Local well posedness for a linear coagulation equation.

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Abstract

In this paper a family of linear coagulation models is solved. These models arise in the analysis of the asymptotic behaviour of coagulation equations yielding gelation for large particles. The tools and techniques that are developed in this paper are based on the definition of a class of weighted Sobolev spaces that take into account the characteristic time scales associated to the coagulation equation for large particles, as well as in the continuation argument introduced by Schauder to prove well posedness of general classes of elliptic and parabolic equations. The estimates derived in this paper will be used in [5] to construct rigorously classical solutions of the coagulation equation exhibiting loss of mass.

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1 Introduction

This paper is part of a program to study the well posedness of classical solutions after the gelation time for a general class of coagulation equations that behave asymptotically as homogeneous kernels.

The classical coagulation equation reads:

$$\partial_t f(x,t) = \frac{1}{2}\int_0^x K(x-y,y)\, f(x-y,t)\, f(y,t)\, dy - \int_0^\infty K(x,y)\, f(x,t)\, f(y,t)\, dy \tag{1.1}$$

This equation can be thought as describing the distribution of sizes for a set of particles with size x that aggregate with particles of size y, independently distributed, with a rate $K(x,y)$.

It is well known that for kernels $K(x,y)$ that behave asymptotically for large x,y as $(xy)^{\frac{\lambda}{2}}$ with $1<\lambda<2$, the solutions of (1.1) exhibit the phenomenon known as “gelation”, that means that the first moment of f, that is formally preserved for the solutions of (1.1), is not any longer preserved after some finite time t^*, due to the fact that a macroscopic fraction of particles “escapes” to infinitely large sizes (cf. [8]).

A detailed description of the asymptotics of the function $f(x,t)$ as $x\to\infty$ for solutions exhibiting gelation behaviour is still lacking, except for the case $K(x,y)=x\cdot y$ where (1.1) can be explicitly solved using integral transforms (cf. [3], [10], [11]) . In order to obtain more information on the asymptotics of the solutions of (1.1) for more general kernels, we have studied in a detailed way in [4] the fundamental solution of the linearized problem obtained considering the evolution

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of functions f that are close to $\bar{f}(x) = Ax^{-\frac{3+\lambda}{2}}$ with kernel $K(x,y) = (x\,y)^{\frac{\lambda}{2}}$, $1 < \lambda < 2$. The reason for considering the evolution near such power law, is that , as it has been proved in [12], the function $\bar{f}(x) = Ax^{-\frac{3+\lambda}{2}}$ is a steady solution of the equation (1.1). The interpretation of this distribution in the particle setting above corresponds to a continuous transport of particles emanating from the origin and being transported towards $x = \infty$ at a constant rate (cf. [12]). The power law $x^{-\frac{3+\lambda}{2}}$ was also obtained in [13] where explicit particular solutions of the discrete coagulation equations yielding loss of mass have been constructed. A precise description of the meaning of this statement can be found in [4].

In order to understand solutions of the coagulation equation (1.1) yielding a "flux" of particles towards infinity, it is natural to linearize around $\bar{f}(x)$, in order to clarify how such transport of particles could take place. We will denote the linearized problem considered in [4] as:

$$g_t = L[g] \quad , \quad g(0) = g_0 \tag{1.2}$$

where

$$\begin{aligned} L(g) &= \int_0^{x/2} \left((x-y)^{-3/2} - x^{-3/2}\right) y^{\lambda/2} g(y)dy \\ &+ \int_0^{x/2} \left((x-y)^{\lambda/2} g(x-y) - x^{\lambda/2} g(x)\right) y^{-3/2} dy \\ &- x^{-3/2} \int_{x/2}^{\infty} y^{\lambda/2} g(y)dy - 2\sqrt{2} x^{(\lambda-1)/2} g(x). \end{aligned} \tag{1.3}$$

A technical difficulty that arises in the study of the linearization of (1.1) around $\bar{f}(x)$ is due to the fact that this function is singular near $x = 0$. This has several relevant consequences. First, the resulting linearized operator becomes singular near $x = 0$, and as a consequence, it has regularizing effects that cannot be expected to take place for the original problem (1.1). As a matter of fact, the linearized operator considered in [4] behaves, locally near a given value of x, as:

$$g_t(x,t) = -(-D_{xx})^{\frac{1}{4}} \left(x^{\lambda/2} g(x,t)\right) + \text{higher order terms}$$

In particular, this problem can be considered as a nonlocal parabolic equation whose generator has Fourier symbol $-\sqrt{2ik}$. Clearly, the regularity properties associated to this problem can be expected to be very different from the ones associated to (1.1).

Something that it is worth noticing is that the same half-derivative operator arises for all the values of λ. This can be seen in the second term on the right-hand side of (1.3) since the same exponent $y^{-\frac{3}{2}}$ appears independently on the value of λ. This term $y^{-\frac{3}{2}}$ is due to the combination of the term $(y)^{\frac{\lambda}{2}}$ and the factor $(y)^{-\frac{\lambda}{2}}$ in $(y)^{-\frac{3+\lambda}{2}}$.

On the other hand, from the physical point of view, the solution $\bar{f}(x)$ is associated to the presence of a constant flux of particles leaving from the origin, as it can be seen in [4]. Bounded solutions of (1.1) do not have such a constant flux of particles with size $x = 0$.

However, in spite of these differences between the linearized problem (1.2) and (1.1) there are good reasons for studying (1.2) in order to understand particle fluxes towards infinity for the nonlinear equation (1.1). The main one is that solutions of (1.1) yielding particle fluxes towards $x = \infty$ can be expected to behave as one of the solutions $\bar{f}$ as indicated above for each given time. Moreover, the problem (1.2) can be solved explicitly using the methods in [1] due to its good properties under rescalings. Moreover it is possible to derive detailed estimates of the corresponding fundamental solution in all the regions of the space-time (x,t) (cf. [4]).

Nevertheless, in order to avoid the shortcomings of (1.2) as an approximation of (1.1), it would be more convenient to study the linearization of (1.1) near a smooth bounded function $f_0(x)$ that behaves asymptotically as $Ax^{-\frac{3+\lambda}{2}}$ as $x \to \infty$. The resulting problem would be:

$$g_t = \mathcal{L}(g) \quad , \quad g(0) = g_0 \tag{1.4}$$

with:

$$\mathcal{L}(g) = \int_0^x (x-y)^{\frac{\lambda}{2}} f_0(x-y) x^{\frac{\lambda}{2}} g(y)\,dy - x^{\frac{\lambda}{2}} f_0(x) \int_0^\infty y^{\frac{\lambda}{2}} g(y)\,dy - x^{\frac{\lambda}{2}} g(x) \int_0^\infty y^{\frac{\lambda}{2}} f_0(y)\,dy$$

Problem (1.4) is in some sense closer to (1.1) than (1.2). Indeed, (1.4) does not have regularizing properties at any $x > 0$. On the other hand, bounded solutions of (1.4) yield a zero flux of particles from the origin.

Unfortunately, the solution of (1.4) cannot be obtained explicitly as it has been made in [4]. Moreover, to prove even local solvability in time of (1.4) is not an easy task due to the presence of the integral term $\int_0^\infty y^{\frac{\lambda}{2}} g(y)\,dy$. In the absence of this term the local solvability of (1.4) could be easily obtained using a fixed point argument. However, the presence of this integral term makes this problem much harder to solve.

The key idea that will be used in this paper is to solve (1.4) approximating it by means of (1.2) for $x \to \infty$. The operator on the right hand side can be thought as an operator having half derivative at $x = \infty$. As a consequence, the equation (1.4) has some kind of "smoothing effects" for $x = \infty$. The presence of these regularizing effects is cleaner in the equation (1.2). Nevertheless, this last equation has regularizing effects for all the values of x. Therefore, to approximate the regularizing effects of (1.4) by means of those of (1.2) is something that must be given a precise meaning and it will be made in this paper. Regularising effects in kinetic equations with singular kernels have been previously obtained for Boltzmann equation cf. [2] and [14].

There is another feature of the approximation of (1.4) using (1.2) that is worth mentioning. As indicated above, the function $\bar{f}(x)$ can be thought as describing a flux of particles coming from $x = 0$ that are transported towards $x = \infty$. On the contrary, the funtion $f_0(x)$ does not provide any flux of particles from the origin, although it is associated to a flux of particles transported towards $x = \infty$. If we rewrite these two functions using the change of variables $F = R^{\frac{3+\lambda}{2}} f(R\xi)$, with ξ of order one and $R \to \infty$, it follows that $\bar{f}(x) = Ax^{-\frac{3+\lambda}{2}}$ becomes $\bar{F}(\xi) = A\xi^{-\frac{3+\lambda}{2}}$ and $f_0(x)$ becomes $F_0(\xi) = R^{\frac{3+\lambda}{2}} f_0(R\xi)$. Notice that $F_0(\xi) \to \bar{F}(\xi)$ as $R \to \infty$, for all $\xi > 0$. Such a convergence fails for $\xi \to 0$ or, more precisely, for x of order one. Actually that is the region where (1.4) cannot be approximated by (1.2). This region can be considered as containing a "boundary layer" where the boundedness of f_0 plays a role, and where the absence of particle fluxes and regularizing effects for (1.4) are seen. The analysis of this paper can be thought as the development of the mathematical techniques to handle such a boundary layer effects, as well as the proof of the fact that the dynamics of (1.4) can be approximated by means of the singular problem (1.2) at least for times of order one.

Let us remark that to solve the problems (1.2), (1.4) is equivalent to the solution of suitable problems with sources and vanishing initial data. Indeed, suppose that $\tilde{g} = g(x,t)$, is a smooth function satisfying $\tilde{g}(x,0) = g_0(x)$. Let us define $h = g - \tilde{g}$. Then:

$$h_t = L[h] + \tilde{\mu} \quad , \quad h(0) = 0 \tag{1.5}$$

$$h_t = \mathcal{L}[h] + \mu \quad , \quad h(0) = 0 \tag{1.6}$$

with $\tilde{\mu} = L[\tilde{g}] - \tilde{g}_t,\ \mu = \mathcal{L}[g] - g_t$.

The method that we will use in this paper to solve (1.4) makes use of a classical continuation method. More precisely, we will embed (1.6) into the family of problems:

$$h_t = (1-\theta) L[h] + \theta \mathcal{L}(h) + \mu \quad , \quad h(0) = 0 \quad , \quad \theta \in [0,1] \tag{1.7}$$

The problem (1.7) can be explicitly solved for $\theta = 0$ using the fundamental solution in [4]. Suppose that (1.7) can be solved for $\theta = \theta^* \in [0,1)$. We will show that (1.7) can be solved for

$\theta > \theta^*$ with $(\theta - \theta^*)$ small enough. This will allow to extend by continuity the solution of (1.7) from $\theta = 0$ to $\theta = 1$, and then to obtain a solution of (1.6). Similar continuity methods have been extensively used in the analysis of PDE's (cf. [6, 7, 15]).

