\frac{\Gamma\left(\frac{\lambda+1}{\lambda-1}\right)}{2\pi i} \frac{\mathcal{V}(2i)}{\mathcal{V}(\frac{\lambda+1}{2} i)} e^{-\frac{3+\lambda}{2}\theta} \\ &\quad + \frac{1}{\pi(\lambda-1)\, i\, \mathcal{V}(\frac{\lambda+1}{2} i)} \int_{Im(\xi)=\beta_5} d\xi\, e^{i\xi\theta} \mathcal{V}(\xi) \Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right), \end{aligned} \tag{9.27}$$

where $\beta_4 \in ((4-\lambda)/2, 3/2)$, $\beta_5 \in ((3+\lambda)/2, 1+\lambda)$. We have derived (9.26), (9.27) deforming the contour of integration upward and downward respectively.

Proposition 13.2 ensures that the function $\mathcal{V}(\xi)\Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right)$ is integrable and then, for $Re\,\theta\le 0$:

$$\left|\int_{Im(\xi)=\beta_4} d\xi\, e^{i\xi\theta}\mathcal{V}(\xi)\Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right)\right|\le Ce^{-\beta_4\,Re(\theta)}, \tag{9.28}$$

while for $\theta\ge 0$:

$$\left|\int_{Im(\xi)=\beta_5} d\xi\, e^{i\xi\theta}\mathcal{V}(\xi)\Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right)\right|\le Ce^{-\beta_5\,Re(\theta)}. \tag{9.29}$$

Proposition 9.8 follows from (9.26), (9.27), (9.28) and (9.29). □

9.2.1. Estimate of $G_1(t,\theta)$ in formula (9.20). We have the following lemma.

Lemma 9.9. *There exist $\delta_0>0$ and $C>0$ such that, for all $t>1$:*

$$|G_1(t,\theta)|\,e^{\frac{3\theta}{2}}\le C\,t^{\frac{2}{\lambda-1}-\delta_0}\qquad \text{for all } \theta\le 0, \tag{9.30}$$

$$|G_1(t,\theta)|\,e^{\frac{3+\lambda}{2}\theta}\le C\,t^{\frac{2}{\lambda-1}-\delta_0}\qquad \text{for all } \theta\ge 0, \tag{9.31}$$

where G_1 is given by (9.20).

Proof of Lemma 9.9. The function G_1 may be written as:

$$G_1(t,\theta)=\frac{i}{\pi(\lambda-1)}\int_{Im(\xi)=\beta_2} d\xi\, e^{i\xi\theta}\,H_1\,(t,\xi), \tag{9.32}$$

$$H_1(t,\xi)=\int_{Im(y)=\beta_3} dy\,\frac{\mathcal{V}(\xi)}{\mathcal{V}(y)}t^{-\frac{2iy}{\lambda-1}}\Gamma\left(-\frac{2i}{\lambda-1}(\xi-y)\right). \tag{9.33}$$

The function H_1 is estimated in the same way as the function $\widehat{G}(t,\xi)$ in Lemma 8.3. To this end we first perform the change of variables: $y=\xi+\sqrt{|\xi|}\,Z$ and obtain:

$$H_1(t,\xi)=\sqrt{|\xi|}\int_{Im(Z)=\frac{\beta_3-\beta_2}{\sqrt{|\xi|}}} dZ\,\frac{\mathcal{V}(\xi)}{\mathcal{V}(\sqrt{|\xi|}Z+\xi)}t^{-\frac{2i\,(\sqrt{|\xi|}Z+\xi)}{\lambda-1}}\Gamma\left(\frac{2i}{\lambda-1}\sqrt{|\xi|}\,Z\right). \tag{9.34}$$

Then we deform the contour of integration in (9.34) to the new contour $\mathcal{D}_1$ (cf. Fig. 4). We then need to bound:

$$\sqrt{|\xi|}\int_{\mathcal{D}_1}\left|\frac{\mathcal{V}(\xi)}{\mathcal{V}(\sqrt{|\xi|}Z+\xi)}t^{-\frac{2i\,(\sqrt{|\xi|}Z+\xi)}{\lambda-1}}\Gamma\left(\frac{2i}{\lambda-1}\sqrt{|\xi|}\,Z\right)\right||dy|$$

that may be estimated following the same arguments as in the proof of Lemma 8.3. The only difference with the argument used there is how to bound the contribution of the time dependent term. However arguing as in the proof of Lemma 8.3 and taking into account that along $\mathcal{D}_1$ we have $Im(\sqrt{|\xi|}Z+\xi)\le\beta_3$ we finally obtain:

$$|H_1(t,\xi)|\le Ct^{\frac{2}{\lambda-1}-\delta}e^{-a\sqrt{|\xi|}}. \tag{9.35}$$

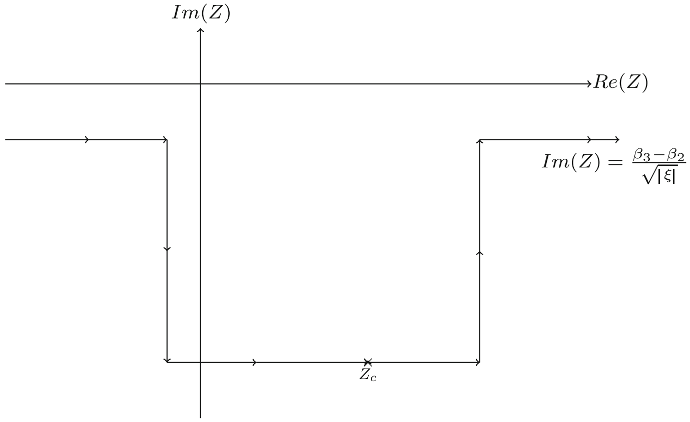

Fig. 4. The curve $\mathcal{D}_1$

Using (9.1) it then follows

$$|G_1(t,\theta)| \le C_R\, t^{\frac{2}{\lambda-1}-\delta}$$

for all $|\theta| \le R$.

To conclude the proof of the Lemma it only remains to obtain estimates as $\theta \to \pm\infty$. To this end we use contour deformation.To obtain the estimate (9.31) as $\theta \to +\infty$ we deform the contour upward. The first singularity of $\widehat{G}_1(t,\xi)$ is located at $\xi = (3+\lambda)\, i/2$ (see Proposition 6.1). Therefore:

$$G_1(t,\theta) = b_2(t)\, e^{-\frac{3+\lambda}{2}\theta} + Q_1(t,\theta), \tag{9.36}$$

$$Q_1(t,\theta) = \int_{Im(\xi)=\beta_6} d\xi\, e^{i\xi\theta}\, H_2(t,\xi), \tag{9.37}$$

$$H_2(t,\xi) = \frac{i}{\pi(\lambda-1)} \int_{Im(y)=\beta_3} dy\, \frac{\mathcal{V}(\xi)}{\mathcal{V}(y)} t^{-\frac{2iy}{\lambda-1}}\, \Gamma\left(-\frac{2i}{\lambda-1}(\xi-y)\right), \tag{9.38}$$

where $\beta_6 \in ((3+\lambda)/2, (1+\lambda))$, and

$$b_2(t) = \frac{\mathcal{V}(2\,i)}{2\pi\, i(\lambda-1)} \int_{Im(y)=\beta_3} dy\, \frac{t^{-\frac{2iy}{\lambda-1}}}{\mathcal{V}(y)}\, \Gamma\left(-\frac{2i}{\lambda-1}\left(\frac{3+\lambda}{2}\, i - y\right)\right).$$

Since $\beta_3 \in ((3-\lambda)/2, 1)$, we deduce:

$$|b_2(t)| \le C t^{\frac{2}{\lambda-1}-\delta_0}, \quad \text{for } t > 1. \tag{9.39}$$

The function H_2 is estimated in the same way as the function $\widehat{G}(t,\xi)$ in Lemma 8.3. We change variables as $y = \xi + \sqrt{|\xi|}\, Z$ to obtain:

$$H_2(t,\xi) = \sqrt{|\xi|} \int_{Im(Z)=\frac{\beta_3-\beta_6}{\sqrt{|\xi|}}} dy\ \frac{\mathcal{V}(\xi)}{\mathcal{V}(\sqrt{|\xi|}Z+\xi)} t^{-\frac{2i\,(\sqrt{|\xi|}Z+\xi)}{\lambda-1}}\, \Gamma\left(\frac{2i}{\lambda-1}\sqrt{|\xi|}\, Z\right). \tag{9.40}$$

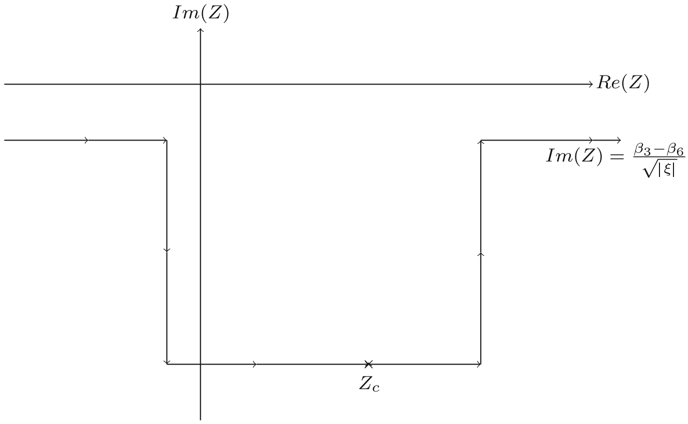

Fig. 5. The curve $\mathcal{D}_2$

Then we deform the integration contour in (9.40) to $\mathcal{D}_2$ (cf. Fig. 5). Since along this new contour $Im(Z) = \frac{\beta_3-\beta_6}{\sqrt{|\xi|}}$, we have $Im(\sqrt{|\xi|}Z + \xi) \le \beta_3$ and then

$$|t^{-2i\,\frac{\sqrt{|\xi|}Z+\xi}{\lambda-1}}| \le Ct^{\frac{2\beta_1}{\lambda-1}} = t^{\frac{2}{\lambda-1}-\delta},$$

whence

$$|H_2(t,\xi)| \le Ct^{\frac{2}{\lambda-1}-\delta}e^{-a\sqrt{|\xi|}}. \tag{9.41}$$

Using now (9.37) we deduce that

$$|Q_1(t,\theta)| \le Ct^{\frac{2}{\lambda-1}-\delta}e^{-\beta_6\theta}. \tag{9.42}$$

Combining this with (9.39), estimate (9.31) follows.

The estimate (9.30) for $\theta \to -\infty$ is obtained in a very similar way. We deform downward the contour of the integral (9.33) and we continue the proof as for (9.31). This concludes the proof of Lemma 9.9. □

9.3. Behaviour as $0 \le t \le 1$, $|X| \to +\infty$. For small values of time the solution is described in the (t, X) variables as follows.

Lemma 9.10. *There exists positive constants* δ *and* δ_1 *such that, for* $0 \le t \le 1$ *, the following estimates hold:*

$$G(t,X) = \begin{cases} e^{-\frac{3}{2}X}\,t + b_3(t)e^{-\frac{3}{2}X} + \mathcal{O}\left(e^{-\left(\frac{3}{2}-\delta_1\right)X}\,t\right) & as\ \ X \to -\infty \\ a_3\,e^{-\frac{3+\lambda}{2}X}\,t + b_4(t)e^{-\frac{3+\lambda}{2}X} + \mathcal{O}\left(e^{-\left(\frac{3+\lambda}{2}+\delta_1\right)X}\,t\right) & as\ \ X \to +\infty, \end{cases} \tag{9.43}$$

where $a_3 = \frac{\mathcal{V}(2i)}{4\pi\,\mathcal{V}\left(\left(1+\frac{\lambda}{2}\right)i\right)}$ *(cf. (3.13)) and* b_3 *and* b_4 *are continuous functions such that* $|b_3(t)| + |b_4(t)| \le Ct^{1+\delta}$.

Proof of Lemma 9.10. Using (8.33) and (9.1) we obtain:

$$G(t,X)=\frac{i}{\pi(\lambda-1)}\int_{Im(\xi)=\beta_2}e^{i\xi X}d\xi\int_{Im(Y)=-\gamma_1}dy\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}t^{-\frac{2i}{\lambda-1}Y}\,\Gamma\left(\frac{2iY}{\lambda-1}\right),$$

where $\beta_0\in\left(\frac{3}{2},\frac{3+\lambda}{2}\right)$ and $\gamma_1>0$ small. Proposition 8.1 implies that $\widehat{G}(t,\cdot)$ decays exponentially in the ξ variable in the region $\xi\in\mathcal{T}_L$. This provides the convergence of all the integrals used in this proof if L is taken sufficiently large.

We can now deform the contour of integration on ξ crossing the poles of $\widehat{G}(t,\xi)$ that are due to the poles of $\mathcal{V}(\xi)$. The closest poles are at $\xi=\frac{3}{2}i,\ \xi=\frac{3+\lambda}{2}i$ respectively. We deform the contour upwards if $X>0$ and downwards if $X<0$. We then obtain using (9.24), (9.25):

$$\begin{aligned}&G(t,X)\\&=-2\pi\left(\frac{i}{\pi(\lambda-1)}\right)\mathcal{V}\left(\left(1+\frac{\lambda}{2}\right)i\right)e^{-\frac{3}{2}X}\int_{Im(Y)=-\gamma_1}dy\frac{t^{-\frac{2i}{\lambda-1}Y}}{\mathcal{V}(\frac{3}{2}i+Y)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)\\&\quad+\frac{i}{\pi(\lambda-1)}\int_{Im(\xi)=\beta_7}e^{i\xi X}d\xi\int_{Im(Y)=-\gamma_1}dy\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}t^{-\frac{2i}{\lambda-1}Y}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)\\&\equiv J_1+J_2,\end{aligned}\tag{9.44}$$

$$\begin{aligned}G(t,X)&=-\frac{\mathcal{V}(2i)\,i}{2\pi(\lambda-1)}e^{-\frac{3+\lambda}{2}X}\int_{Im(Y)=-\gamma_1}dy\frac{t^{-\frac{2i}{\lambda-1}Y}}{\mathcal{V}(\frac{3+\lambda}{2}i+Y)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)\\&\quad+\frac{i}{\pi(\lambda-1)}\int_{Im(\xi)=\beta_8}e^{i\xi X}d\xi\int_{Im(Y)=-\gamma_1}dy\frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+Y)}t^{-\frac{2i}{\lambda-1}Y}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)\\&\equiv J_1+J_2,\end{aligned}\tag{9.45}$$

where $\beta_7=\frac{3}{2}-\delta$, $\beta_8=\frac{3+\lambda}{2}+\delta$ with $\delta>0$ small.

The terms J_2 in (9.44), (9.45) can be estimated easily. Indeed, they can be written in the form:

$$G(t,X)=\frac{1}{\sqrt{2\pi}}\int_{Im(\xi)=\beta_\ell}e^{i\xi X}\widehat{G}(t,\xi)\,d\xi,\quad \ell=7,8.$$

Integrating by parts we obtain:

$$G(t,X)=-\frac{1}{\sqrt{2\pi}X^2}\int_{Im(\xi)=\beta_\ell}d\xi e^{i\xi X}\frac{\partial^2}{\partial\xi^2}\left(\widehat{G}(t,\xi)\right).$$

We can now estimate $\frac{\partial^2}{\partial\xi^2}\left(\widehat{G}(t,\xi)\right)$ using Proposition 8.1. It then follows that:

$$\begin{aligned}|J_2|&\le\frac{C}{X^2}\int_{Im(\xi)=\beta_\ell}\left|e^{i\xi X}\right|\frac{t}{\left(1+|\xi|^{3/2}\right)}e^{-a\sqrt{|\xi|t}}\,|d\xi|\\&\le\frac{Ce^{\beta_\ell X}}{X^2}\int_{Im(\xi)=\beta_\ell}\frac{t}{\left(1+|\xi|^{3/2}\right)}e^{-a\sqrt{|\xi|t}}\,|d\xi|\,.\end{aligned}$$

In order to estimate the integral we split it as follows:

$$\int_{Im(\xi)=\beta_\ell} \frac{t}{\left(1+|\xi|^{3/2}\right)} e^{-a\sqrt{|\xi|}t}\,|d\xi| = \int_{Im(\xi)=\beta_\ell,\ |\xi|\le \frac{1}{t^2}} [\ldots]\,|d\xi| + \int_{Im(\xi)=\beta_\ell,\ |\xi|\ge \frac{1}{t^2}} [\ldots]\,|d\xi| \le Ct + Ct^2.$$

Then

$$|J_2| \le Cte^{\beta_\ell X} \quad \text{for} \quad |X| \ge 1,$$

where $\ell = 7$ for $X < 0$ and $\ell = 8$ for $X > 0$. In order to compute the terms J_1 in (9.44), (9.45) we deform the contour on Y upwards. We cross the first pole of the function $\Gamma\left(\frac{2iY}{\lambda-1}\right)$ for $Y = 0$. However, this point is not a pole of the integrand, because the functions $\mathcal{V}\left(\frac{3}{2}i+Y\right)$, $\mathcal{V}\left(\frac{3+\lambda}{2}i+Y\right)$ have also a pole at $Y = 0$. Therefore the first pole of the integrand that is found is the one at $Y = \frac{\lambda-1}{2}i$. Notice that $\mathcal{V}\left(\frac{3}{2}i+Y\right)$ does not have a zero before, since $\frac{3}{2}+\frac{\lambda-1}{2} = 1+\frac{\lambda}{2} < \frac{3}{2}+\frac{\lambda}{2}$. On the other hand $\frac{3+\lambda}{2}+\frac{\lambda-1}{2} = 1+\lambda$, while the first zero of $\mathcal{V}(\eta)$ is at $\eta = 2+\frac{\lambda}{2}$ and $1+\lambda < 2+\frac{\lambda}{2}$. Then, after deforming the integral contour as indicated, the terms J_1 can be written as:

$$J_1 = e^{-\frac{3}{2}X}\,t - \frac{2i}{\lambda-1}\mathcal{V}\left(\left(1+\frac{\lambda}{2}\right)i\right)e^{-\frac{3}{2}X}\int_{Im(Y)=\gamma_2} dy\,\frac{t^{-\frac{2i}{\lambda-1}Y}}{\mathcal{V}(\frac{3}{2}i+Y)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)$$
$$= e^{-\frac{3}{2}X}\,(t+J_3) \quad \text{for } X < 0,$$

and

$$J_1 = e^{-\frac{3+\lambda}{2}X}\frac{\mathcal{V}(2i)\,t}{4\pi\,\mathcal{V}\left(\left(1+\frac{\lambda}{2}\right)i\right)} - \frac{i\,\mathcal{V}(2i)}{2\pi\,(\lambda-1)}e^{-\frac{3+\lambda}{2}X}\int_{Im(Y)=\gamma_2} dy\,\frac{t^{-\frac{2i}{\lambda-1}Y}}{\mathcal{V}(\frac{3+\lambda}{2}i+Y)}\,\Gamma\left(\frac{2iY}{\lambda-1}\right)$$
$$= e^{-\frac{3+\lambda}{2}X}\left(\frac{\mathcal{V}(2i)\,t}{4\pi\,\mathcal{V}\left(\left(1+\frac{\lambda}{2}\right)i\right)} + J_3\right) \quad \text{for } X > 0,$$

where $\gamma_2 > (\lambda-1)/2$ is such that $\gamma_2 - (\lambda-1)/2$ is small. In both cases there exist positive constants δ and C_δ such that for all $t > 1$:

$$|J_3| \le C_\delta t^{1+\delta},$$
$$|J_1| \le Cte^{\beta_\ell X} \quad \text{for} \quad |X| \ge 1,$$

where $\ell = 7$ for $X < 0$ and $\ell = 8$ for $X > 0$.

It then follows that:

$$|J_1| + |J_2| \le Cte^{\beta_\ell X} \quad \text{for} \quad |X| \ge 1,$$

where, as before, $\ell = 7$ for $X < 0$ and $\ell = 8$ for $X > 0$, and Lemma 9.10 follows. □

10. The Initial Value Problem

Using the fundamental solution g obtained in Theorem 3.5 we can obtain a solution of the initial value problem

$$\frac{\partial h}{\partial t} = L[h], \tag{10.1}$$

$$h(0, x) = h_0(x), \tag{10.2}$$

with $L[\cdot]$ defined in (3.2). Assuming that there are not difficulties with the integrals written below, we would expect, due to the linearity of the problem (10.1), (10.2) the following representation formula for their solutions (cf. Theorem 3.5):

$$h(t, x) = \int_0^\infty h_0(y)\, g\left(ty^{\frac{\lambda-1}{2}}, \frac{x}{y}, 1\right) \frac{dy}{y}. \tag{10.3}$$

We first precise sufficient conditions on h_0 that allow to define $h(t, x)$ in (10.3).

Theorem 10.1. *Suppose that the function $h_0 \in C(\mathbb{R}^+)$ satisfies*

$$\int_0^1 |h_0(y)|\, y^\lambda dy + \int_1^\infty |h_0(y)|\, dy < \infty. \tag{10.4}$$

Then the function $h(t, x)$ defined for $t \geq 0,\ \ x > 0$ by means of (10.3) solves the initial value problem (10.1), (10.2).

The proof of this theorem reduces to a detailed analysis of the conditions on h_0 yielding integrability of the right-hand side of (10.3).

Under more stringent assumptions on h_0 it is possible to use Theorem 3.5 to derive more detailed information on the asymptotics of the solutions of (10.1), (10.2) for $x \to 0$ and $x \to \infty$. The meaning of this asymptotics will be explained in the next section.

Theorem 10.2. *Suppose that*

$$|h_0(x)| \leq Cx^{-\frac{3}{2}+\delta}, \quad 0 < x \leq, \quad \varepsilon > 0, \tag{10.5}$$

$$|h_0(x)| \leq Cx^{-(1+\delta)}, \quad x \geq 1, \quad \varepsilon > 0. \tag{10.6}$$

Then the function $h(t, x)$ given in (10.3) satisfies for any $t > 0$,

$$\left| h(t, x) - A_-(t)\, x^{-\frac{3}{2}} \right| \leq B_-(t)\, x^{-\frac{3}{2}+\delta} \quad \text{for } 0 < x \leq 1, \tag{10.7}$$

$$\left| h(t, x) - A_+(t)\, x^{-\frac{3+\lambda}{2}} \right| \leq B_+(t)\, x^{-\frac{3+\lambda}{2}-\delta} \quad \text{for } x \geq 1 \tag{10.8}$$

for suitable functions $A_-(t),\ A_+(t),\ B_-(t),\ B_+(t)$.