The structure of this paper is the following. Section 2 contains the definition of the functional spaces that will be used in the rest of the paper. These spaces have been chosen in order to capture the main features of the asymptotics of the solutions of (1.4) as $x \to \infty$, as well as suitable regularity properties that is convenient to control in the solutions due to the fact that the $L(\cdot)$ can be thought as a rescaled half-derivative operator as $x \to \infty$ as indicated above. Sections 3 and 4 are devoted to the proof of regularizing effects for the evolution problem associated to the most singular part of the operator $\mathcal{L}(\cdot)$. Section 3 contains some technical preliminary Lemmas and Section 4 contains the actual proof of the required regularity results which are summarized in Theorem 3.1. Section 5 contains similar regularizing effects for the evolution problem associated to the operator $\mathcal{L}(\cdot)$. These results are derived from those obtained in Sections 3 and 4 treating the less singular terms in $\mathcal{L}(\cdot)$ in a perturbative manner (cf. Theorem 5.1). Section 6, which contains some of the most technical arguments of the paper, derives detailed estimates for the difference $(\mathcal{L} - L)$. The terms in the difference $(\mathcal{L} - L)(\varphi)$ can be classified in two types. The terms denoted in Section 6 as A_1 (cf. (6.1)) are those whose estimate does not require good regularity estimates for the function φ. On the contrary in the terms denoted as A_2 (cf. (6.2)) the operator $(\mathcal{L} - L)$ acts like a half derivative operator and strict regularity assumptions are required for φ. The terms in A_1 which are much easier to control are estimated in Lemmas 6.1, 6.2. The terms in A_2 are estimated in Lemma 6.5, with the help of some auxiliary technical results (Lemmas 6.3 and 6.4). Section 7 derives regularity estimates for the solutions of the family of interpolating problems (1.7) (cf. Lemma 7.4). These estimates are obtained treating the interpolated problem (1.7) as a perturbation of the problem with $\theta = 1$ which has been explicitly solved in [4]. Lemma 7.2 contains an estimate for the solutions of (1.7) with $\theta = 1$ and Lemma 7.3 estimates the effect of the terms in the difference $(\mathcal{L} - L)$. Finally Section 8 contains the proofs of the main results of the paper (namely Theorems 2.1, 2.2) by means of the continuation argument in the solutions of (1.7) sketched above.

The results obtained in this paper will be used in [5] to construct rigorously classical solutions of the coagulation system exhibiting loss of mass.

2 Functional Framework and statement of the main results.

We introduce now the set of initial data, $f_0 \in \mathbf{C}^{1,\gamma}(\mathbb{R}^+)$, $\gamma \in (0,1)$, that we shall consider in this paper. We will assume that the function f_0 is close to the function $Ax^{-(3+\lambda)/2)}$ for some constant $A \in \mathbb{R}$. To this end define

$$h_0(x) = f_0(x) - Ax^{-(3+\lambda)/2)}\xi(x) \tag{2.1}$$

where ξ is a smooth cutoff function such that $\xi(x) = 1$ for $x \geq 1$ and $\xi(x) = 0$ if $0 \leq x \leq 1/2$. We then require in all this paper that for some positive constants B and δ, the following condition holds :

$$y^{\frac{3+\lambda}{2}+\delta}|h_0(y)| + y^{\frac{3+\lambda}{2}+1+\delta}|h_0'(y)| + \sup_{R\geq 1} R^{\frac{3+\lambda}{2}+\gamma+\delta}[h_0']_{C^{0,\gamma}[R/2,2R]} + [h_0']_{C^{0,\gamma}(0,1)} \leq B. \tag{2.2}$$

All the estimates in the rest of paper will depend on the constants A, B, γ and δ. For the sake of shortedness this dependence will not be indicated.

The operator $\mathcal{L}$ is given by:

$$\mathcal{L}(g) = \int_0^x (x-y)^{\lambda/2} f_0(x-y)\, y^{\lambda/2}\, g(y)\, dy - x^{\lambda/2}\, f_0(x) \int_0^\infty y^{\lambda/2}\, g(y)\, dy - \\ -x^{\lambda/2}\, g(x) \int_0^\infty y^{\lambda/2}\, f_0(y). \tag{2.3}$$

Our first goal is to study the solutions of the Cauchy problem:

$$\frac{\partial h}{\partial \tau} = \mathcal{L}(h) + \mu(\tau, x) \tag{2.4}$$

$$h(0,x) = 0 \tag{2.5}$$

for some initial data h_0 and non homogeneous term μ.
We shall also use repeatedly the following "localised version" of this equation. To this end, for all $R > 1$ fixed, let $\chi(x) \in \mathbf{C}_0^\infty(0,+\infty)$ be such that:

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \left(R - \frac{R}{8}, R + \frac{R}{8}\right) , \\ 0 & \text{if } x \notin \left(R - \frac{R}{4}, R + \frac{R}{4}\right) . \end{cases} \tag{2.6}$$

If we multiply the equation (2.4) by $\chi(x)$ and call $\tilde{g} = \chi(x)\, g(x)$ we obtain:

$$\frac{\partial \tilde{g}}{\partial t} = \mathcal{L}(\tilde{g}) + \mathcal{R}(g) \tag{2.7}$$

$$\begin{aligned} \mathcal{R}(g) &= \chi(x) \int_0^{x/2} \Big((x-y)^{\lambda/2} f_0(x-y) - x^{\lambda/2} f_0(x) \Big)\, y^{\lambda/2} g(y) dy \\ &- x^{\lambda/2} \tilde{g}(x) \int_{x/2}^\infty y^{\lambda/2} f_0(y) dy - x^{\lambda/2} f_0(x) \chi(x) \int_{x/2}^\infty y^{\lambda/2} g(y) dy \\ &+ \int_0^{x/2} (\chi(x) - \chi(x-y))\, (x-y)^{\lambda/2} g(x-y) y^{\lambda/2} f_0(y)\, dy. \end{aligned} \tag{2.8}$$

For any $p \geq 1$, L^p will denote the usual Lebesgue space. For any $\sigma > 0$ and any interval $I \subset (0,+\infty)$ we denote by $H^\sigma(I)$ the usual Sobolev space $W^{\sigma,2}(I)$. The corresponding norms will be denoted $||\cdot||_{L^p}$ and $||\cdot||_{H^\sigma}$. When dealing with functions depending on variables x and t we will write H^σ_x or L^p_t in order to indicate the argument with respect to which the norm is taken.

In order to define the functional spaces that will be needed we first introduce

$$N_\infty(h;\, t_0, R) = \left(R^{\frac{\lambda-1}{2}} \int_{t_0}^{\min(t_0 + R^{-(\lambda-1)/2}, T)} ||h(t)||^2_{L^\infty(R/2,2R)} dt \right)^{1/2} \tag{2.9}$$

$$N_{2;\,\sigma}(h;\, t_0, R) = \left(R^{\frac{\lambda-1}{2}+2\sigma-1} \int_{t_0}^{\min(t_0 + R^{-(\lambda-1)/2}, T)} ||D^\sigma_x h(t)||^2_{L^2(R/2,2R)} dt \right)^{1/2} \tag{2.10}$$

$$M_\infty(h;\, R) = \left(\int_0^T ||h(t)||^2_{L^\infty(R/2,2R)} dt \right)^{1/2} \tag{2.11}$$

$$M_{2;\sigma}(h;\, R) = \left(R^{2\sigma-1} \int_0^T ||D^\sigma_x h(t)||^2_{L^2(R/2,2R)} dt \right)^{1/2} \tag{2.12}$$

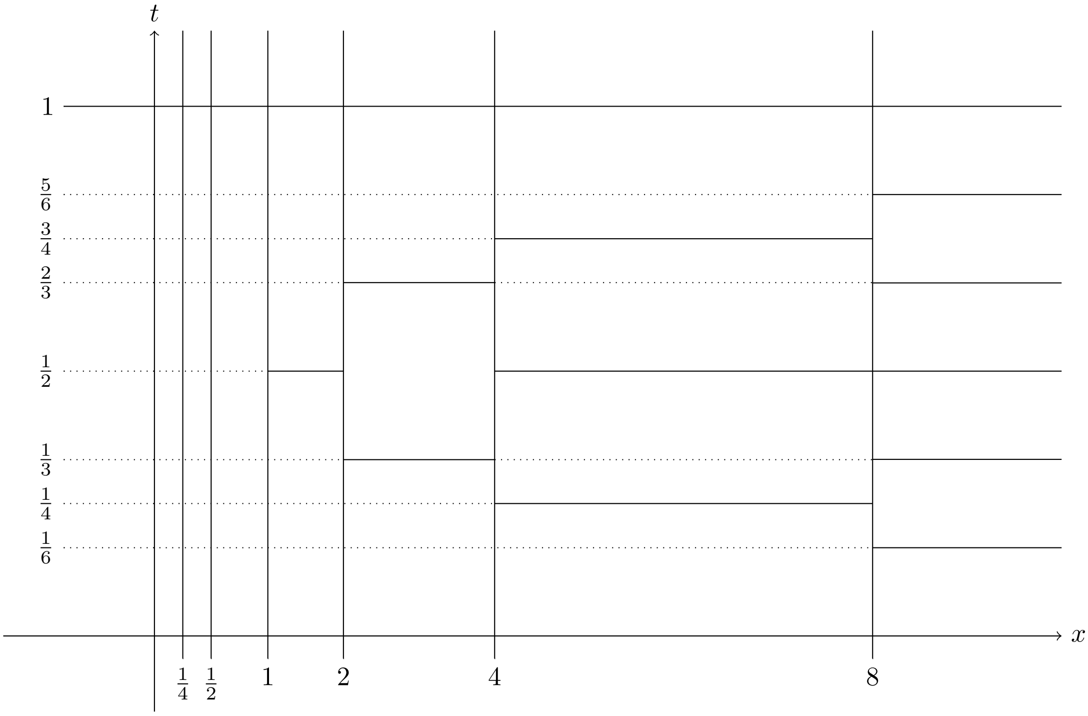

Figure 1: Domain decomposition for $\lambda = 1.5$, $t_0 = n\,R^{-(\lambda-1)/2}$ and $R = 2^n$.