Detailed proofs of these two results will be given in [12].

11. Particle Fluxes for Singular Solutions of the Coagulation Equation

It is well known that (1.1), (1.2) may be written as

$$\frac{\partial}{\partial t}(x f) = -\frac{\partial}{\partial x} j(f), \tag{11.9}$$

$$j(f)(t,x) = \int_0^x \int_{x-y}^{\infty} y K(y,z) f(y) f(z)\, dz\, dy \tag{11.10}$$

see [20], as well as [5] for similar formulas in the self similar regime and [15] for an application of the function j to handle with solutions which are singular near the origin.

The number $j(f)(t,x)$ represents the flux of particles at x produced by the collisions. In particular:

$$\frac{d}{dt}\left(\int_{R_1}^{R_2} x f(t,x)\, dx\right) = j(f)(t,R_1) - j(f)(t,R_2). \tag{11.11}$$

Using the formula (11.10) it follows that a solution f of (1.1), (1.2) with the asymptotics $f(t,x) \sim \frac{A(t)}{x^{\frac{3+\lambda}{2}}}$ as $x \to \infty$ can be interpreted as a particle distribution yielding a flux of particles to infinity:

$$\lim_{R\to+\infty} j(f)(t,R) = 2\pi A^2(t).$$

Notice in particular that $f(t,x) = o(x^{-(3+\lambda)/2})$ as $x \to +\infty$ implies that the mass of the particle distribution is conserved. The function $f_s(x) = Ax^{-(3+\lambda)/2}$ can be thought as a singular steady state of (1.1), (1.2) because $j(f_s) = 2\pi A^2$.

We recall that (3.1), (3.2) has been obtained using

$$f(x,t) = x^{-\frac{3+\lambda}{2}} + g(x,t) \tag{11.12}$$

in (1.1), (1.2) and keeping just linear terms on g. Then, we can derive formulas for the particle fluxes associated to (3.1), (3.2) linearizing (11.10), (11.14):

$$j_{lin}(g)(t,x) = \int_0^x \int_{x-y}^{\infty} \left[y^{-1/2} z^{\lambda/2} g(z) + y^{1+\lambda/2} z^{-3/2} g(y)\right] dz\, dy \tag{11.13}$$

and

$$\frac{d}{dt}\left(\int_{R_1}^{R_2} x\, g(t,x)\, dx\right) = j_{lin}(g)(t,R_1) - j_{lin}(g)(t,R_2). \tag{11.14}$$

Using $g(t,x) \sim a(t)\, x^{-3/2}$ as $x \to 0$ it follows that for all $t > 0$, $\lim_{R\to 0^+} j_{lin}(g)(t,R) = 0$. Therefore the perturbation g does not modify the incoming flux of mass that is the one of $f_s(x) = x^{-(3+\lambda)/2}$. Moreover

$$\frac{d}{dt}\left(\int_0^R xg(t,x)\, dx\right) = -j_{lin}(g)(t,R). \tag{11.15}$$

Integrating (11.15) we obtain:

$$\int_0^R x\, g(0,x) dx = \int_0^t j_{lin}(g)(s,R) ds + \int_0^R x\, g(t,x) dx. \tag{11.16}$$

Since the solution $g(t, x)$ obtained in Theorem 3.5 satisfies $g(t,x) \sim a(t)\, x^{-(3+\lambda)/2}$ as $x \to +\infty$, taking the limit $R \to \infty$ in (11.16) it follows that:

$$\int_0^\infty x\, g(0,x)dx = 4\pi \int_0^t a(s)\, ds + \int_0^\infty x\, g(t,x)dx. \tag{11.17}$$

The self-similar asymptotics (3.9)-(3.12) implies $\lim_{t\to\infty} \int_0^\infty xg(t,x)\,dx = 0$. Then:

$$\int_0^\infty x\, g(0,x)dx = 4\pi \int_0^\infty a(s)\, ds. \tag{11.18}$$

The left-hand side of (11.18) is the initial total mass of the perturbation. The right hand side of (11.18) is the total amount of particles contained in clusters of infinite size. Equation (11.18) means that all the excess of particles initially introduced in the system move as $t \to +\infty$ to an infinitely large cluster.

12. The Function Φ

We take now the Mellin transform on both hands of the equation. We recall that the Mellin transform of a function $g(y)$ is defined as:

$$\mathcal{M}(g)(s) = \int_0^\infty y^{s-1} g(y)dy. \tag{12.1}$$

Taking the Mellin transform of the right hand side of (3.1), (3.2) we obtain after straightforward calculations:

$$\frac{\partial}{\partial t}\mathcal{M}(g)(s) = \mathcal{M}(g)\left(s + \frac{\lambda-1}{2}\right) P\left(-s+\frac{3}{2}\right), \tag{12.2}$$

$$P(s) = \int_2^\infty \theta^{1/2-s}\left((\theta-1)^{-3/2} - \theta^{-3/2}\right) d\theta$$
$$+ \int_{1/2}^1 (1-\theta)^{-3/2}\left(\theta^{s-1}-1\right) d\theta + \frac{2^{-s}}{s} - 2\sqrt{2} \tag{12.3}$$
$$= I_1(s) + I_2(s) + I_3(s) + I_4. \tag{12.4}$$

Remark 12.1. If the function g satisfies the estimates (3.5), (3.6) for some $r > 1$ and $\rho < 2$ such that $\rho < r$ we will have that its Mellin transform is well defined in the strip $Res \in (\lambda/2 + \rho, \lambda/2 + r)$.

12.1. The function $P(s)$. We consider in this section the auxiliary function P obtained by taking the Mellin transform of Eq. (3.1), (3.2) and first rewrite it in terms of Gamma functions.

Proposition 12.2. *The function $P(s)$ defined in (12.3) can be written as*

$$P(s) = -\frac{2\sqrt{\pi}\,\Gamma(s)}{\Gamma(s-1/2)}. \tag{12.5}$$

The function $P(s)$ is meromorphic on the whole complex plane. It has simple zeros and poles at the points

$$s_z(n) = \frac{1}{2} - n, \quad n = 0, 1, 2, \ldots, \tag{12.6}$$

and

$$s_p(n) = -n, \quad n = 0, 1, 2, \ldots,$$

respectively

Proof. We can write the term I_1 of (12.4) as

$$I_1(s) = \int_2^\infty \theta^{1/2-s}\theta^{-3/2}\left\{\left(1-\frac{1}{\theta}\right)^{-3/2} - 1\right\} d\theta.$$

Using the Binomial Theorem to expand $\left(1-\frac{1}{\theta}\right)^{-3/2}$ and integrating each term of the resulting series we obtain:

$$I_1(s) + I_3(s) = 2^{-s}\sum_{n=0}^{\infty}\binom{-3/2}{n}(-1)^n\frac{2^{-n}}{s+n}. \tag{12.7}$$

Integrating by parts we obtain:

$$I_2(s) = 2\sqrt{2} - 2\sqrt{2}(1/2)^{s-1} - 2(s-1)\int_{1/2}^{1}(1-\theta)^{-1/2}\theta^{s-2}d\theta. \tag{12.8}$$

In order to compute the last term in (12.8) we write:

$$\int_{1/2}^{1}(1-\theta)^{-1/2}\theta^{s-2}d\theta = \int_0^1(1-\theta)^{-1/2}\theta^{s-2}d\theta - \int_0^{1/2}(1-\theta)^{-1/2}\theta^{s-2}d\theta. \tag{12.9}$$

Expanding $(1-\theta)^{-1/2}$ using the Binomial Theorem and integrating each term of the resulting series:

$$\int_0^{1/2}(1-\theta)^{-1/2}\theta^{s-2}d\theta = 2^{-s}\left\{\frac{2}{s-1} + \sum_{\ell=0}^{\infty}\binom{-1/2}{\ell+1}\frac{(-1)^{\ell+1}}{\ell+s}2^{-\ell}\right\}. \tag{12.10}$$

Moreover,

$$\frac{2}{s-1} + \sum_{n=0}^{\infty}\binom{-1/2}{n+1}\frac{(-1)^{n+1}}{n+s}2^{-n} = \frac{2(1-1/2)^{-1/2}}{s-1} - \frac{1}{2(s-1)}\sum_{n=0}^{\infty}\binom{-3/2}{n+1}\frac{(-1)^n}{n+s}2^{-n}, \tag{12.11}$$

where we have used that $\binom{-1/2}{n+1} = -\frac{1}{2(n+1)}\binom{-3/2}{n}$ as well as the Binomial Theorem. Combining (12.10) and (12.11) we deduce

$$\int_0^{1/2} (1-\theta)^{-1/2}\theta^{s-2}d\theta = 2^{-s}\left\{\frac{2\sqrt{2}}{s-1} - \frac{1}{2(s-1)}\sum_{n=0}^{\infty}\binom{-3/2}{n+1}\frac{(-1)^n 2^{-n}}{n+s}\right\}. \tag{12.12}$$

Therefore, using (12.3), (12.7), (12.8), (12.9) and (12.12),

$$P(s) = -2(s-1)\int_0^1 (1-\theta)^{-1/2}\theta^{s-2}d\theta = -\frac{2\sqrt{2}\,\Gamma(s)}{\Gamma(s-1/2)}. \tag{12.13}$$

The properties of zeros and poles are consequence of the properties of the Gamma function. □

12.2. Behaviour of Φ at infinity.

Proposition 12.3. *The following asymptotic formulas hold:*

$$P(s) = -2\sqrt{\pi s}\left(1 - \frac{3}{8s} + O\left(\frac{1}{s^2}\right)\right) \quad as \quad |Im(s)| \to \infty \tag{12.14}$$

uniformly in sets where $\arg(s) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ *for any* $\varepsilon_0 > 0$.

Proof. Formula (12.14) is a consequence of (12.5) as well as the asymptotic formula:

$$\Gamma(z) \sim \sqrt{2\pi}\,(z)^{z-\frac{1}{2}}\,e^{-z}\left(1 + \frac{1}{12z} + O\left(\frac{1}{z^2}\right)\right) \quad \text{as} \quad |z| \to \infty \tag{12.15}$$

that is uniformly valid in sets $\arg(z) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ for any $\varepsilon_0 > 0$.
Then

$$P(s) = -\frac{2\sqrt{\pi}\,(s)^{s-\frac{1}{2}}\left(s-\frac{1}{2}\right)^{\frac{1}{2}}}{\left(s-\frac{1}{2}\right)^{s-\frac{1}{2}}e^{\frac{1}{2}}}\left(1 + O\left(\frac{1}{s^2}\right)\right) \quad \text{as} \quad |s| \to \infty,$$

uniformly in sets $\arg(s) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ and $\arg(s - 1/2) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ for any $\varepsilon_0 > 0$. Notice that

$$\frac{(s)^{s-\frac{1}{2}}}{\left(s-\frac{1}{2}\right)^{s-\frac{1}{2}}} = \sqrt{e}\left[1 - \frac{1}{8\left(s-\frac{1}{2}\right)} + O\left(\frac{1}{s^2}\right)\right] \quad \text{as} \quad |s| \to \infty,$$

uniformly in the same sets as above, whence

$$P(s) = -2\sqrt{\pi s}\left(1 - \frac{3}{8s} + O\left(\frac{1}{s^2}\right)\right) \quad \text{as} \quad |s| \to \infty,$$

uniformly in sets $\arg(s) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ for any $\varepsilon_0 > 0$. and (12.14) follows. □

13. On the Properties of the Function $\mathcal{V}$

We prove now some auxiliary results used to prove the results of the paper.

Lemma 13.1. *Suppose that f and h are two analytic functions in the cone*

$$C(2\varepsilon_0) = \left\{\zeta \in \mathbb{C};\ \zeta = |\zeta|e^{i\theta},\ \theta \in (-2\varepsilon_0, 2\varepsilon_0)\right\}$$

for some $\varepsilon_0 > 0$ and real valued in $\mathbb{R}^+$. Let us also assume that

$$\int_0^\infty \frac{|f(re^{i\theta})| + |h(re^{i\theta})|}{1+r^2} dr < +\infty, \quad \textit{for any} \quad \theta \in (-2\varepsilon_0, 2\varepsilon_0), \tag{13.1}$$

$$\lim_{|\zeta|\to 0} f(\zeta) = \theta_1 \quad \textit{and} \quad \lim_{|\zeta|\to\infty} f(\zeta) = \theta_2, \tag{13.2}$$

$$|f'(\zeta)| = o\left(\frac{1}{|\zeta|}\right) \quad \textit{as} \quad |\zeta| \to 0,\ |\zeta| \to \infty,\ \zeta \in C(2\varepsilon_0). \tag{13.3}$$

Then, the function

$$F(\zeta) = \frac{1}{2\pi i}\int_0^\infty (h(s) + if(s))\left(\frac{1}{s-\zeta} - \frac{1}{s+1}\right) ds \tag{13.4}$$

is analytic in the domain:

$$D(\varepsilon_0) = \left\{\zeta \in \mathcal{R};\ \zeta = |\zeta|e^{i\theta},\ |\zeta| > 0,\ \theta \in (-\varepsilon_0, 2\pi + \varepsilon_0)\right\}, \tag{13.5}$$

where $\mathcal{R}$ is the Riemann surface associated to the function $\ln \zeta$. *Moreover,*

$$F(\zeta) = -\frac{\theta_1}{2\pi}\ln\zeta + iH(\zeta) + o(\ln|\zeta|), \quad \textit{as} \ \ \zeta \to 0,\ \zeta \in D(\varepsilon_0), \tag{13.6}$$

$$F(\zeta) = -\frac{\theta_2}{2\pi}\ln\zeta + iH(\zeta) + o(\ln|\zeta|), \quad \textit{as} \ \ |\zeta| \to +\infty,\ \zeta \in D(\varepsilon_0), \tag{13.7}$$

where the function $H(\zeta)$ is a real valued function defined by

$$H(\zeta) = -\frac{1}{2\pi}\int_0^\infty h(s)\left(\frac{1}{s-\zeta} - \frac{1}{s+1}\right) ds. \tag{13.8}$$

Proof of Lemma 13.1. Using Lemma C.2 of [10] with the function f we obtain (13.6) and (13.8). On the other hand, the condition (13.1) ensures that the function F is well defined. □

Proposition 13.2. *Let $\mathcal{V}(\xi)$ defined by (6.6) and (6.12). Then, for any $\varepsilon > 0$ arbitrarily small and all $M > 0$ arbitrarily large, there exist two positive constants $C_{1,\varepsilon,M}$, $C_{2,\varepsilon,M}$ such that*

$$C_{1,\varepsilon,M}e^{-\frac{1}{2}(\frac{\pi}{\lambda-1}+\varepsilon)|\xi|} \le |\mathcal{V}(\xi)| \le C_{2,\varepsilon,M}e^{-\frac{1}{2}(\frac{\pi}{\lambda-1}-\varepsilon)|\xi|} \tag{13.9}$$

uniformly for $Im(\xi)$ in compact sets of $(3/2, (3+\lambda)/2)$ as well as for all ξ such that $|Re(\xi)| \ge 1$, $|Im(\xi)| \le M$.

Proof of Proposition 13.2. Given $\xi \in \mathcal{S}$ we can represent the function $\mathcal{V}$ by (6.6) and (6.12) for β_1 satisfying (6.13). In order to simplify some of the calculations we use the following change of variables.

$$\zeta = e^{\frac{4\pi}{\lambda-1}(\xi-\beta_1)}, \tag{13.10}$$

$$\nu(\zeta) = \mathcal{V}(\xi), \tag{13.11}$$

$$\varphi(\zeta) = \Phi(\xi). \tag{13.12}$$

The function $\ln(-\varphi(s))$ may be written as

$$\ln(-\varphi(s)) = \ln(|\varphi(s)|) + i \arg(-\varphi(s)) \,. \tag{13.13}$$

The functions $\ln(|\varphi(s)|)$ and $\arg(\varphi(s))$ satisfy the hypothesis required to h and f respectively in Lemma 13.2. In particular, by Proposition 4.1 and the fact that $Re(-\Phi(y)) > 0$ for all y such that $Im(y) \in ((2+\lambda)/2, (3+\lambda)/2)$ we may normalize the argument of the function $\ln(-\varphi(s))$ such that:

$$\lim_{\zeta\to 0} \arg(-\varphi(\zeta)) = -\frac{\pi}{4}, \qquad \lim_{\zeta\to\infty} \arg(-\varphi(\zeta)) = \frac{\pi}{4}. \tag{13.14}$$

Applying Lemma 13.1 it follows that

$$\frac{1}{2\pi i}\int_0^\infty \ln(-\varphi(s))\left(\frac{1}{s-\zeta} - \frac{1}{s+1}\right) ds = \frac{1}{8}\ln(-\zeta) + iH(\zeta) + o(\ln|\zeta|) \quad \text{as } \zeta \to 0,\ \zeta \in D(\varepsilon_0), \tag{13.15}$$

$$\frac{1}{2\pi i}\int_0^\infty \ln(-\varphi(s))\left(\frac{1}{s-\zeta} - \frac{1}{s+1}\right) ds = -\frac{1}{8}\ln(-\zeta) + iH(\zeta) + o(\ln|\zeta|) \quad \text{as } \zeta \to \infty,\ \zeta \in D(\varepsilon_0). \tag{13.16}$$

The two estimates in (13.9) follow, for $Im(\xi)$ in compact sets of $(3/2, (3+\lambda)/2)$, by taking exponentials in both sides of (13.15) and (13.16) and inverting the change of variables (13.10)-(13.12).

In order to prove the estimate for ξ in the region $|Re(\xi)| \geq 1$ and $|Im(\xi)| \leq M$, we extend analytically the function $\mathcal{V}(\xi)$ to such regions using (5.11) as well as the fact that, by Proposition 4.1, we have for some positive constants C_1 and C_2:

$$C_1|\xi|^{1/2} \leq |\Phi(\xi)| \leq C_2|\xi|^{1/2}$$

for $|Re(\xi)| \geq 1$ and $|Im(\xi)| \leq M$. □

14. Contour Deformation Estimates

We must estimate in Sects. 8 and 9 several integral expressions of the form

$$\int_{Im\,Y=-\gamma_1} e^{\Psi(\xi,Y,t)} m(\xi, Y)\, dY$$

for a given function Ψ but different functions m. This is done using contour deformation combined with Laplace's method. We collect in this section some technical results about the function $\Psi(\xi, Y, t)$ and its critical points.

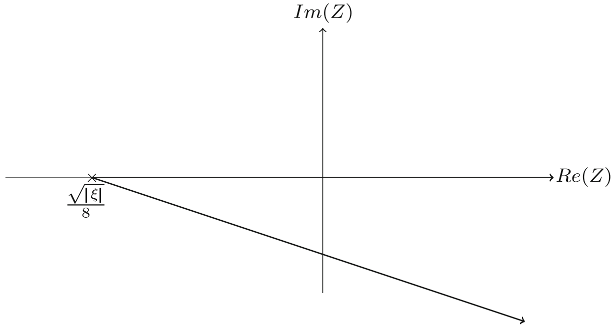

Fig. 6. $D(\xi, B)$ when $Re(\xi) > 0$ and then $Q = 1$

14.1. The critical point. We first compute the critical points of the function $\Psi(\xi, Y, t)$ defined by (8.4) or equivalently, those of $\widetilde{\Psi}(\xi, Z, t)$ (cf. (8.12)).

14.1.1. The case $0 < t < 1$ We start with the following:

Lemma 14.1. *Consider the function $F(\xi, Z)$ defined by means of*

$$F(\xi, Z) = \frac{2}{(\lambda - 1)\, i} \int_{Im(\eta)=\beta_1} \ln(-\Phi(\eta))\, \Theta(\eta - \xi, \sqrt{|\xi|} Z) d\eta \tag{14.1}$$

with β_1 as in (6.13), $\xi \in \mathbb{C}$ and

$$\frac{1}{\sqrt{|\xi|}}(\beta_1 - Im(\xi)) - \frac{\lambda - 1}{2} < Im(Z) \leq \frac{(\beta_1 - Im(\xi))}{\sqrt{|\xi|}}. \tag{14.2}$$

For any constant $B > 0$ there exists $L > 0$, and ξ_0, both depending on B, such that, for all $\xi \in \mathcal{T}_L \cap (B_{\xi_0}(0))^c$, the function $F(\xi, \cdot)$ can be extended analytically on the variable Z to the domain $Z \in D(\xi, B) \cap B_{\sqrt{|\xi|}/8}(0)$, where $D(\xi, B)$ is as in (8.6). Moreover, there exists a positive constant C depending on B such that

$$\left| F(\xi, Z) + \frac{2i}{(\lambda - 1)} \ln(-\Phi(\xi))\, \sqrt{|\xi|}\, Z \right| \leq C \left(Z^2 + \mathcal{O}\left(\frac{1}{|\xi|} \right) \right) \tag{14.3}$$

for $Z \in D(\xi, B) \cap B_{\sqrt{|\xi|}/8}(0)$ and , $\xi \in \mathcal{T}_L \cap (B_{\xi_0}(0))^c$.

Proof. The function F is well defined in (14.2) since in that domain the variable Z is in the region where the function Θ is analytic. The function $F(\xi, \cdot)$ given by (14.1) is then analytic in the strip $|Im(Z)| \leq \frac{\delta_0(Im(\xi))}{\sqrt{|\xi|}}$ for some $\delta_0(Im(\xi))$ sufficiently small.

We now claim that for any fixed constant $B > 0$, there exists $L > 0$ (large) depending on B such that for $\xi \in \mathcal{T}_L$ the function $F(\xi, \cdot)$ may be extended analytically to the region $D(\xi, B)$ Fig. 6. To prove this we derive new representation formulas of the function F performing suitable contour deformations in the variable η in (14.1).