Then, we define the spaces:

$$||f||_{X_{q,p}(T)} = \sup_{0<R<1} R^q\, M_\infty(f;R) + \sup_{0\le t_0\le T}\sup_{R\ge 1} R^p\, N_\infty(f;t_0,R) \tag{2.13}$$

$$X_{p,q}(T) = \left\{f; ||f||_{X_{q,p}(T)} < \infty\right\} \tag{2.14}$$

$$||f||_{Y^\sigma_{q,p}(T)} = \sup_{0<R\le 1} R^q\, M_{2;\,\sigma}(f;R) + \sup_{0\le t_0\le T}\sup_{R\ge 1} R^p\, N_{2;\,\sigma}(f;t_0,R) \tag{2.15}$$

$$Y^\sigma_{q,p}(T) = \left\{f; ||f||_{Y^\sigma_{q,p}(T)} < \infty\right\}. \tag{2.16}$$

We also use the following norms defined for functions $\varphi = \varphi(x)$:

$$|||\varphi|||_{q,p} = \sup_{0\le x\le 1}\{x^q|\varphi(x)|\} + \sup_{x>1}\{x^p|\varphi(x)|\}$$

and the next one defined for functions $\psi = \psi(\cdot,t)$ and any $T > 0$:

$$|||\psi|||_\sigma = \sup_{0\le t\le T} |||\psi(\cdot,t)|||_{3/2,(3+\lambda)/2} + ||\psi||_{Y^\sigma_{3/2,(3+\lambda)/2}}.$$

We define the space $\mathcal{E}_{T;\sigma}$ as

$$\mathcal{E}_{T;\sigma} = \{f; |||f|||_\sigma < \infty\}$$

endowed with the norm $|||\cdot|||_\sigma$. Unless it is explicitly stated, we assume in all the remaining of the paper that σ is a fixed number satisfying

$$\sigma \in (1,2). \tag{2.17}$$

In order to discharge the notation we will not write explicitly the dependence of the space $\mathcal{E}_{T;\sigma}$ and the norm $|||\cdot|||_\sigma$ on σ unless it is needed. We will then write:

$$\mathcal{E}_{T;\,\sigma} \equiv \mathcal{E}_T; \quad |||\cdot|||_\sigma \equiv |||\cdot||| \tag{2.18}$$

We introduce a functional seminorm that measures in a natural way the regularising effect of the operator $\mathcal{L}$ as $x \to \infty$. Consider a cutoff function $\eta(x)$ defined as

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \left(\frac{3}{4}, \frac{5}{4}\right) \,, \\ 0 & \text{if } x \notin \left(\frac{1}{8}, 4\right) \,. \end{cases} \tag{2.19}$$

Given then a function $f \in \mathcal{E}_{T,\sigma}$, for all $R > 0$ and $t_0 \in [0,T]$ we define

$$F_{R,t_0}(X,\tau) = \eta(X)\, f\left(R\,X, t_0 + \tau\, R^{-(\lambda-1)/2}\right) \tag{2.20}$$

$$[f] = \sup_{R\geq 1}\ \sup_{0\leq t_0\leq T}\ R^{(3+\lambda)/2} \times \tag{2.21}$$

$$\times \left(\int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2},T)} \int_{\mathbb{R}} |\widehat{F}_{R,t_0}(k,\tau)|^2\, \left(1+|k|^{2\,\sigma}\min\{|k|,\, R\}\right)\, dk\, dt\right)^{1/2}.$$

Some explanation about the meaning of these complicated norms is in order. Roughly speaking the norms in (2.13)-(2.16) , estimate functions behaving like the power law x^{-p} as $x \to \infty$ and x^{-q} as $x \to 0$. In particular the space $\mathcal{E}_{T;\sigma}$ consists of functions which can be estimated as the power law $x^{-\frac{3+\lambda}{2}}$ as $x \to \infty$ and $x^{-\frac{3}{2}}$ as $x \to 0$. The exponent $x^{-\frac{3+\lambda}{2}}$ appears in a natural manner in the analysis of the equations (1.1) (or (1.2), (1.4)) because this is the expected behaviour of the solutions yielding mass transfer as $x \to \infty$. The role of the exponent $x^{-\frac{3}{2}}$ is more technical, and it could be probably be replaced by other power laws. However the value $x^{-\frac{3}{2}}$ appears in a natural way because this is the singular behaviour that arises for the solutions of (1.2).

Notice that we have controlled the behaviour of the functions f using L^2 and Sobolev spaces H^σ_x (cf. (2.9), (2.12)). The reason for using this functional framework, instead of L^∞, C^k, $C^{k,\alpha}$ or similar is because we will prove many regularity results using Fourier analysis, for which the L^2 framework is more convenient. The control of this regularity is needed because, as indicated in the Introduction, the operator L can be thought as some kind of rescaled $\frac{1}{2}-$derivative operator as $x \to \infty$, and therefore, the control of these derivatives becomes needed. We use Sobolev spaces H^σ_x with $\sigma > \frac{1}{2}$ because we need L^∞ estimates for some of the functions in the corrective terms of $(\mathcal{L} - L)$ as $x \to \infty$. The embedding $H^\sigma_x \hookrightarrow L^\infty$ for $\sigma > \frac{1}{2}$ provides such estimates. These estimates will also be needed when considering the complete non linear coagulation equation (cf. [5])

Finally, a point that deserves comment is the choice of time scale for the decomposition of the space (x,t) in the set of rectangles shown in Figure 1. While the dyadic decomposition in space is natural, the choice of the length scale $R^{-\frac{\lambda-1}{2}}$ is not so obvious. The reason for this is that due to the rescaling properties of the kernel $K(x,y)$ in (1.1), the natural time scale producing changes in f comparable to itself for functions f behaving like $x^{-\frac{3+\lambda}{2}}$ as $x \to \infty$, due to the coalescence of particles of size x is $x^{-\frac{\lambda-1}{2}}$.

The main results of this paper are the following

Theorem 2.1 *For any $\sigma \in (1,2)$, $\delta > 0$ and for any f_0 satisfying (2.1) and (2.2), there exists $T > 0$ such that for all $\mu \in Y^\sigma_{3/2,\,2+\delta}$ the Cauchy problem (2.4) (2.5) has a unique solution h in $\mathcal{E}_{T;\,\sigma}$. Moreover,*

$$|||h||| \leq C\, ||\mu||_{Y^\sigma_{3/2,\,2+\delta}}$$

for some positive constant C depending on T, σ, δ as well as A, B and γ in (2.1) and (2.2) but not on μ.

Theorem 2.2 *For any $\sigma \in (1,2)$, $\delta > 0$ and for any f_0 satisfying (2.1) and (2.2), the solution of the Cauchy problem (2.4) (2.5) satisfies*

$$[h] \leq C||\mu||_{Y^{\sigma}_{3/2,\,2+\delta}}$$

for some positive constant C depending on T, σ, δ as well as A, B and γ in (2.1) and (2.2) but not on μ.

Theorem 2.2 is a regularising effect for the solutions of (2.4) (2.5). The operator $\mathcal{L}$ can be thought as half a derivative as $x \to \infty$. However the solutions of (2.4), (2.5) do not gain any regularity for any finite value of x. The norm (2.21) can be thought heuristically as a measure of

$$R^{\frac{3+\lambda}{2}} \left| \frac{f(x+\varepsilon) - f(x)}{\varepsilon^{1/2}} \right|$$

with $\varepsilon \geq 1/R$. Theorem 2.2 then states that for the function h this quantity may be estimated by $||\mu||_{Y^{\sigma}_{3/2,\,2+\delta}}$.

We end this Section with two warning remarks. The first one is that all along the paper we are going to use freely the letters $I_1, I_2, \cdots$ and $J_1, J_2, \cdots$ to denote different integrals. These letters will be used in different arguments. They will be used consistently within each argument. The second remark is that, in several arguments, we shall need to extend to a given interval suitable regularity estimates that have already been proved in smaller intervals. This is done following a standard and well known procedure involving decomposition of the identity and is not detailed in the paper.

3 Auxiliary regularity results.

In order to study the regularity properties of the solutions to the equation (1.7) we define, for all ε such that $0 \leq \varepsilon \leq 1$:

$$T_{\varepsilon,R}(f)(x) = \int_0^{\infty} (f(x) - f(x-y))\,\Phi(y,R,\varepsilon)dy \tag{3.1}$$

$$\Phi(y,R,\varepsilon) = \frac{\varepsilon}{y^{3/2}} + (1-\varepsilon)\,R^{(3+\lambda)/2}\,y^{\lambda/2}\,f_0(Ry). \tag{3.2}$$

The operator $T_{\varepsilon,R}\,(f)$ with $\varepsilon = 1$ is the most singular term of the operator $\mathcal{L}$ in (2.3). Moreover, as indicated in the Introduction, we will solve (2.4), (2.5) by means of a continuation argument, changing the parameter θ in (1.7) from zero to one. This will require to obtain uniform regularity estimates for all the family of interpolated operators. The family of operators $T_{\varepsilon,R}\,(f)$ yields the most singular term of those interpolated operators for suitable choices of $\varepsilon \in [0,1]$. We also define for further references the operators

$$\left(M_{\lambda/2} f\right)(x) = x^{\frac{\lambda}{2}}\,f(x). \tag{3.3}$$

$$\widehat{\Lambda\,\varphi}(\xi) = -\sqrt{2\pi}\,|\xi|^{1/2}\,\widehat{\varphi}(\xi). \tag{3.4}$$

We now consider the interior regularity properties of the linear semigroup generated by the operator $T_{\varepsilon,R} \circ M_{\lambda/2}$.

Theorem 3.1 *(i) Suppose that $Q \in L^2_t(0,1;H^{\sigma}_x(1/2,2))$, $P \in L^2_t(0,1;H^{\sigma-1/2}_x(1/2,2))$ with $\sigma \in (1/2,2)$, $\kappa \in (0,1]$ and $f \in L^{\infty}((1/4,2)\times(0,1)) \cap L^2(0,1;H^{1/2}(1/4,2)) \cap H^1(0,1;L^2(1/4,2))$ is such that $f = 0$ if $x < 1/8$ or $x > 4$ and satisfies*

$$\frac{\partial f}{\partial t} = \kappa\,T_{\varepsilon,R}\left(M_{\lambda/2}\,f\right) + Q + P \tag{3.5}$$

for all $x \in (1/4, 2)$, $t \in (0,1)$ and $f(x,0)=0$. Then:

$$||f||_{L^2_t(0,1;H^\sigma_x(3/4,5/4))} \leq C \left(||Q||_{L^2_t(0,1;H^\sigma_x(1/2,2))} + \frac{1}{\varepsilon\,\kappa}||P||_{L^2_t(0,1;H^{\sigma-1/2}_x(1/2,2))} + ||f||_{L^\infty((1/4,2)\times(0,1))}\right) \tag{3.6}$$

for some positive constant C independent of ε and R.