Notice that the singularities of the integrand in (14.1) are contained in the set $Re(\eta) = 0$, $\eta = \xi + (\lambda - 1)\ell/2$ and $\eta = \xi + \sqrt{|\xi|}\, Z + (\lambda - 1)\ell/2$ for $\ell \in \mathbb{Z}$. Given Z_0 in the region above, let $\widehat{Z}_0$ be such that:

$$Re(\widehat{Z}_0) = Re(Z_0), \quad |Im(\widehat{Z}_0)| \leq \frac{\delta_0(Im(\xi))}{\sqrt{|\xi|}}$$

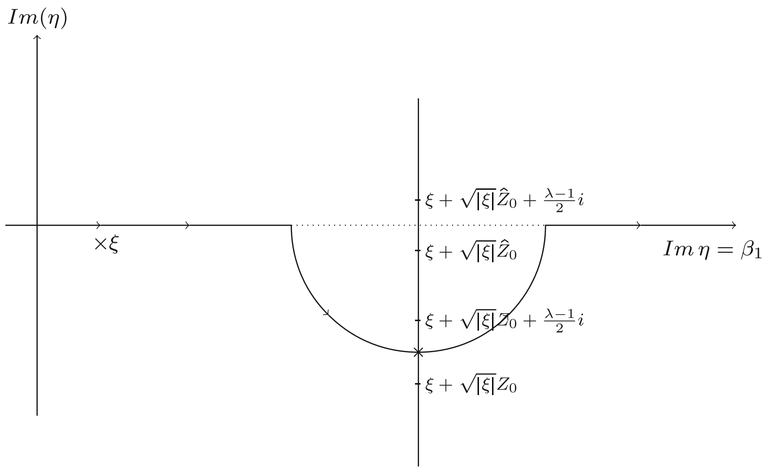

Fig. 7. The curve $\mathcal{C}_2$

and the integral curve $Im\,(\eta) = \beta_1$ lies between the two points $\xi + \widehat{Z}_0\sqrt{|\xi|}$ and $\xi + \widehat{Z}_0\sqrt{|\xi|} + (\lambda - 1)i/2$. (Notice that this is possible since $\delta_0(Im(\xi))$ may be made as small as we need and $Im(\eta) > Im(\xi)$).

Consider the vertical segment of the complex plane connecting Z_0 and $\widehat{Z}_0$: $Z_\theta = (1-\theta)\widehat{Z}_0 + \theta Z_0$, $\theta \in [0, 1]$. We then obtain an analytic extension of $F(\xi, \cdot)$ varying θ continuously from 0 to one and deforming continuously the contour $Im\,\eta = \beta_0 + \frac{\theta-1}{2} - \varepsilon$ to a new contour $\mathcal{C}_2$ (cf. Fig. 7), in such a way that:

- it always passes between $\xi + \widehat{Z}_\theta\sqrt{|\xi|}$ and $\xi + \widehat{Z}_\theta\sqrt{|\xi|} + (\lambda - 1)i/2$
- we do not change the original integration contour for

$$\left|Re\left(\eta - \xi - Z_0\sqrt{|\xi|}\right)\right| \geq \frac{|Re(Z_0)\sqrt{|\xi|}}{2}.$$

The first condition ensures that the integration contour never crosses any of the singularities of the function $\left(1 - e^{-\frac{4\pi}{\lambda-1}(\eta-\xi-Z\sqrt{|\xi|})}\right)^{-1}$. The second one ensures that it does not cross either of the singularities of $\left(1 - e^{\frac{4\pi}{\lambda-1}(\eta-\xi)}\right)^{-1}$.

Finally, since $sgn(Re(Z_0)) = sgn(Re(\xi))$, the new integration contour never crosses the line $Re(\eta) = 0$, where the singularities of $\ln(-\Phi(\eta))$ are located.

To estimate this integral we write

$$\begin{aligned}
&\frac{2}{(\lambda-1)\,i}\int_{\mathcal{C}_2} \ln(-\Phi(\eta))\,\Theta(\eta-\xi, \sqrt{|\xi|}Z)d\eta \\
&= \frac{2\,\ln(-\Phi(\xi))}{(\lambda-1)\,i}\int_{\mathcal{C}_2} \Theta(\eta-\xi, \sqrt{|\xi|}Z)d\eta \\
&\quad + \frac{2}{(\lambda-1)\,i}\int_{\mathcal{C}_2} \ln\left(\frac{\Phi(\eta)}{\Phi(\xi)}\right)\Theta(\eta-\xi, \sqrt{|\xi|}t\,Z)d\eta \\
&= I_1 + I_2.
\end{aligned} \tag{14.4}$$

The first integral, I_1 is computed explicitly :

$$I_1 = \frac{2\,\ln(-\Phi(\xi))}{(\lambda-1)\,i}\sqrt{|\xi|}\,Z = -\frac{2\,i}{(\lambda-1)}\ln(-\Phi(\xi))\,\sqrt{|\xi|}\,Z. \tag{14.5}$$

In order to compute the second integral we have to distinguish the cases $Re(\xi)\to-\infty$ and $Re(\xi)\to+\infty$. Since both may be treated using similar arguments let us treat only the case $Re(\xi)\to+\infty$. In that case we decompose I_2 as follows

$$\begin{aligned} I_2 = {} & \frac{2}{(\lambda-1)\,i}\int_{\mathcal{C}_2,Re(\eta)>0,|\eta-\xi|\le\frac{|\xi|}{4}}[\cdots]\,d\eta + \int_{\mathcal{C}_2,Re(\eta)>0,|\eta-\xi|>\frac{|\xi|}{4}}[\cdots]\,d\eta \\ & + \frac{2}{(\lambda-1)\,i}\int_{\mathcal{C}_2,Re(\eta)<0}[\cdots]\,d\eta = I_{2,1}+I_{2,2}+I_{2,3}. \end{aligned} \tag{14.6}$$

In $I_{2,1}$ we use Proposition 4.1 and Taylor's expansion to obtain:

$$\frac{\Phi(\eta)}{\Phi(\xi)} = \frac{\sqrt{\eta}\left(1-\frac{(2\lambda+1)i}{8\eta}+O(|\eta|^{-2}\right)}{\sqrt{\xi}\left(1-\frac{(2\lambda+1)i}{8\xi}+O(|\xi|^{-2}\right)}.$$

Therefore:

$$\ln\left(\frac{\Phi(\eta)}{\Phi(\xi)}\right) = \mathcal{O}\left(\frac{\eta-\xi}{\xi}\right)+\mathcal{O}\left(\frac{1}{\xi^2}\right),\quad \text{as } Re(\xi)>1,\ |\eta-\xi|\le\frac{|\xi|}{4}.$$

We now estimate the two following integrals for all $Z\in D(\xi,B)$ and $|Z|\le\frac{\sqrt{|\xi|}}{8}$. The first one can be bounded as:

$$\begin{aligned} & \int_{\mathcal{C}_2,Re(\eta)>0,|\eta-\xi|\le\frac{|\xi|}{4}}\left(\frac{|\eta-\xi|}{|\xi|}\right)\Theta(\eta-\xi,\sqrt{|\xi|}Z)d\eta \\ & = \frac{1}{|\xi|}\int_{\widetilde{\mathcal{C}}_2,Re(\sigma)>-Re(\xi),\,|\sigma|\le\frac{|\xi|}{4}}\left|\sigma\Theta(\sigma,\sqrt{|\xi|}Z)\right|d\sigma \\ & \le \frac{1}{|\xi|}\int_{\widetilde{\mathcal{C}}_2,Re(\sigma)>-Re(\xi),\,|\sigma|\le 2|\xi|^{1/2}Z}\left|\sigma\Theta(\sigma,\sqrt{|\xi|}Z)\right|d\sigma \\ & \quad + \frac{1}{|\xi|}\int_{\widetilde{\mathcal{C}}_2,Re(\sigma)>-Re(\xi),\,|\sigma|\ge 2|\xi|^{1/2}Z}\left|\sigma\Theta(\sigma,\sqrt{|\xi|}Z)\right|d\sigma \\ & \le C\,Z^2 + C\frac{e^{-a|\xi|^{1/2}|Z|}}{|\xi|}\quad \text{for } Re(\xi)>\xi_0, \end{aligned}$$

and the second one:

$$\begin{aligned}
&\int_{\mathcal{C}_2, Re(\eta)>0, |\eta-\xi|\leq\frac{|\xi|}{4}} \frac{1}{|\xi|^2}\left|\Theta(\eta-\xi,\sqrt{|\xi|}Z)\right| d\eta \\
&\quad = \frac{1}{|\xi|^2}\int_{\widetilde{\mathcal{C}}_2, Re(\sigma)>-Re(\xi), |\sigma|\leq\frac{|\xi|}{4}} \left|\Theta(\sigma,\sqrt{|\xi|}Z)\right| d\sigma \\
&\quad \leq \frac{1}{|\xi|^2}\int_{\widetilde{\mathcal{C}}_2, Re(\sigma)>-Re(\xi), |\sigma|\leq 2|\xi|^{1/2}Z} \left|\Theta(\sigma,\sqrt{|\xi|}Z)\right| d\sigma \\
&\qquad + \frac{1}{|\xi|^2}\int_{\widetilde{\mathcal{C}}_2, Re(\sigma)>-Re(\xi), |\sigma|\geq 2|\xi|^{1/2}Z} \left|\Theta(\sigma,\sqrt{|\xi|}Z)\right| d\sigma \\
&\quad \leq C\frac{|Z|}{|\xi|^{3/2}} + C\frac{e^{-a|\xi|^{1/2}|Z|}}{|\xi|^2} \leq C\left(|Z|^2+\frac{1}{|\xi|^3}\right) + C\frac{e^{-a|\xi|^{1/2}|Z|}}{|\xi|^2} \quad \text{for } Re(\xi)>\xi_0,
\end{aligned}$$

where in both cases $\widetilde{\mathcal{C}}_2 = \mathcal{C}_2 - \xi$, ξ_0 is a positive constant sufficiently large, depending on B and a and C are positive constants which may depend on B but are independent on $Re(\xi)$ and $Z \in D(\xi, B)$. We then deduce that, for all $Re(\xi) > \xi_0$:

$$|I_{2,1}| \leq C\left(|Z|^2+\frac{1}{|\xi|^3}\right) + C\frac{e^{-a|\xi|^{1/2}|Z|}}{|\xi|}. \tag{14.7}$$

In the integral $I_{2,2}$ we use the fact that when $|\eta-\xi| > \frac{|\xi|}{4}$ and $Re(\xi) > \xi_0$, the function $\Theta(\eta-\xi, Y)$ has an exponential decay $Ce^{-a|\xi|}$ with C and a as above as well as the inequality $|\ln(-\Phi(\eta))| \leq C|\ln(|\eta-\xi|+|\xi|)| \leq C|(\ln(|\eta-\xi|)|+|\ln(|\xi|)|)$ for η large. For η of order one we use that $\ln(-\Phi(\eta))$ is of order one to derive a similar estimate. Then

$$|I_{2,2}| \leq \int_{|\eta-\xi|>\frac{|\xi|}{4}} e^{-a|\eta-\xi|}\left(|\ln(|\eta-\xi|)|+|\ln(|\xi|)|\right) d\eta = \mathcal{O}\left(e^{-a|\xi|}\right). \tag{14.8}$$

Finally, the estimate of $I_{2,3}$ follows using the same cut-off properties of the function Θ since $Re(\eta) < 0$ and $Re(\xi) \to +\infty$ implies that $Re(\eta-\xi) > C|\xi|$. The final estimate of I_2, by (14.6), is then,

$$|I_2| \leq C\left(|Z|^2+\frac{1}{|\xi|^3}\right) + C\frac{e^{-a|\xi|^{1/2}|Z|}}{|\xi|}. \tag{14.9}$$

Using Proposition 4.1, (8.12), (14.4), (14.5) and (14.9) the lemma follows. □

Lemma 14.2. *Given $B > 0$, let ξ_0, L be the ones given by Lemma 14.1. Then there exists a constant $C > 0$ depending on B such that the function h defined by means of:*

$$\begin{aligned}
\widetilde{\Psi}(\xi, Z, t) = -\sqrt{|\xi|}\frac{2iZ}{\lambda-1}\left[1+\ln t - \ln\left(\frac{2iZ}{\lambda-1}\right) + \ln\left(2\sqrt{\pi}e^{iQ\frac{\pi}{4}}\right)\right] \\
-\frac{1}{2}\ln\left(|\xi|^{1/2}\right) - \frac{1}{2}\ln\left(\frac{2iZ}{\lambda-1}\right) + h(\xi, Z, t)
\end{aligned} \tag{14.10}$$

satisfies

$$|h(\xi, Z, t)| \leq C\left(Z^2 + \mathcal{O}\left(\frac{1}{|\xi|}\right)\right)$$

for $\xi \in \mathcal{T}_L \cap (B_{\xi_0}(0))^c$, $Z \in D(\xi, B) \cap B_{\sqrt{|\xi|}/8}(0)$.

Proof. This lemma is a direct consequence of Lemma (14.1). The only difficulty in order to estimate the function $\widetilde{\Psi}$ defined in (8.12) comes from the term $\mathcal{V}(\xi)/\mathcal{V}(\xi+Y)$ which corresponds to the integral term. This term, given by formula (6.15) may also be written as:

$$\begin{aligned}\Lambda(\xi, Z) &= \frac{\mathcal{V}(\xi)}{\mathcal{V}(\xi+\sqrt{|\xi|}Z)}\\ &= \exp\left[\frac{2}{(\lambda-1)\,i}\int_{Im\,\eta=\beta_1} \ln\left(-\Phi(\eta)\right)\Theta(\eta-\xi,\sqrt{|\xi|}Z)d\eta\right].\end{aligned}$$

□

As for the critical point Z_c of $\widetilde{\Psi}(\xi,\cdot,t)$, the precise result is the following.

Lemma 14.3. *Suppose that $B > 2\sqrt{\pi}$ let ξ_0, L be the ones given by Lemma 14.1. Let us define $\delta_0 = (\lambda-1)\sqrt{\pi}/4$. There exists $R > 0$ depending on B, such that for all $0 \le t \le 1$ and $\xi \in \mathcal{T}_L \cap (B_{\xi_0}(0))^c$, $|\xi|\,t^2 \ge R$, there exists a unique point $Z_c \in D(\xi, B)\backslash B_{\delta_0 t}(0)$ such that $\partial\widetilde{\Psi}(\xi, Z_c, t)/\partial Z = 0$. Moreover the following asymptotics holds uniformly for $0 \le t \le 1$:*

$$\frac{2i\,Z_c}{\lambda-1} = \sqrt{2\pi}\,t(1+iQ))\left(1+\mathcal{O}\left(\frac{1}{\sqrt{|\xi|}\,t}\right)\right) \quad as\ \ |Re(\xi)|\,t^2 \to \infty. \tag{14.11}$$

Proof. Computing the derivative of the function $\widetilde{\Psi}$ given by (8.12) gives for all $Z \in D(\xi, B)$:

$$\begin{aligned}\frac{\partial\widetilde{\Psi}}{\partial Z}(\xi, Z, t) = &-\frac{8\pi\sqrt{|\xi|}\,i}{(\lambda-1)^2}\int_{\mathcal{C}_2}\ln\left(-\Phi(\eta)\right)\frac{e^{\frac{4\pi}{\lambda-1}(\sqrt{|\xi|}Z-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(\sqrt{|\xi|}Z-\eta+\xi)}\right)^2}d\eta\\ &+\sqrt{|\xi|}\left(+\left(\frac{2i}{\lambda-1}\right)\ln\left(\frac{2i(Z/t)}{\lambda-1}\right)-\frac{1}{2\sqrt{|\xi|}Z}\right.\\ &\left.+\frac{2i}{\lambda-1}\ln|\xi|^{1/2}\right).\end{aligned} \tag{14.12}$$

We compute the leading term of the integral in the right hand side of (14.12) as $|\xi| \to \infty$:

$$\begin{aligned}I(\xi, Z) &= \int_{\mathcal{C}_2}\ln\left(-\Phi(\eta)\right)\frac{e^{\frac{4\pi}{\lambda-1}(\sqrt{|\xi|}Z-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(\sqrt{|\xi|}Z-\eta+\xi)}\right)^2}d\eta\\ &= \int_{\widehat{\mathcal{C}}_2}\ln\left(-\Phi(\sigma+\xi+\sqrt{|\xi|}Z)\right)\frac{e^{-\frac{4\pi}{\lambda-1}\sigma}}{\left(1-e^{-\frac{4\pi}{\lambda-1}\sigma}\right)^2}d\sigma,\end{aligned}$$

where $\widehat{\mathcal{C}}_2 = C_2 - \xi - \sqrt{|\xi|}\,Z$. Using Proposition 4.1 we have that, uniformly for $Z \in D(\xi, B)$, $|Z| \le B$:

$$\ln\left(-\Phi(\sigma+\xi+\sqrt{|\xi|}Z)\right) = \ln\left(-\Phi(\xi+\sqrt{|\xi|}Z)\right) + A(\xi)\frac{\sigma}{|\xi|} + \mathcal{O}\left(\frac{\sigma^2}{|\xi|^2}\right)$$

for $|\xi| \to +\infty$, where $A(\xi)$ is a bounded function of $sgn(Re(\xi))$. It then follows:

$$I(\xi, Z) = -\frac{\lambda-1}{4\pi}\ln\left(-\Phi(\xi+\sqrt{|\xi|}Z)\right) + \frac{A(\xi)}{|\xi|}\int_{\widehat{\mathcal{C}}_2}\sigma\frac{e^{-\frac{4\pi}{\lambda-1}\sigma}}{\left(1-e^{-\frac{4\pi}{\lambda-1}\sigma}\right)^2}d\sigma + \int_{\widehat{\mathcal{C}}_2}\mathcal{O}\left(\frac{\sigma^2}{|\xi|^2}\right)\frac{e^{-\frac{4\pi}{\lambda-1}\sigma}}{\left(1-e^{-\frac{4\pi}{\lambda-1}\sigma}\right)^2}d\sigma + \mathcal{O}\left(e^{-a|\xi|}\right) \text{ as } |\xi| \to +\infty \tag{14.13}$$

uniformly for $Z \in D(\xi, B)$, $|Z| \le B$, where $a > 0$ is independent on ξ, Z, where we have used $\int_{\widehat{\mathcal{C}}_2}\frac{e^{-\frac{4\pi}{\lambda-1}\sigma}}{\left(1-e^{-\frac{4\pi}{\lambda-1}\sigma}\right)^2}d\sigma = -\frac{\lambda-1}{4\pi}$.

On the other hand, in order to estimate the second term in the right-hand side of (14.13) we deform the integration contour $\widehat{\mathcal{C}}_2$ to a horizontal line at a bounded distance of the real axis. The resulting integral can then be bounded by a positive constant independent of ξ, Z. The third term in the right hand side of (14.13) can be bounded using the specific form of $\widehat{\mathcal{C}}_2$ as:

$$C\left(\frac{|Z|^2}{|\xi|}+\frac{|Z|^3}{|\xi|^{1/2}}\right).$$

We notice that by Proposition 4.1, we have:

$$\ln\left(-\Phi(\xi+\sqrt{|\xi|}Z)\right) = \ln\left(-\Phi(\xi)\right) + \mathcal{O}\left(\frac{|Z|}{\sqrt{|\xi|}}\right), \text{ as } |\xi| \to +\infty$$

uniformly for $Z \in D(\xi, B)$, $|Z| \le B$. Combining everything we deduce:

$$I(\xi, Z) = -\frac{\lambda-1}{4\pi}\ln\left(-\Phi(\xi)\right) + \mathcal{O}\left(\frac{|Z|}{|\xi|^{1/2}}\right), \text{ as } |\xi| \to +\infty \tag{14.14}$$

uniformly for $Z \in D(\xi, B)$, $|Z| \le B$. Combining (14.12) and (14.14) it follows:

$$\frac{\partial\widetilde{\Psi}}{\partial Z}(\xi, Z, t) = \frac{2i\sqrt{|\xi|}}{(\lambda-1)}\left(\ln\left(-\Phi(\xi)\right) + \ln\left(\frac{2i(Z/t)}{\lambda-1}\right) + \frac{(\lambda-1)i}{4\sqrt{|\xi|}Z} + \ln\left(|\xi|^{1/2}\right) + \mathcal{O}\left(\frac{|Z|}{|\xi|^{1/2}}\right)\right) \text{ as } |\xi| \to +\infty$$

uniformly for $Z \in D(\xi, B)$, $|Z| \le B$.

Using Rouché's Theorem it then follows that for $|\xi|t^2$ sufficiently large, $\xi \in \mathcal{T}_L \cap (B_{\xi_0}(0))^c$ there exists a unique root of $(\partial\widetilde{\Psi}/\partial Z)(\xi, Z, t) = 0$ in $Z \in D(\xi, B)\backslash B_{\delta_0 t}(0)$ satisfying the asymptotics (14.3) and the lemma follows. □

We now derive estimates for higher order derivatives of $\widetilde{\Psi}$.

Lemma 14.4. *Under the same conditions as in Lemma 14.3 the following asymptotics holds:*

$$\frac{\partial^2\widetilde{\Psi}}{\partial Z^2}(\xi, Z_c, t) = \frac{2\,i\,\sqrt{|\xi|}}{(\lambda-1)\,Z_c}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{|\xi|}t}\right)\right) \text{ as } |\xi|\,t^2 \to +\infty,$$

uniformly in $0 \le t \le 1$.