(ii) Suppose moreover that, for some $T_{max} > 0$, $Q \in L^2_t(0,T_{max}; H^\sigma_x(1/2,2))$, $P \in L^2_t(0,T_{max}; H^{\sigma-1/2}_x(1/2,2))$, $f \in L^\infty((1/4,2)\times(0,T_{max})) \cap C^1_t(0,T_{max}; H^{1/2}_x(1/4,2)))$ is such that $f = 0$ if $x < 1/8$ or $x > 4$ and satisfies

$$\frac{\partial f}{\partial t} = T_{\varepsilon,R}\left(M_{\lambda/2}\, f\right) + Q + P - a(x,t)\, f, \qquad x \in (1/4,2),\, t > 0) \tag{3.7}$$

$$f(x,0) = 0 \tag{3.8}$$

for some function $a \in L^\infty(0,T_{max}; H^\sigma(1/2,2))$, $a \geq A > 0$. Then, for all $t \in [0, T_{max}-1]$:

$$\sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||f(t)||^2_{H^\sigma(3/4,5/4)}\, dt\right)^{1/2} \leq \tag{3.9}$$
$$C \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||Q(t)||^2_{H^\sigma(1/2,2)}\, dt\right)^{1/2}$$
$$+\frac{C}{\varepsilon} \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||P(t)||^2_{H^{\sigma-1/2}(1/2,2)}\, dt\right)^{1/2}$$
$$+C\,||f||_{L^\infty((1/4,2)\times(0,T_{max}))}$$

(iii) Suppose that for some $T_{max} > 0$, $Q \in L^2_t(0,T_{max}; H^\sigma_x(1/2,2))$, $f \in L^\infty((1/4,2)\times(0,T_{max})) \cap C^1_t(0,T_{max}; H^{1/2}_x(1/4,2)))$ is such that $f = 0$ if $x < 1/8$ or $x > 4$ and satisfies (3.7) (3.8) with $P = 0$ and $\varepsilon = 0$. Then

$$\left(\int_T^{\min(T+1,T_{max})} \int_{\mathbb{R}} |\widehat{F}(k,t)|^2 |k|^{2\sigma} \min\{|k|, R\} dk\right)^{1/2} \leq$$
$$C \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||Q(t)||^2_{H^\sigma(1/2,2)}\, dt\right)^{1/2} + C\,||f||_{L^\infty((1/4,2)\times(0,T_{max}))} \tag{3.10}$$

where $F(x,t) = \eta(x)\, f(x,t)$, η is defined in (2.19) and C is independent of R.

Remark 3.2 *It will be used repeatedly in the paper that the condition $\sigma > 1/2$ ensures that the space H^σ_x is an algebra under the multiplication.*

Remark 3.3 *Roughly speaking, the part (i) of Theorem 3.1 provides regularity estimates for times t of order one. While part (ii) provides regularity estimates for arbitrary long times. It is important to notice that in the Theorem 3.1, the time T_{max} can be arbitrarily large.*

The proof of Theorem 3.1 is based on the classical freezing coefficients method that reduces the problem to the case of a constant coefficient operator. Let us then define, for all $x_0 \in \mathbb{R}^+$ the operator:

$$S_{\varepsilon,R}(t) = \exp\left[t\, x_0^{\frac{\lambda}{2}}\, T_{\varepsilon,R}\right]. \tag{3.11}$$

We also define the operators:

$$\widehat{T_1(\varphi)}(\xi) = ReW(\xi,\varepsilon,R)\widehat{\varphi}(\xi) \tag{3.12}$$

$$\widehat{T_2(\varphi)}(\xi) = i\, ImW(\xi,\varepsilon,R)\widehat{\varphi}(\xi) \tag{3.13}$$

where

$$W(\xi,\varepsilon,R) = \int_0^\infty \Phi(y,\varepsilon,R)\left(e^{-i\xi y}-1\right)dy \tag{3.14}$$

$$\widetilde{W}(\xi) = \int_0^\infty f_0(y)\, y^{\lambda/2}\left(e^{-i\xi y}-1\right)dy. \tag{3.15}$$

We now collect several estimates on the operators T_1 and T_2 which are used in order to obtain bounds on the operator $S_{\varepsilon,R}$.

Lemma 3.4 *The function $W(\xi,\varepsilon,R)$ defined in (3.14) may be rewritten as follows:*

$$W(\xi,\varepsilon,R) = -\varepsilon\sqrt{2}\,\Gamma(1/2)\,(1+isign(\xi))|\xi|^{1/2} + (1-\varepsilon)\sqrt{R}\,\widetilde{W}(\xi/R)\,, \tag{3.16}$$

where the function $\widetilde{W}$ satisfies:

$$Re\widetilde{W} \le 0 \;\; \textit{with } Re\widetilde{W} = 0 \;\; \textit{if and only if } \xi = 0, \tag{3.17}$$

$$\lim_{z\to 0}\frac{\widetilde{W}(z)}{(1+isign(z))|z|^{1/2}} = -\sqrt{2}\,\Gamma(1/2)\,, \tag{3.18}$$

$$\lim_{z\to+\infty}\widetilde{W}(z) = -\int_0^\infty y^{\lambda/2} f_0(y)\,, \tag{3.19}$$

$$|\widetilde{W}'(\xi)| \le \frac{C}{1+|\xi|^{1+\gamma}} \;\; \textit{for all } \xi > 0. \tag{3.20}$$

As a consequence of these properties, the function W satisfies:

$$ReW \le 0 \;\; \textit{with } ReW = 0 \;\; \textit{if and only if } \xi = 0, \tag{3.21}$$

and is such that, for all $\varepsilon > 0$ and ξ fixed,

$$\lim_{R\to+\infty} W(\xi,\varepsilon,R) = -\sqrt{2}/2(1+isign(\xi))|\xi|^{1/2}. \tag{3.22}$$

Proof of Lemma 3.4. Using formulas (3.2), (3.14) and (3.15) properties (3.17)–(3.19) follow. In order to prove (3.20) we may write:

$$\begin{aligned}\widetilde{W}'(\xi) &= -i\int_0^\infty y^{\lambda/2+1} f_0(y)\, e^{-iy\xi}dy = \frac{i}{\xi}\int_0^\infty \partial_y\left(y^{\lambda/2+1} f_0(y)\right)e^{-iy\xi}\,dy\\ &= \frac{i}{\xi}\int_0^\infty h(y)e^{-iy\xi}\,dy \qquad \text{with } h(y) = \partial_y\left(y^{\lambda/2+1} f_0(y)\right).\end{aligned}$$

We now split the integral $\int_0^\infty h(y)\, e^{-iy\xi}dy$ in the intervals of integration $\left[\frac{2\pi n}{\xi},\frac{2\pi(n+1)}{\xi}\right)$, $n = 0,1,2,\ldots.$ Writing in each of the intervals $h(y) = h\left(\frac{2\pi n}{\xi}\right) + \left[h(y) - h\left(\frac{2\pi n}{\xi}\right)\right]$, using that $\int_{\frac{2\pi n}{\xi}}^{\frac{2\pi(n+1)}{\xi}} e^{-iy\xi}dy = 0$ as well as the Hölderianity properties of f_0 stated in (2.2) we obtain:

$$\left|\int_0^\infty h(y)e^{-iy\xi}\,dy\right| \;\le\; C\sum_{n=0}^\infty\int_{\frac{2\pi n}{\xi}}^{\frac{2\pi(n+1)}{\xi}}\left(\frac{1}{|\xi|^\gamma}\frac{1}{(1+|y|)^{3/2+\gamma}}\right)dy \le \frac{C}{|\xi|^\gamma}$$

(3.20) follows. Finally, properties (3.21) and (3.22) directly follow from (3.17)–(3.20).

□

We collect now some regularising properties of the semigroups generated by the operators $S_{\varepsilon,R}$ and $x_0^{\lambda/2}\,T_1$ defined in (3.11) and (3.12).

Proposition 3.5 *For all $\sigma > 0$ and $\kappa \in (0,1]$:*

$$\left|\left|\int_0^t S_{\varepsilon,R}(t-s)\,h(s)\,ds\right|\right|^2_{H^\sigma(\mathbb{R})} \leq C\int_0^t ||h(s)||^2_{H^\sigma(\mathbb{R})}\,ds \tag{3.23}$$

$$\int_0^1 \left|\left|\int_0^t \kappa\, T_1\, e^{x_0^{\lambda/2}\kappa\, T_1(t-s)} h(s)ds\right|\right|^2_{H^\sigma(\mathbb{R})} dt \leq C\int_0^1 ||h(t)||^2_{H^\sigma}(\mathbb{R})\,dt \tag{3.24}$$

$$\int_0^1 \left|\left|\int_0^t e^{x_0^{\lambda/2}\varepsilon\Lambda(t-s)} h(s)ds\right|\right|^2_{H^\sigma(\mathbb{R})} dt \leq \frac{C}{\varepsilon^2}\int_0^1 ||h(t)||^2_{H^{\sigma-1/2}}(\mathbb{R})\,dt. \tag{3.25}$$

Moreover, for all $\beta \in (0,1]$ and η a C^∞ function of compact support, there exists $0 < \rho < \min(\sigma, \beta/2)$ such that

$$||S_{\varepsilon,R}(t)\,[\eta, T_{\varepsilon,R}]\,h||_{H^\sigma(\mathbb{R})} \leq C\,t^{-\beta}||h||_{H^{\sigma-\rho}(\mathbb{R})}, \tag{3.26}$$

$$\int_0^1 \left|\left|\int_0^t T_1 e^{x_0^{\lambda/2}T_1(t-s)}[\eta,\, T_1]h(s)\,ds\right|\right|^2_{H^\sigma(\mathbb{R})} ds \leq C\int_0^1 ||T_1 h||^2_{H^{\sigma-\rho}(\mathbb{R})} \tag{3.27}$$

where C denotes a generic positive constant independent of the function h of R and ε but depending on σ, β, ρ and η.

In the proof of Proposition 3.5 we will use the following result.

Lemma 3.6 *There exists a positive constant C such that for all $R > 1$, $\varepsilon > 0$ and $\alpha \geq 0$, $\beta \geq 0$ satisfying $\alpha + \beta = 1$:*

$$|W(\xi,\varepsilon,R) - W(z,\varepsilon,R)| \leq C\frac{|\xi - z|}{|z|^\alpha|\xi|^\beta}(1 + |W(z,\varepsilon,R)|)^\alpha(1 + |W(\xi,\varepsilon,R)|)^\beta \tag{3.28}$$

for all $z \in \mathbb{R}$ and $\xi \in \mathbb{R}$ such that $|z| \geq 1$ and $|\xi| \geq 1$.

Proof of Lemma 3.6. By definition:

$$W(\xi,\varepsilon,R) = -\varepsilon\sqrt{2\,\pi}\,(1 + isign(\xi))|\xi|^{1/2} + (1-\varepsilon)\sqrt{R}\,\widetilde{W}(\xi/R).$$

The following estimate can be readily obtained studying separately the cases $sign(\xi) = sign(z)$, $sign(\xi) = -sign(z)$

$$|sign(\xi)|\xi|^{1/2} - sign(z)|z|^{1/2}| \leq 2\,\frac{|\xi - z|}{|\xi|^{1/2} + |z|^{1/2}}$$

Therefore:

$$|\varepsilon\sqrt{2\,\pi}\,(1 + isign(\xi))|\xi|^{1/2} - \varepsilon\sqrt{2\,\pi}\,(1 + isign(z))|z|^{1/2}| \leq C\varepsilon\frac{|\xi - z|}{|\xi|^{1/2} + |z|^{1/2}}.$$

Using then $|W(z)| \geq 2\,\sqrt{\pi}\,\varepsilon\,|z|^{1/2}$ we obtain:

$$\frac{|\varepsilon\sqrt{2\,\pi}\,(1 + isign(\xi))|\xi|^{1/2} - \varepsilon\sqrt{2\,\pi}\,(1 + isign(z))|z|^{1/2}|}{(1 + |W(z)|)^\alpha\,(1 + |W(\xi)|)^\beta} \leq C\,\frac{|\xi - z|}{|z|^\alpha\,|\xi|^\beta}.$$

In order to prove a similar estimate for $\widetilde{W}$ we consider the following cases:

(i) $|\xi| \leq 2R$ and $|z| \leq 2R$,

(ii) $|\xi| \geq R/2$ and $|z| \geq R/2$

(iii) $|\xi| \geq 2R$ and $|z| \leq R/2$

(iv) $|z| \geq 2R$ and $|\xi| \leq R/2$.