Proof. The second derivative of $\widetilde{\Psi}$ with respect to Z is:

$$\frac{\partial^2 \widetilde{\Psi}}{\partial Z^2} = -\frac{8\pi |\xi| i}{(\lambda-1)^2} \int_{\mathcal{C}_2} [\ln(-\Phi(\eta))]' \frac{e^{\frac{4\pi}{\lambda-1}(Z\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta + \frac{2\sqrt{|\xi|}\, i}{(\lambda-1)\, Z} + \frac{1}{2Z^2}$$
$$= -\frac{8\pi |\xi| i}{(\lambda-1)^2} \int_{\mathcal{C}_2} \frac{\Phi'(\eta)}{\Phi(\eta)} \frac{e^{\frac{4\pi}{\lambda-1}(Z\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta + \frac{2 i\sqrt{|\xi|}}{(\lambda-1)\, Z} + \frac{1}{2Z^2}.$$

Taking $Z = Z_c$ and using Lemma 14.3, we deduce, uniformly in $0 \le t \le 1$,

$$\frac{\partial^2 \widetilde{\Psi}}{\partial Z^2}(\xi, Z_c, t) = \frac{2 i\sqrt{|\xi|}}{(\lambda-1)\, Z_c} - \frac{8\pi |\xi| i}{(\lambda-1)^2} \int_{\mathcal{C}_2} \frac{\Phi'(\eta)}{\Phi(\eta)} \frac{e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta + \mathcal{O}\left(\frac{1}{t}\right), \tag{14.15}$$

as $|\xi| t^2 \to +\infty$. Using Proposition 4.1:

$$J(\xi) = \int_{\mathcal{C}_2} \frac{\Phi'(\eta)}{\Phi(\eta)} \frac{e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta = \frac{1}{2}\int_{\mathcal{C}_2} \frac{1}{\eta} \frac{e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta$$
$$+ \int_{\mathcal{C}_2} \mathcal{O}\left(\frac{1}{|\xi|^2}\right) \frac{e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta + \mathcal{O}\left(e^{-a|\xi|}\right), \text{ as } |\xi| \to +\infty, \tag{14.16}$$

where a is a positive constant independent on ξ. The first term in the right hand side is estimated as:

$$\int_{\mathcal{C}_2} \frac{1}{\eta} \frac{e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta = \frac{1}{\xi}\int_{\mathcal{C}_2} \frac{e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta$$
$$+ \int_{\mathcal{C}_2} \mathcal{O}\left(\frac{\eta-\xi}{|\xi|^2}\right) \frac{e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z_c\sqrt{|\xi|}-\eta+\xi)}\right)^2} d\eta$$
$$+ \mathcal{O}\left(e^{-a|\xi|}\right) \text{ as } |\xi| \to +\infty.$$

The first term in the right hand side can be computed explicitly and gives $-(\lambda - 1)/4\pi\xi$. The second one is estimated using the form of the contour $\mathcal{C}_2$ and is bounded by $\mathcal{O}(|Z_c|/|\xi|)$.
Therefore, the first and second terms in the right hand side of (14.16) can be estimated respectively as $\mathcal{O}(1/|\xi|)$, $\mathcal{O}(1/|\xi|^{3/2})$as $|\xi| \to +\infty$. Then

$$J(\xi) = \mathcal{O}\left(\frac{1}{|\xi|}\right) \text{ as } |\xi| \to +\infty.$$

Using (14.15) we obtain

$$\frac{\partial^2 \widetilde{\Psi}}{\partial Z^2}(\xi, Z_c, t) = \frac{2\,i\,\sqrt{|\xi|}}{(\lambda-1)\,Z_c} + \mathcal{O}\left(\frac{1}{t}\right) \quad \text{as } |\xi|t^2 \to +\infty$$

whence Lemma 14.4 follows. □

Lemma 14.5. *Suppose that Z_c, B and δ_0 are as in Lemma 14.3. Then the following asymptotics holds:*

$$\left|\frac{\partial^3 \widetilde{\Psi}}{\partial Z^3}(\xi, Z, t)\right| = \mathcal{O}\left(\frac{\sqrt{|\xi|}}{t^2}\right) \quad as \;\; |\xi|\,t^2 \to +\infty,$$

uniformly in $0 \le t \le 1$, $Z \in D(\xi, B)$, $|Z| \le B$.

Proof of Lemma 14.5.

$$\frac{\partial^3 \widetilde{\Psi}}{\partial Z^3}(\xi, Z, t) = -\frac{8\pi|\xi|^{3/2} i}{(\lambda-1)^2}\int_{\mathcal{C}_2}\left(\frac{\Phi'(\eta)}{\Phi(\eta)}\right)' \frac{e^{\frac{4\pi}{\lambda-1}(Z\sqrt{|\xi|}-\eta+\xi)}}{\left(1-e^{\frac{4\pi}{\lambda-1}(Z\sqrt{|\xi|}-\eta+\xi)}\right)^2}d\eta$$
$$-\frac{2\,i\sqrt{|\xi|}}{(\lambda-1)\,Z^2} - \frac{1}{Z^3}. \tag{14.17}$$

The last two terms of (14.17) are bounded as $C(\sqrt{|\xi|}/t^2 + 1/t^3)$. and this can be estimated as $C\sqrt{|\xi|}/t^2$ for $|\xi|t^2 >> 1$. On the other hand we may bound the first term in the right hand side of (14.17) using

$$\left|\left(\frac{\Phi'(\eta)}{\Phi(\eta)}\right)'\right| \le \frac{C}{1+|\eta|^2},$$

the form of the contour $\mathcal{C}_2$. The term under consideration is then bounded by a constant. Combining all these estimates for the terms in the right hand side of (14.17) the lemma follows. □

Lemma 14.1 and Lemma 14.3 yield

Corollary 14.6. *For all* $0 < t < 1$:

$$\widetilde{\Psi}(\xi, Z_c, t) = -\sqrt{|\xi|}\sqrt{2\pi}\,t\,(1+iQ) - \frac{1}{2}\ln\left(|\xi|^{1/2}\right) - \frac{1}{2}\ln\left(\frac{2\,i\,Z_c}{\lambda-1}\right) + \mathcal{O}(1),$$

as $|\xi|t^2 \to +\infty$.

Proof. Using (14.11) and (14.10) we deduce

$$\widetilde{\Psi}(\xi, Z_c, t) = -\sqrt{|\xi|}\frac{2iZ_c}{\lambda-1}\left[1+\ln(t) - \ln\left(\frac{2iZ_c}{\lambda-1}\right) + \ln\left(2\sqrt{\pi}e^{iQ\frac{\pi}{4}}\right)\right]$$
$$-\frac{1}{2}\ln\left(\frac{2\,i\,Z_c}{\lambda-1}\right) + \mathcal{O}(1)$$
$$= -\sqrt{|\xi|}\sqrt{2\pi}\,t\,(1+iQ) - \frac{1}{2}\ln\left(|\xi|^{1/2}\right) - \frac{1}{2}\ln\left(\frac{2\,i\,Z_c}{\lambda-1}\right) + \mathcal{O}(1) \tag{14.18}$$

as $|\xi|t^2 \to +\infty$. □

Remark 14.7. Notice that the sign of the real part of the main term in the asymptotic expansion of $\widetilde{\Psi}$ as $t^2\,|\xi| \to \infty$ is essential in these arguments.

In the next lemma we prove that the function $\widetilde{\Psi}(\xi, Z, t)$ is "well behaved" in a region $|Z - Z_c| \geq \varepsilon t$ and $|Z| \leq M\,t$ for $M > 0$ large.

Lemma 14.8. *For all $\varepsilon > 0$ and $M > 0$ large, there exists $\delta_0 > 0$ and $R > 0$ such that the function $\widetilde{\Psi}(\xi, Z, t)$ satifies:*

$$Re\,\widetilde{\Psi}(\xi, Z, t) \leq Re\,\widetilde{\Psi}(\xi, Z_c, t) - \delta_0\sqrt{|\xi|}\,t$$

when Z lies in the curve

$$\gamma_t(M) = \gamma_{1,t}(M) \cup \gamma_{2,t}(M) \cup \gamma_{3,t}(M) \backslash \{Z;\ |Z - Z_c| \leq \varepsilon\,t\},$$

where

$$\gamma_{1,t}(M) = \{Z;\ Z = Z_c + \lambda,\ \lambda \in \mathbb{R}, |\lambda| \leq M\,t\}, \tag{14.19}$$

$$\gamma_{2,t}(M) = \left\{Z;\ Z = Z_c + M\,t + \lambda\,i,\ \lambda \in \left[0,\ |Im(Z_c)| + \gamma_1\right]\right\}, \tag{14.20}$$

$$\gamma_{3,t}(M) = \left\{Z;\ Z = Z_c - M\,t + \lambda\,i,\ \lambda \in \left[0,\ |Im(Z_c)| + \gamma_1\right]\right\} \tag{14.21}$$

for all ξ and t such that $|\xi|\,t^2 > R$.

Proof. Let us consider the auxiliary function:

$$\Theta(\xi, \Omega, t) = -2\sqrt{\pi}\sqrt{|\xi|}\,\Omega\,t\,[1 - \ln(\Omega) + \ln(\Omega_0)], \tag{14.22}$$

$$\Omega = \frac{iZ}{\lambda - 1}\frac{1}{\sqrt{\pi}\,t}, \tag{14.23}$$

$$\Omega_0 = e^{iQ\frac{\pi}{4}}. \tag{14.24}$$

By Lemma 14.1:

$$\widetilde{\Psi}(\xi, Z, t) = \Theta(\xi, \Omega, t) - \frac{1}{2}\ln\left(|\xi|^{1/2}\right) + h(\xi, Z, t).$$

Moreover, it is easily checked that Ω_0 is the critical point of the function $\Theta(\xi, \Omega, t)$. By Lemma 14.3 we already know that Z_c, the critical point of the function $\widetilde{\Psi}$, converges to Ω_0 as $|\xi|\,t^2 \to +\infty$. We now study the behaviour of the function Θ along the curve obtained from $\gamma(M)$ using the change of variable (14.23). Due to the convergence properties of the function $\widetilde{\Psi}$ and its critical point Z_c when $|\xi|\,t^2 \to +\infty$ this will be enough in order to prove the statement in Lemma 14.8. We first consider the curve corresponding to $\gamma_{1,t}(M)$. It is then enough to consider $Re\,\Theta(\xi, \Omega, t)$ along the points: $\Omega = \Omega_0 + \frac{\lambda}{\sqrt{2}}\,i, \quad \lambda \in \mathbb{R}$. A straightforward calculation yields:

$$Re\,\Theta(\xi, \Omega, t) = -\sqrt{2\pi}\sqrt{|\xi|}\,t\,\psi(\sigma),$$
$$\sigma = 1 + Q\lambda$$
$$\psi(\sigma) = 1 - \frac{1}{2}\ln\left(1 + \sigma^2\right) + \frac{1}{2}\ln 2 - \sigma\left(\frac{\pi}{4} - arctg\sigma\right).$$

Since $\psi'(\sigma) = -\frac{\pi}{4} + arctg(\sigma)$ the point $\sigma_0 = 1$ is a strict minimum for the function ψ. It is also easily checked that

$$\psi(\sigma) \sim \frac{\pi}{4} \quad \text{as } \sigma \to +\infty,$$
$$\psi(\sigma) \sim -\frac{3\pi}{4} \quad \text{as } \sigma \to -\infty.$$

It follows that, for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that if $|\sigma - 1| > \varepsilon$,

$$\psi(\sigma) - \psi(1) \geq \delta_0.$$

Arguing by continuity this yields the statement of the lemma when Z lies in the curve $\gamma_{1,t}(M)$.