In the case (i) we have $C_1|\xi|^{1/2} \leq \left|\sqrt{R}\,\widetilde{W}\left(\frac{\xi}{R}\right)\right| + |\xi|\left|\frac{\partial}{\partial\xi}\widetilde{W}\left(\frac{\xi}{R}\right)\right| \leq C_2|\xi|^{1/2}$ and similar estimates also for z. Defining $g = \sqrt{R}\,\widetilde{W}\left(\frac{\xi}{R}\right)$ we have by Taylor's theorem:

$$|g^2(\xi) - g^2(z)| \leq \int_z^\xi |g(\eta)||g'(\eta)|\,d\eta \leq C\,|\xi - z|.$$

Then $|g(\xi) - g(z)| \leq \frac{|\xi - z|}{g(\xi)+g(z)}$, whence $\sqrt{R}\left|\widetilde{W}\left(\frac{\xi}{R}\right) - \widetilde{W}\left(\frac{z}{R}\right)\right| \leq C\frac{|\xi - z|}{|\xi|^{1/2} + |z|^{1/2}}$ and the conclusion follows as above.

If condition (ii) holds, suppose first that $sign\,\xi = -sign\,z$. Then, $|\xi - z| = |\xi| + |z|$ and

$$\left|\sqrt{R}\widetilde{W}\left(\frac{\xi}{R}\right) - \sqrt{R}\widetilde{W}\left(\frac{z}{R}\right)\right| \leq C\sqrt{R}.$$

Using the Young's inequality $\frac{|\xi| + |z|}{|\xi|^\beta |z|^\alpha} \geq C > 0$ for a positive constant $C = C(\alpha, \beta)$ we deduce

$$\left|\sqrt{R}\widetilde{W}\left(\frac{\xi}{R}\right) - \sqrt{R}\widetilde{W}\left(\frac{z}{R}\right)\right| \leq C\,R^{\alpha/2}R^{\beta/2}\,\frac{|\xi| + |z|}{|\xi|^\beta |z|^\alpha} \leq C\frac{|\xi - z|}{|\xi|^\beta |z|^\alpha}(1 + |W(z)|)^\alpha (1 + |W(\xi)|)^\beta.$$

Suppose now, still under assumption (ii), that $sign\,\xi = sign\,z$. Then, using (3.20) we have

$$\left|\frac{\partial}{\partial\xi}\left(\sqrt{R}\,\widetilde{W}\left(\frac{\eta}{R}\right)\right)\right| \leq C\,\frac{R^{1/2+\alpha}}{|\eta|^{\alpha+1}}$$

for all $\eta \geq R/2$. Using Taylor's theorem it then follows that:

$$\frac{|\sqrt{R}\widetilde{W}\left(\frac{\xi}{R}\right) - \sqrt{R}\widetilde{W}\left(\frac{z}{R}\right)|}{\sqrt{R}\left(\widetilde{W}\left(\frac{\xi}{R}\right)\right)^\alpha \left(\widetilde{W}\left(\frac{z}{R}\right)\right)^\beta} \leq R^\alpha \int_\xi^z \frac{d\eta}{\eta^{1+\alpha}}.$$

Therefore, if $|\xi|/2 \leq |z| \leq 2|\xi|$ then,

$$R^\alpha \int_\xi^z \frac{d\eta}{\eta^{1+\alpha}} \leq R^\alpha \frac{|\xi - z|}{|z|^{\alpha+1}} \leq \frac{|\xi - z|}{|z|} \leq C\frac{|\xi - z|}{|z|^\alpha |\xi|^\beta}.$$

Otherwise,

$$R^\alpha \int_\xi^z \frac{d\eta}{\eta^{1+\alpha}} \leq R^\alpha \int_{R/2}^\infty \frac{d\eta}{\eta^{1+\alpha}} = \frac{2^\alpha}{\alpha} \leq C\frac{|\xi - z|}{|z|} \leq C\frac{|\xi - z|}{|z|^\alpha |\xi|^\beta}.$$

The cases (iii) and (iv) can be treated equivalently. In both cases we have $\frac{|\xi - z|}{|z|} \geq C_1 > 0$. Moreover, in the case (iv):

$$\frac{|\sqrt{R}\widetilde{W}\left(\frac{\xi}{R}\right) - \sqrt{R}\widetilde{W}\left(\frac{z}{R}\right)|}{\sqrt{R}\left(\widetilde{W}\left(\frac{\xi}{R}\right)\right)^\alpha \left(\widetilde{W}\left(\frac{z}{R}\right)\right)^\beta} \leq C\,\frac{R^{\alpha/2}}{|\xi|^{\alpha/2}} \leq C\,\frac{|z|^{\alpha/2}}{|\xi|^{\alpha/2}} \leq C\,\frac{|z|^\alpha}{|\xi|^\alpha} \leq C\frac{|z|}{|z|^{1-\alpha}|\xi|^\alpha} \leq C\frac{|z - \xi|}{|z|^{1-\alpha}|\xi|^\alpha}.$$

And this ends the proof of Lemma 3.6. ◻

Proof of Proposition 3.5. In order to prove (3.23) we write:

$$\left|\left|\int_0^t S_{\varepsilon,R}(t-s)\,h(s)\,ds\right|\right|^2_{H^\sigma(\mathbb{R})} =$$

$$C\int_{\mathbb{R}}\int_0^t\int_0^t (1+|\xi|^s)^2\, e^{x_0^{\lambda/2}(t-s_1)T_1(\xi)} e^{x_0^{\lambda/2}(t-s_1)T_2(\xi)}\widehat{h}(\xi,s_1)\times$$

$$\times e^{x_0^{\lambda/2}(t-s_2)T_1(\xi)} e^{-x_0^{\lambda/2}(t-s_2)T_2(\xi)}\overline{\left(\widehat{h}(\xi,s_2)\right)}\,ds_1\,ds_2\,d\xi$$

$$\le C\int_{\mathbb{R}}(1+|\xi|^s)^2\left(\int_0^t e^{x_0^{\lambda/2}(t-s_1)T_1(\xi)}\left|\widehat{h}(\xi,s_1)\right|ds_1\right)^2 d\xi \le C\int_0^t ||h||^2_{H^\sigma(\mathbb{R})}ds_1,$$

where T_1 and T_2 are defined in (3.12) and (3.13). This proves (3.23).

We prove now (3.24). To this end let us define the function $\varphi(x,t) = \displaystyle\int_0^t e^{x_0^{\lambda/2}\kappa\,T_1(t-s)}h(s)ds$ which satisfies:

$$\frac{\partial\varphi}{\partial t} = x_0^{\lambda/2}\kappa\,T_1(\varphi) + h(x,t),\quad t>0,\ x>0;\quad \varphi(x,0)=0.$$

Multiplying this equation by $-\kappa\,T_1\,M^{2\sigma}$ in $L^2(\mathbb{R})$ where M is the multiplier operator associated to the symbol $|\xi|$ we obtain:

$$\begin{aligned}\frac{\kappa}{2}\frac{\partial}{\partial t}||M^\sigma(-T_1)^{1/2}\varphi||^2_{L^2(\mathbb{R})} \quad &+\quad \kappa^2\,x_0^{\lambda/2}||M^\sigma(T_1\varphi)||^2_{L^2(\mathbb{R})} \le \kappa\,||M^\sigma T_1(\varphi)||_{L^2(\mathbb{R})}||M^\sigma h||_{L^2(\mathbb{R})}\\ &\le \quad \frac{1}{2x_0^{\lambda/2}}||h(s)||^2_{H^\sigma_x(\mathbb{R})} + \frac{\kappa^2\,x_0^{\lambda/2}}{2}||M^\sigma(T_1\varphi)||^2_{L^2(\mathbb{R})}\end{aligned}$$

whence:

$$\frac{\kappa}{2}\frac{\partial}{\partial t}||M^\sigma(-T_1)^{1/2}\varphi||^2_{L^2(\mathbb{R})} + \frac{\kappa^2\,x_0^{\lambda/2}}{2}||M^\sigma(T_1\varphi)||_{L^2(\mathbb{R})} \quad\le\quad \frac{1}{2\,x_0^{\lambda/2}}||h(s)||^2_{H^\sigma_x(\mathbb{R})}.$$

The result follows integrating in time and adding the corresponding inequality for $\sigma=0$. The proof of (3.25) is similar. We multiply the equation by $-M^{2(\sigma-1/2)}\,\Lambda$ in $L^2(\mathbb{R})$ to obtain:

$$\frac{1}{2}\frac{d}{dt}||(-\Lambda)^{1/2}\varphi||^2_{H^{\sigma-1/2}} + \varepsilon||\Lambda\varphi||^2_{H^{\sigma-1/2}} \le ||h||_{H^{\sigma-1/2}}||\Lambda\varphi||_{H^{\sigma-1/2}}.$$

Using Young's inequality and integrating in time we obtain (3.25).

We prove now (3.26). By definition:

$$\widehat{[\eta, T_{\varepsilon,R}]}\,\varphi(\xi) = \int_{\mathbb{R}} K(\xi-z)\,(W(z,\varepsilon,R)-W(\xi,\varepsilon,R))\,\hat{\varphi}(z)dz$$

where $K(z)=\hat{\eta}(z)$. Since the function η is C^∞, for any $m>0$ there is a constant C_m such that:

$$|K(\xi)| \le \frac{C_m}{1+|\xi|^m}\quad\text{for all }\ \xi\in\mathbb{R}. \tag{3.29}$$

Therefore:

$$||S_{\varepsilon,R}(t)\,[\eta,T_{\varepsilon,R}]\,h||^2_{H^\sigma(\mathbb{R})} = \int_{\mathbb{R}} e^{-2\,t\,|ReW(\xi)|}(1+|\xi|^\sigma)^2\left|\int_{\mathbb{R}}K(\xi-z)\,(W(\xi)-W(z))\,\hat{\varphi}(z)\,dz\right|^2 d\xi$$

We split the integral in two pieces:

$$||S_{\varepsilon,R}(t)\,[\eta,T_{\varepsilon,R}]\,h||^2_{H^\sigma(\mathbb{R})}=\int_{|\xi|\le 1}[\cdots]d\,\xi+\int_{|\xi|\ge 1}[\cdots]d\,\xi$$

The first term is estimated by:

$$\left|\int_{|\xi|\le 1}[\cdots]d\,\xi\right|\le C\int_{|\xi|\le 1}\left(\int_{\mathbb{R}}\frac{C_m}{1+|\xi-z|^m}(1+|z|^{1/2})|\hat{\varphi}(z)|\,dz\right)^2 d\xi$$
$$\le C\left(\int_{\mathbb{R}}\frac{C_m}{1+|z|^{m-1/2}}|\hat{\varphi}(z)|\,dz\right)^2\le C||\varphi||^2_{L^2}. \quad (3.30)$$

In the second term we have:

$$\int_{|\xi|\ge 1}[\cdots]d\,\xi\le 2\int_{|\xi|\ge 1}e^{-2\,t\,|ReW(\xi)|}(1+|\xi|^\sigma)^2\Big(\int_{|z|\le 1}[\cdots]dz)^2d\,\xi+$$
$$+2\int_{|\xi|\ge 1}e^{-2\,t\,|ReW(\xi)|}(1+|\xi|^\sigma)^2\Big(\int_{|z|\ge 1}[\cdots]dz)^2d\,\xi=J_1+J_2.$$

We estimate J_1 follows:

$$J_1\le C\int_{|\xi|\ge 1}(1+|\xi|^\sigma)^2\Big(\int_{|z|\le 1}\frac{C_m}{1+|\xi-z|^m}(1+|\xi|^{1/2})|\hat{\varphi}(z)|\,dz)^2d\,\xi$$
$$\le C\int_{|\xi|\ge 1}\frac{1}{(1+|\xi|^{2m-1-2\sigma})}\left(\int_{|z|\le 1}|\hat{\varphi}(z)|\,dz\right)^2d\,\xi\le C||\varphi||^2_{L^2}. \quad (3.31)$$

It only remains to estimate J_2.

$$\begin{aligned}J_2 &\le \int_{|\xi|\ge 1}e^{-2\,t\,|ReW(\xi)|}(1+|\xi|^\sigma)^2\left(\int_{|z|\ge 1}K(\xi-z)\,(W(\xi)-W(z))\,\hat{\varphi}(z)\,dz\right)^2d\xi\\ &\le C_m||\varphi||^2_{H^{\sigma-\rho}}\times\\ &\quad\times\int_{|\xi|\ge 1}e^{-2\,t\,|ReW(\xi)|}(1+|\xi|^\sigma)^2\left(\int_{|z|\ge 1}\frac{(W(\xi)-W(z))^2}{(1+|\xi-z|^m)^2}\,\frac{dz}{(1+|z|^{\sigma-\rho})^2}\right)d\xi\end{aligned}$$

Using Lemma 3.6:

$$\begin{aligned}J_2 &\le C||\varphi||_{H^{\sigma-\rho}}\int_{|\xi|\ge 1}e^{-2\,t\,|ReW(\xi)|}(1+|\xi|^\sigma)^2\left(\int_{|z|\ge 1}\frac{|\xi-z|^2}{(1+|\xi-z|^m)^2}\times\right.\\ &\qquad\left.\times\frac{|W(\xi)|^{2\beta}|W(z)|^{2\alpha}}{|z|^{2\alpha}|\xi|^{2\beta}}\frac{dz}{(1+|z|^{\sigma-\rho})^2}\right)d\xi\\ &\le C\,||\varphi||_{H^{\sigma-\rho}}t^{-2\beta}\times\\ &\quad\times\int_{|\xi|\ge 1}|\xi|^{2\sigma}\left(\int_{|z|\ge 1}\frac{1}{(1+|\xi-z|^{m-1})^2}\frac{|W(z)|^{2\alpha}}{|z|^{2\alpha}|\xi|^{2\beta}}\,\frac{dz}{(1+|z|^{\sigma-\rho})^2}\right)d\xi\end{aligned}$$

where we have used that, for all $\xi\in\mathbb{R}$ and all $t>0$:

$$e^{-2\,t\,|ReW(\xi)|}|w(\xi)|^{2\beta}\le\frac{C}{t^{2\,\beta}}. \quad (3.32)$$

Using now that $|W(z)|/|z|\le|z|^{-1/2}$ we deduce

$$J_2\le C||\varphi||_{H^{\sigma-\rho}}t^{-2\beta}\int_{|\xi|\ge 1}|\xi|^{2(\sigma-\beta)}\left(\int_{|z|\ge 1}\frac{1}{(1+|\xi-z|^{m-1})^2}\frac{1}{|z|^\alpha}\,\frac{dz}{(1+|z|^{\sigma-\rho})^2}\right)d\xi.$$

We change the order of integration and rewrite the resulting integral as

$$\begin{aligned}\int_{|\xi|\ge 1}\frac{|\xi|^{2(\sigma-\beta)}\,d\xi}{(1+|\xi-z|^{m-1})^2} &= \int_{|\xi|\ge 1,\ |\xi|\le 8|z|}[\cdots]\,d\xi+\int_{|\xi|\ge 1,\ |\xi|\ge 8|z|}[\cdots]\,d\xi\\ &= I_1+I_2.\end{aligned}$$

In the second integral we have $|\xi-z|\ge C|\xi|$ and therefore:

$$\begin{aligned}\int_{|\xi|\ge 1,\ |\xi|\ge 8|z|}\frac{|\xi|^{2(\sigma-\beta)}\,d\xi}{(1+|\xi-z|^{m-1})^2} &\le C\int_{|\xi|\ge 1,\ |\xi|\ge 8|z|}\frac{|\xi|^{2(\sigma-\beta)}\,d\xi}{|\xi|^{2(m-1)}}\\ &\le C|z|^{2\sigma-2\beta-2m+3}\le C|z|^{2(\sigma-\beta)},\end{aligned}$$

assuming that m is large. In the first integral we have

$$\int_{|\xi|\ge 1,\ |\xi|\le 8|z|}\frac{|\xi|^{2(\sigma-\beta)}\,d\xi}{(1+|\xi-z|^{m-1})^2}\le C|z|^{2(\sigma-\beta)}\int_{|\xi|\le 1,\ |\xi|\le 8|z|}\frac{d\xi}{(1+|\xi-z|^{m-1})^2}.$$

Then $I_1+I_2\le C\,|z|^{2(\sigma-\beta)}$ and

$$\int_{|\xi|\ge 1}|\xi|^{2(\sigma-\beta)}\left(\int_{|z|\ge 1}\frac{1}{(1+|\xi-z|^{m-1})^2}\,\frac{1}{|z|^{\alpha}}\,\frac{dz}{(1+|z|^{\sigma-\rho})^2}\right)d\xi\le C\int_{|z|\ge 1}|z|^{-1-\beta+2\rho}\,dz.$$

This integral is bounded as soon as $2\rho<\beta$. This concludes the proof of (3.26).

In order to prove (3.27) we estimate its left hand side as:

$$\int_0^1 dt\int_0^t ds\int_{\mathbb{R}} e^{-2(t-s)ReW(\xi)}(1+|\xi|^{2\sigma})|W(\xi)|^2\left|\int_{\mathbb{R}}K(\xi-z)(W(\xi)-W(z))h(z,s)dz\right|^2 d\xi$$

arguing as in the proof of (3.30) and (3.31) we obtain that

$$\begin{aligned}&\int_0^1 dt\int_0^t ds\int_{\mathbb{R}} e^{-2(t-s)ReW(\xi)}(1+|\xi|^{2\sigma})|W(\xi)|^2\times\\ &\times\left|\int_{\mathbb{R}}\mathbf{1}_{\min(|\xi|,|z|)\le 1}(\xi,z)K(\xi-z)(W(\xi)-W(z))h(z,s)dz\right|^2 d\xi\le C\int_0^1 ||h(s)||^2_{L^2(\mathbb{R})}ds.\end{aligned}\tag{3.33}$$

On the other hand,

$$\begin{aligned}&\int_0^1 dt\int_0^t ds\int_{|\xi|\ge 1} e^{-2(t-s)ReW(\xi)}(1+|\xi|^{2\sigma})|W(\xi)|^2\times\\ &\times\left|\int_{|z|\ge 1}K(\xi-z)(W(\xi)-W(z))h(z,s)dz\right|^2 d\xi\\ &\le\int_0^1 dt\int_0^t ds\int_{|\xi|\ge 1} e^{-2(t-s)ReW(\xi)}(1+|\xi|^{2\sigma})\times\\ &\times\left|\int_{|z|\ge 1}|K(\xi-z)||W(\xi)-W(z)||W(z)||h(z,s)|dz\right|^2 d\xi+\\ &+\int_0^1 dt\int_0^t ds\int_{|\xi|\ge 1} e^{-2(t-s)ReW(\xi)}(1+|\xi|^{2\sigma})\times\\ &\times\left|\int_{|z|\ge 1}|K(\xi-z)||W(\xi)-W(z)|^2|h(z,s)|dz\right|^2 d\xi=I_1+I_2.\end{aligned}$$

Arguing as in the derivation of (3.26) we obtain

$$I_1 \le C \int_0^1 dt \int_0^t ds \frac{1}{(t-s)^\beta} ||T_1 h(s)||^2_{H^\sigma(\mathbb{R})} \le C \int_0^1 ||T_1 h(s)||^2_{H^\sigma(\mathbb{R})} ds. \qquad (3.34)$$

On the other hand, in I_2 we use (3.28) and formula (3.32), to obtain:

$$\begin{aligned} I_2 &\le \int_0^1 dt \int_0^t ds \, ||T_1 h(s)||^2_{H_x^{\sigma-\rho}} \int_{|\xi|\ge 1} e^{-2(t-s)Re(W(\xi))} |\xi|^{2\sigma} d\xi \times \\ &\times \left(\int_{|z|\ge 1} \frac{|\xi - z|^4}{(1+|\xi - z|^n)^2} \frac{|W(\xi)|^{4\beta}}{|\xi|^{4\beta}} \frac{|W(z)|^{4\alpha-2}}{|z|^{4\alpha}} \frac{dz}{|z|^{2\sigma-2\rho}} \right) \\ &\le \int_0^1 dt \int_0^t ds \, ||T_1 h(s)||^2_{H_x^{\sigma-\rho}} \int_{|\xi|\ge 1} \frac{|\xi|^{2\sigma}}{(t-s)^{4\beta}} d\xi \times \\ &\times \left(\int_{|z|\ge 1} \frac{1}{(1+|\xi - z|^{n'})^2} \frac{1}{|\xi|^{4\beta}} \frac{|z|^{2\alpha-1}}{|z|^{4\alpha+2\sigma-2\rho}} dz \right) \\ &\le \int_0^1 dt \int_0^t ds \, \frac{1}{(t-s)^{4\beta}} ||T_1 h(s)||^2_{H_x^{\sigma-\rho}} \int_{|\xi|\ge 1} \frac{|\xi|^{2\rho}}{|\xi|^{3+2\beta}} d\xi \\ &\le C \int_0^1 ||T_1 h(s)||^2_{H_x^{\sigma-\rho}} ds \end{aligned} \qquad (3.35)$$

Combining (3.33), (3.34) and (3.35), (3.27) follows and then Proposition 3.5. □

We will also use the following Lemma.