The evolution of the function θ along the two curves corresponding to $\gamma_{2,t}(M)$ and $\gamma_{3,t}(M)$ is studied with very similar arguments. We then only consider the case of the curve $\gamma_{2,t}(M)$. It is then enough to consider $Re\,\Theta(\xi, \Omega, t)$ along the points:

$$\Omega = r\, e^{i\varphi},$$
$$Im\left(r\, e^{i\varphi} - \Omega_0\right) = \pm\frac{M}{\pi(\lambda - 1)}.$$

A straightforward calculation gives:

$$Re\,\Theta(\xi, \Omega, t) = -2\sqrt{\pi}\sqrt{|\xi|}\,\Omega\, t\left[r\,\cos\varphi(1 - \ln r) - r\,\sin\varphi\left(\frac{\pi Q}{4} - \varphi\right)\right].$$

For $Z \in \gamma_{2,t}(M)$ and the constant M sufficiently large, we have that $\varphi > \pi/4 + \delta$, $r\cos\varphi \leq 2$ and $r > M/2$. Similarly, if $Z \in \gamma_{3,t}(M)$ and the constant is M sufficiently large, we have that $\varphi < -\pi/4 - \delta$, $r\cos\varphi \leq 2$ and $r > M/2$. Therefore, we have in both cases:

$$r\,\cos\varphi(1 - \ln r) - r\,\sin\varphi\left(\frac{\pi Q}{4} - \varphi\right) > \delta_0 > 0,$$

for M large enough.

Using again the convergence properties of the function $\widetilde{\Psi}$ and its critical point Z_c when $|\xi|\,t^2 \to +\infty$ this yields the statement in Lemma 14.8 when Z lies in the curves $\gamma_{2,t}(M)$, $\gamma_{3,t}(M)$. □

In the following Lemma we extend the behaviour of $\widetilde{\Psi}(\xi, Z, t)$ to the region $|Z - Z_c| \geq \varepsilon t$ and $|Z| \leq \delta_1\sqrt{|\xi|}$ for some $\delta_1 > 0$ sufficiently small.

Lemma 14.9. *For all $M > 0$ large, there exists $a > 0$, $\varepsilon_1 > 0$ and $R > 0$ such that the function $\widetilde{\Psi}(\xi, Z, t)$ satifies:*

$$Re\,\widetilde{\Psi}(\xi, Z, t) \leq -a\sqrt{|\xi|}\,|Z|,$$

for all $Z \in \mathcal{C}_1$ such that $M\,t \leq |Z| \leq \varepsilon_1\sqrt{|\xi|}$ and all ξ and t such that $|\xi|\,t^2 > R$.

Proof. We only need to check that the function Θ defined in the proof of the previous lemma behaves linearly when $|\Omega| \to +\infty$ and $Re(\Omega)$ remains constant. This follows from

$$Re(\Theta(\xi, \Omega, t)) \leq -\frac{\pi^{3/2}}{2}\sqrt{|\xi|}\, t\, |\Omega|$$

uniformly for $\Omega = i\,|\Omega| + \mathcal{O}(1)$. Using (14.23) we deduce

$$Re(\Theta(\xi, \Omega, t)) \leq -\frac{\pi}{2}\sqrt{|\xi|}\,\frac{|Z|}{\lambda - 1}$$

for all $Z \in \mathcal{C}_1$ such that $|Z| \geq M\,t$ assuming that $M > 0$ is sufficiently large. Using Lemma 14.1 we have:

$$\begin{aligned} Re\,\widetilde{\Psi}(\xi, Z, t) &\leq Re(\Theta(\xi, \Omega, t)) + C\left(Z^2 + \mathcal{O}\left(\frac{1}{|\xi|}\right)\right) \\ &\leq -\frac{\pi}{2}\sqrt{|\xi|}\,\frac{|Z|}{\lambda - 1} + C\left(Z^2 + \mathcal{O}\left(\frac{1}{|\xi|}\right)\right) \\ &\leq -\sqrt{|\xi|}\,|Z|\left(\frac{\pi}{2(\lambda - 1)} - \varepsilon_1 + \mathcal{O}\left(\frac{1}{|\xi|^{3/2}\,t}\right)\right) \end{aligned}$$

for all $Z \in \mathcal{C}_1$ such that $M\,t \leq |Z| \leq \varepsilon_1\sqrt{|\xi|}$. The result follows for ε_1 small enough and $|\xi|^{1/2}\,t \to +\infty$. $\square$

Lemma 14.10. *For all $B > 0$ there exists ξ_0 and $C > 0$ such that*

$$\left|\frac{\partial^{\ell}\Psi}{\partial\xi^{\ell}}(\xi, Y, t)\right| \leq C\,\frac{|Y|}{|\xi|^{\ell}}, \quad \ell = 1, 2 \tag{14.25}$$

for $Y = Z\sqrt{|\xi|}$, $|Y| \leq \frac{|\xi|}{8}$, $Z \in D(\xi, B)$ and $|Re(\xi)| > \xi_0$.

Proof. Let Ψ be given:

$$\begin{aligned} \Psi(\xi, Y, t) = &\frac{2}{(\lambda - 1)\,i}\int_{Im\,\eta=\beta_1} \ln(-\Phi(\eta))\,\Theta(\eta - \xi, Y)d\eta \\ &-\frac{2iY}{\lambda - 1}\ln(t) - \frac{2iY}{\lambda - 1} + \left(\frac{2iY}{\lambda - 1} - \frac{1}{2}\right)\ln\left(\frac{2iY}{\lambda - 1}\right). \end{aligned}$$

Differentiating with respect to ξ we obtain:

$$\begin{aligned} \frac{\partial\Psi}{\partial\xi}(\xi, Y, t) &= \frac{2}{(\lambda - 1)\,i}\int_{Im\,\eta=\beta_1} \ln(-\Phi(\eta))\,\frac{\partial\Theta}{\partial\xi}(\eta - \xi, Y)d\eta \\ &= \frac{2}{(\lambda - 1)\,i}\int_{Im\,\eta=\beta_1} \frac{\Phi'(\eta)}{\Phi(\eta)}\Theta(\eta - \xi, Y)d\eta. \end{aligned}$$

By Proposition 4.1 we have

$$\left|\frac{\Phi'(\eta)}{\Phi(\eta)}\right| \leq \frac{C}{1 + |\eta|}$$

in the domain $Y/\sqrt{|\xi|} \in D(\xi, B)$. We deform the contour of integration to $\mathcal{C}_2$ defined in Fig. 4. Then we split the integral in two pieces:

$$|\frac{\partial \Psi}{\partial \xi}(\xi, Y, t)| \le C \int_{\eta \in \mathcal{C}_2, |\eta-\xi| \le \frac{|\xi|}{4}} \frac{1}{1+|\eta|} |\Theta(\eta-\xi, Y)||d\eta|$$
$$+C \int_{\eta \in \mathcal{C}_2, |\eta-\xi| \ge \frac{|\xi|}{4}} \frac{1}{1+|\eta|} |\Theta(\eta-\xi, Y)||d\eta| = J_1 + J_2.$$

By the exponential decay of the function Θ:

$$J_2 \le Ce^{-a|\xi|}$$

for some positive constant a. On the other hand,

$$\Theta(\eta-\xi, Y) = \frac{1}{1-e^{-\frac{4\pi}{\lambda-1}(\eta-\xi)}} - \frac{1}{1-e^{\frac{4\pi}{\lambda-1}(-(\eta-\xi)+Y)}}.$$

The integral J_1 is then divided in two parts. The first, $J_{1,1}$ is the integral along the "vertical part of the curve" $\mathcal{C}_2$. The second, $J_{1,2}$ is along the horizontal part of that curve, where $Im(\eta) = \beta_1$. In the integral $J_{1,1}$, $Re(\eta)$ is bounded and therefore $|\Theta(\eta-\xi, Y)| \le C$.

Since the total length of the integration curve of $J_{1,1}$ is of order $|Y|$ we deduce that $J_{1,1} \le C|Y|/(1+|\xi|)$. We split the integral $J_{1,2}$ as follows:

$$J_{1,2} \le \frac{C}{1+|\xi|} \left(\int_{Im(\eta)=\beta_1, |\eta-\xi| \le 2|Y|} |\Theta(\eta-\xi, Y)| d\eta \right.$$
$$\left. + \int_{Im(\eta)=\beta_1, |\eta-\xi| \ge 2|Y|} |\Theta(\eta-\xi, Y)| d\eta \right),$$
$$\int_{Im(\eta)=\beta_1, |\eta-\xi| \ge 2|Y|} |\Theta(\eta-\xi, Y)| d\eta$$
$$\le \int_{Im(\eta)=\beta_1, |\sigma| \ge 2|Y|} \left| \frac{e^{-\frac{4\pi}{\lambda-1}\sigma} - e^{\frac{4\pi}{\lambda-1}(Y-\sigma)}}{(1-e^{-\frac{4\pi}{\lambda-1}\sigma})(1-e^{\frac{4\pi}{\lambda-1}(Y-\sigma)})} \right| d\eta.$$

We use now that, if $Re(\sigma) > 2|Y|$:

$$\left| \frac{e^{-\frac{4\pi}{\lambda-1}\sigma} - e^{\frac{4\pi}{\lambda-1}(Y-\sigma)}}{(1-e^{-\frac{4\pi}{\lambda-1}\sigma})(1-e^{\frac{4\pi}{\lambda-1}(Y-\sigma)})} \right| \le Ce^{-\frac{2\pi}{\lambda-1}|\sigma|},$$

and if $Re(\sigma) < -2|Y|$,

$$\left| \frac{e^{-\frac{4\pi}{\lambda-1}\sigma} - e^{\frac{4\pi}{\lambda-1}(Y-\sigma)}}{\left(1-e^{-\frac{4\pi}{\lambda-1}\sigma}\right)\left(1-e^{\frac{4\pi}{\lambda-1}(Y-\sigma)}\right)} \right| \le Ce^{\frac{4\pi}{\lambda-1}Y} e^{-\frac{6\pi}{\lambda-1}|\sigma|}.$$

The last remaining term is easily estimated by:

$$\int_{Im(\eta)=\beta_1, |\eta-\xi| \le 2|Y|} |\Theta(\eta-\xi, Y)||d\eta| \le C \int_{Im(\eta)=\beta_1, |\eta-\xi| \le 2|Y|} |d\eta| \le C|Y|.$$

It then follows that $J_{1,2} \leq Ce^{-a|Y|}/(1+|\xi|)$ for positive constant C and a. This ends the proof of (14.25) for $\ell = 1$. Similarly,

$$\frac{\partial^2 \Psi}{\partial \xi^2}(\xi, Y, t) = \frac{2}{(\lambda - 1)\, i} \int_{Im\,\eta=\beta_1} \left(\frac{\Phi'(\eta)}{\Phi(\eta)}\right)' \Theta(\eta - \xi, Y) d\eta$$

with

$$\left| \left(\frac{\Phi'(\eta)}{\Phi(\eta)}\right)' \right| \leq \frac{C}{(1+|\eta|)^2}$$

for $Y/\sqrt{|\xi|} \in D(\xi, B)$, again by Proposition 4.1. The proof of (14.25) follows then from the same arguments as those of the proof of (14.25) for $\ell = 2$. $\square$

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