Lemma 3.7 *Let $\alpha \in C_0^\infty(0, +\infty)$ and $\varepsilon_0 > 0$ such that supp $\alpha \subset (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$ and*

$$\varepsilon_0^n \left| \frac{d^n \alpha(x)}{dx^n} \right| \le C_n \, \varepsilon_0, \quad \textit{for all } x > 0 \qquad (3.36)$$

for some positive constant C_n independent of ε_0. Then, there exists positive constants K and C_{ε_0}, with K independent of ε_0, such that

$$||\alpha \, f||_{H^\sigma(\mathbb{R}^+)} \le K\varepsilon_0 \, ||f||_{H^\sigma(\mathbb{R}^+)} + C_{\varepsilon_0} ||f||_{L^\infty(\mathbb{R}^+)} \qquad (3.37)$$

$$||T_1 \alpha \, f||_{H^\sigma(\mathbb{R}^+)} \le K\varepsilon_0 \, ||T_1 f||_{H^\sigma(\mathbb{R}^+)} + C_{\varepsilon_0} ||f||_{L^\infty(\mathbb{R}^+)}. \qquad (3.38)$$

Proof of Lemma 3.7 Let us consider the function $m(k)$ that will be equal to one in the proof of (3.37) and $|Re(W(k, \varepsilon, R))|$ in the proof of (3.38) where $W(k, \varepsilon, R)$ is defined in (3.16). Due to the hypothesis (3.36):

$$|\widehat{\alpha}(k)| \le \frac{C_n \, \varepsilon_0^2}{1 + |k\varepsilon_0|^n} \quad \text{for all } \ k \in \mathbb{R} \qquad (3.39)$$

We proceed to estimate

$$J = \int_{\mathbb{R}} |m(k)|^2 |k|^{2\sigma} \left| \widehat{\alpha}(k-\xi) \, \widehat{f}(\xi) \, d\xi \right|^2 dk = \int_{|k|\le 1} [\cdots] \, dk + \int_{|k|\ge 1} [\cdots] \, dk = J_1 + J_2 \qquad (3.40)$$

The term J_1 is estimated as follows

$$|J_1| \le C ||f||^2_{L^2(\mathbb{R})} ||\widehat{\alpha}||_{L^1} \le C\varepsilon_0 \, ||f||^2_{L^2(\mathbb{R})} \qquad (3.41)$$

for some positive constant C independent on ε_0. On the other hand, we split J_2 as follows:

$$|J_2| \leq J_{2,1} + J_{2,2} \tag{3.42}$$

$$J_{2,1} = \int_{|k|\geq 1} |m(k)|^2 |k|^{2\sigma} \left| \int_{|\xi|\leq 1} \widehat{\alpha}(k-\xi)\, \widehat{f}(\xi)\, d\xi \right|^2 dk \tag{3.43}$$

$$J_{2,2} = \int_{|k|\geq 1} |m(k)|^2 |k|^{2\sigma} \left| \int_{|\xi|\geq 1} \widehat{\alpha}(k-\xi)\, \widehat{f}(\xi)\, d\xi \right|^2 dk. \tag{3.44}$$

To estimate $J_{2,1}$ we use that, for $|k| \geq 1$ and $|\xi| \leq 1$, one has $|m(k) - m(\xi)| \leq C(1 + m(k-\xi))$. We deduce that in the same range of k and ξ:

$$|m(k)|k|^\sigma - m(\xi)|\xi|^\sigma| \leq C(1 + |k-\xi|^\sigma)(1 + m(k-\xi)).$$

Then, since $|m(k-\xi)| \leq C(1 + \sqrt{|k-\xi|})$ we obtain:

$$J_{2,1} \leq C_{\varepsilon_0,n} \int_{|k|\geq 1} \left(\int_{|\xi|\leq 1} \frac{|\widehat{f}(\xi)|}{1 + |k-\xi|^n} d\xi \right)^2 dk \leq C'_{\varepsilon_0,n} ||f||_{L^2(\mathbb{R})} \tag{3.45}$$

where $C_{\varepsilon_0,n}$ and $C'_{\varepsilon_0,n}$ are constants depending on n and ε and using Young's inequality in the last step. Consider finally $J_{2,2}$. To this end we notice that, using (3.28) for the case when $m(k) = |Re(W(k,\varepsilon,R))$:

$$||k|^\sigma m(k) - |\xi|^\sigma m(\xi)| \leq |k|^\sigma |m(k) - m(\xi)| + ||k|^\sigma - |\xi|^\sigma|\, m(\xi) \leq |k|^\sigma \frac{|\xi - k|}{|\xi|} + |k-\xi|^\sigma m(\xi)$$

whence, using once again $|k| \leq |k-\xi| + |\xi|$,

$$||k|^\sigma m(k) - |\xi|^\sigma m(\xi)| \leq C \left(\frac{|k-\xi|^{\sigma+1}}{|\xi|} + \frac{|k-\xi||\xi|^\sigma}{|\xi|} + |k-\xi|^\sigma \right) m(\xi)$$

and then,

$$\begin{aligned} J_{2,2} &\leq \int_{|k|\geq 1} \left| \int_{|\xi|\geq 1} \widehat{\alpha}(k-\xi)\, m(\xi)\, |\xi|^\sigma \widehat{f}(\xi)\, d\xi \right|^2 dk + \\ &+ C_{\varepsilon_0,n'} \int_{|k|\geq 1} \left| \int_{|\xi|\geq 1} \frac{1 + |\xi|^{\sigma-1}}{1 + |k-\xi|^{n'}} m(\xi)\, |\xi|^\sigma \widehat{f}(\xi)\, d\xi \right|^2 dk. \end{aligned}$$

Using Young's inequality we obtain

$$J_{2,2} \leq K\varepsilon_0\, ||T_1 f||_{H^\sigma(\mathbb{R})} + C_{\varepsilon_0} ||T_1 f||_{H^{(\sigma-1)_+}(\mathbb{R})} \quad \text{if } m(k) = |ReW| \tag{3.46}$$

$$J_{2,2} \leq K\varepsilon_0\, ||f||_{H^\sigma(\mathbb{R})} + C_{\varepsilon_0} ||f||_{H^{(\sigma-1)_+}(\mathbb{R})} \quad \text{if } m(k) = 1. \tag{3.47}$$

Combining (3.41), (3.45), (3.46), (3.47) and a classical interpolation argument to estimate the norm $H^{(\sigma-1)_+}(\mathbb{R})$ by the L^∞ and $H^\sigma(\mathbb{R})$ norms the Lemma follows. □

Lemma 3.8 *Let η be a C^∞ compactly supported function in $\mathbb{R}^+$. Then, for any $\sigma > 0$ there exists a positive constant C such that for any $h \in H^\sigma(\mathbb{R})$, for any $R > 0$ and any $\varepsilon > 0$:*

$$\left|\left| \int_0^\infty h(x-y)\, (\eta(x) - \eta(x-y))\, \Phi(y,R,\varepsilon)\, dy \right|\right|_{H^{\sigma+1/2}(\mathbb{R})} \leq C||h||_{H^\sigma(\mathbb{R})}.$$

where $\Phi(y,R,\varepsilon)$ is defined by (3.2).

Proof of Lemma 3.8 We define three functions $M(x,y)$, $P(x,y,R,\varepsilon)$ and $Q(x,R,\varepsilon)$ as follows

$$Q(y) = y\,\Phi(y,R,\varepsilon), \quad M(x,y) = \frac{\eta(x)-\eta(x-y)}{y}$$
$$P(x,y) = (\eta(x)-\eta(x-y))\,\Phi(y,R,\varepsilon) = M(x,y)\,Q(y). \tag{3.48}$$

Where the dependence of P and Q on R and ε is not explicitly written by shortedness. Notice that $M(x,y) \in C^\infty(\mathbb{R}\times\mathbb{R})$. If we suppose that the support of η is contained in an interval $I \subset \mathbb{R}^+$, then the support of M is such that:

$$\mathrm{supp}\,(M) \subset I \times \mathbb{R}^+ \cup \left\{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+;\ x-y \in I\right\}. \tag{3.49}$$

Our goal is then to estimate estimate the the $H^{\sigma+1/2}$ norm of

$$B(h) = \int_0^\infty h(x-y,s)\,(\eta(x)-\eta(x-y))\,\Phi(y,R,\varepsilon)\,dy$$

which we write:

$$\begin{aligned}
||B(h)||^2_{H_x^{\sigma+1/2}(\mathbb{R})} &= \int_{\mathbb{R}} (1+|\xi|^{\sigma+1/2})^2 |\widehat{B(h)}(\xi)|^2\,d\xi \\
\widehat{B(h)}(\xi) &= \frac{1}{2\,\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty e^{i\,\eta(x-y)} M(x,y) Q(y) e^{-i\,x\,\xi}\,\hat{h}(\eta) dy\,d\eta\,dx \\
&= \int_{\mathbb{R}} \widehat{P}(\xi-\eta,\eta)\widehat{h}(\eta)\,d\eta
\end{aligned}$$

where

$$\widehat{P}(\zeta_1,\zeta_2) = \frac{1}{2\,\pi}\int_{\mathbb{R}^2} e^{-i(x\,\zeta_1+y\,\zeta_2)} P(x,y)\,dx\,dy \tag{3.50}$$

is the Fourier of the function P with respect to the two variables x and y.
Notice that:

$$||B(h)||^2_{H_x^{\sigma+1/2}(\mathbb{R})} = \int_{\mathbb{R}} (1+|\xi|^{\sigma+1/2})^2 \left|\int_{\mathbb{R}} \widehat{P}(\eta-\xi,-\eta)\hat{h}(\eta) d\eta\right|^2\,d\xi$$

We now proceed to estimate the function M. For any $m = 0, 1, \cdots$ there is a positive constant C_m, independent of R and ε, such that

$$\left|\frac{\partial^m M(x,y)}{\partial x^m}\right| \le \frac{C_m}{1+|y|} \qquad \text{for all } (x,y) \in \text{ in supp}(M). \tag{3.51}$$

On the other hand, there exists a positive constant C independent on R and ε such that for all $y \in \mathbb{R}^+$:

$$|Q(y,R,\varepsilon)| \le \frac{C}{\sqrt{|y|}} \tag{3.52}$$

Combining (3.48), (3.50), (3.51) and (3.52) we deduce that $P(x,y)$ is integrable in $\mathbb{R}^2$ and then $\widehat{P}$ is a well defined and bounded function on $\mathbb{R}^2$.

Moreover, we can also deduce decay estimate for $\widehat{P}$ for $|\zeta_1|+|\zeta_2| \to +\infty$. To this end we integrate by parts in formula (3.50)

$$\widehat{P}(\zeta_1,\zeta_2) = \frac{1}{2\pi\,i^n\,\zeta_1^n}\int_0^\infty e^{-i\zeta_2\,y} Q(y,R,\varepsilon) S_n(\zeta_1,y) dy \tag{3.53}$$

where

$$S_n(\zeta_1, y) = \int_{\mathbb{R}} e^{-i\zeta_1 x} \frac{\partial^n M(x,y)}{\partial x^n}\, dx. \tag{3.54}$$

Differentiating (3.54) with respect to y and integrating by parts, it easily follows that the function S_n are such that, for all $m = 0, 1, \cdots$, $k = 0, 1, \cdots$ there is a positive constant $C_{k,m,n}$, independent on R and ε satisfying, for all $\zeta_1 \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$:

$$\left| \frac{\partial^k S_n(\zeta_1, y)}{\partial y^k} \right| \leq \frac{C_{k,m,n}}{(1+|y|)\,(1+|\zeta_1|)^m} \tag{3.55}$$

Let us consider the behaviour of $\widehat{P}$ with respect to ζ_2. Using (3.53)

$$\begin{aligned} \widehat{P}(\zeta_1, \zeta_2) &= \frac{\varepsilon}{2\pi\, i^n\, \zeta_1^n} \int_0^1 e^{-i\zeta_2 y} \frac{1}{y^{1/2}} S_n(\zeta_1, y) dy + \\ &+ \frac{(1-\varepsilon)}{2\pi\, i^n\, \zeta_1^n} \int_0^1 e^{-i\zeta_2 y}\, R^{(3+\lambda)/2}\, y^{1+\lambda/2}\, f_0(Ry) S_n(\zeta_1, y) dy \\ &+ \frac{1}{2\pi\, i^n\, \zeta_1^n} \int_1^\infty e^{-i\zeta_2 y} Q(y, R, \varepsilon) S_n(\zeta_1, y) dy = \frac{1}{2\pi\, i^n\, \zeta_1^n} (J_1 + J_2 + J_3). \end{aligned} \tag{3.56}$$

In order to estimate the term J_1 we rewrite it as follows:

$$\begin{aligned} J_1 &= \int_0^1 e^{-i\zeta_2 y} \frac{\varepsilon}{y^{1/2}} S_n(\zeta_1, 0) dy + \int_0^1 e^{-i\zeta_2 y} \frac{\varepsilon}{y^{1/2}} (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) dy \\ &= \frac{\varepsilon\, S_n(\zeta_1, 0)}{\zeta_2^{1/2}} \int_0^{\zeta_2} e^{-i z} \frac{dz}{z^{1/2}} + \int_0^1 e^{-i\zeta_2 y} \frac{\varepsilon}{y^{1/2}} (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) dy. \end{aligned}$$

The integral $\int_0^{\zeta_2} e^{-i z} \frac{dz}{z^{1/2}}$ is uniformly bounded for $\zeta_2 \in \mathbb{R}$. On the other hand, due to (3.55) we have that

$$\left| \frac{\partial}{\partial y} \left(\frac{1}{y^{1/2}} (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) \right) \right| \leq \frac{C}{\sqrt{y}}.$$

Integrating by parts we obtain the existence of a constant C such that for all $\zeta_2 \in \mathbb{R}$:

$$\left| \int_0^1 e^{-i\zeta_2 y} \frac{\varepsilon}{y^{1/2}} (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) dy \right| \leq \frac{C}{1+|\zeta_2|} \qquad \text{for all } \zeta_2 \in \mathbb{R}.$$

Therefore $|J_1| \leq \dfrac{C}{1+|\zeta_2|^{1/2}}$. We use similar arguments to estimate J_2 that we write as follows

$$\begin{aligned} J_2 &= S_n(\zeta_1, 0) \int_0^1 e^{-i\zeta_2 y} \left(R^{(3+\lambda)/2}\, y^{1+\lambda/2}\, f_0(Ry) \right) dy \\ &\quad + \int_0^1 e^{-i\zeta_2 y} \left(R^{(3+\lambda)/2}\, y^{1+\lambda/2}\, f_0(Ry) \right) (S_n(\zeta_1, y) - S_n(\zeta_1, 0))\, dy = I_1 + I_2. \end{aligned}$$

The term I_2 may be estimated as above since (3.55) gives:

$$\left| \frac{\partial}{\partial y} \left[\left(R^{(3+\lambda)/2}\, y^{1+\lambda/2}\, f_0(Ry) \right) (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) \right] \right| \leq C\, y^{-1/2},\ y \in [0,1]$$

using that

$$\begin{aligned} &\frac{\partial}{\partial y} \left[\left(R^{(3+\lambda)/2}\, y^{1+\lambda/2}\, f_0(Ry) \right) (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) \right] \\ &= (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) \frac{\partial}{\partial y} \left[\left(R^{(3+\lambda)/2}\, y^{1+\lambda/2}\, f_0(Ry) \right) \right] + \\ &\quad \left(R^{(3+\lambda)/2}\, y^{1+\lambda/2}\, f_0(Ry) \right) \frac{\partial}{\partial y} \left[(S_n(\zeta_1, y) - S_n(\zeta_1, 0)) \right] \end{aligned}$$

as well as the bounds on f_0 and f_0'. We deduce $|I_2| \leq \dfrac{C}{1+|\zeta_2|}$.
We use in I_1 the change of variables $\eta = \zeta_2 y$ and the auxiliary function $g(y) = y^{1+\lambda/2} f_0(y)$ to obtain

$$I_1 = \frac{R^{1/2}}{\zeta_2} \int_0^{\zeta_2} e^{-i\eta} g\left(\frac{R\eta}{\zeta_2}\right) d\eta = \frac{R^{1/2}}{i\,\zeta_2} \int_0^{\zeta_2} (e^{-i\eta}-1) \frac{R}{\zeta_2} g'\left(\frac{R\eta}{\zeta_2}\right) d\eta$$

after integrating by parts. Then, there is a positive constant C independent on R and ε such that for all $\zeta_2 \in \mathbb{R}$ $|I_1| \leq \dfrac{C}{1+|\zeta_2|^{1/2}}$. Combining the estimates for I_1 and I_2:

$$|J_2| \leq \frac{C}{1+|\zeta_2|^{1/2}}.$$

We estimate J_3 integrating by parts and using (3.55) to obtain $|J_3| \leq \dfrac{C}{1+|\zeta_2|}$, whence

$$|\widehat{P}(\zeta_1,\zeta_2)| \leq \frac{C}{(1+|\zeta_1|^m)(1+|\zeta_2|^{1/2})}.$$

To conclude the proof of Lemma 3.8 we bound the norm of $B(h)$ in $H^{\sigma+1/2}(\mathbb{R})$ as follows:

$$\begin{aligned}
&||B(h)||^2_{H_x^{\sigma+1/2}(\mathbb{R})} \leq \int_{\mathbb{R}} (1+|\xi|^{\sigma+1/2})^2 \left| \int_{\mathbb{R}} \widehat{P}(\xi-\eta,\eta)\hat{h}(\eta) d\eta \right|^2 d\xi \\
&\leq \int_{\mathbb{R}} (1+|\xi|^{\sigma+1/2})^2 \left| \int_{\mathbb{R}} \frac{\hat{h}(\eta)}{(1+|\eta-\xi|^m)(1+|\eta|^{1/2})} d\eta \right|^2 d\xi \\
&\leq C \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\hat{h}(\eta)}{(1+|\eta-\xi|^{m'})} d\eta \right|^2 d\xi + C \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{(1+|\eta|^\sigma)\hat{h}(\eta)}{(1+|\eta-\xi|^m)} d\eta \right|^2 d\xi \\
&\leq C\,||h||^2_{L^2}\,||(1+|\cdot|^{-m'})||^2_{L^1(\mathbb{R})} + C\,||h||^2_{H^\sigma}\,||(1+|\cdot|^{-m})||^2_{L^1(\mathbb{R})}
\end{aligned}$$

where we have used Young's inequality in the last step. □

4 Proof of Theorem 3.1.

We start with the proof of Theorem 3.1.

Proof of (i) of Theorem 3.1. We apply now the classical method of freezing coefficients. To this end let us call χ a C^∞ function such that

$$\chi(x) = \begin{cases} 1 & \text{if } x \in (5/8, 11/8) , \\ 0 & \text{if } x \notin (1/2, 2) . \end{cases} \tag{4.1}$$

We define $\tilde f(x) = \chi(x)\, f(x)$. Then, for all $x \in \mathbb{R}$:

$$
\begin{aligned}
\frac{\partial \tilde f}{\partial t} &= \kappa\, T_{\varepsilon,R}(M_{\lambda/2}\tilde f) + \kappa \int_0^\infty (x-y)^{\lambda/2} f(x-y)\,(\chi(x)-\chi(x-y))\,\Phi(y,R,\varepsilon))\,dy + \\
&\quad + \chi(x)\,Q + \chi(x)\,P \\
&= \kappa T_{\varepsilon,R}(M_{\lambda/2}\tilde f) + \widetilde Q + \widetilde P \\
\widetilde Q &= \widetilde Q_1 + \widetilde Q_2 \\
\widetilde Q_1 &= \kappa \int_0^\infty (x-y)^{\lambda/2} f(x-y)\,(\chi(x)-\chi(x-y))\,\Phi(y,R,\varepsilon)\,dy \\
\widetilde Q_2 &= \chi(x)\,Q \\
\widetilde P &= \chi(x)\,P.
\end{aligned}
$$

Using the change of variables $x-y \to y$ in the definition of $\tilde Q_1$ in order to compute its derivatives differentiating the function Φ and using the fact that $(\chi(x) - \chi(y))$ vanishes if y is near x we obtain:

$$||\widetilde Q_1||_{L^\infty((0,1);W^{1,\infty}(\mathbb{R}))} \leq C\,\kappa\,||f||_{L^\infty((1/4,2)\times(0,1))} \tag{4.2}$$

Equation (3.5) may be written as

$$\frac{\partial \tilde f}{\partial t} = x_0^{\lambda/2}\,\kappa\, T_{\varepsilon,R}\left(\tilde f\right) + \kappa\, T_{\varepsilon,R}\left((M_{\lambda/2} - M_{\lambda/2,0})\tilde f\right) + \widetilde Q + \widetilde P \tag{4.3}$$

where $M_{\lambda/2,0}\tilde f(x) = x_0^{\lambda/2}\,\tilde f(x)$.

Fix now a new cutoff function η such that

$$\eta(x) = \begin{cases} 1 & \text{if } |x-x_0| \leq \delta, \\ 0 & \text{if } |x-x_0| \geq 2\delta\,. \end{cases} \tag{4.4}$$

with δ such that $|x^{\lambda/2} - x_0^{\lambda/2}| \leq \varepsilon_0$, for $|x-x_0| \leq 2\delta$ and ε_0 small enough to be chosen later. If we multiply the equation (4.3) by η and denote $\overline f = \eta \tilde f$ we obtain:

$$
\begin{aligned}
&\frac{\partial \overline f}{\partial t} - x_0^{\lambda/2}\,\kappa\, T_{\varepsilon,R}\left(\overline f\right) = \kappa\,\eta(x) T_{\varepsilon,R}\left((M_{\lambda/2} - M_{\lambda/2,0})\tilde f\right) + \eta(x)(\widetilde Q + \widetilde P) \\
&+ \kappa\, x_0^{\lambda/2} \int_0^\infty \tilde f(x-y)\,(\eta(x) - \eta(x-y))\,\Phi(y,R,\varepsilon)\,dy
\end{aligned} \tag{4.5}
$$

We have the following representation formula for the solution $\overline f$ of (4.5) in $L^\infty((1/4,2)\times(0,1)) \cap L^2(0,1;H^{1/2}(1/4,2)) \cap H^1(0,1;L^2(1/4,2))$:

$$
\begin{aligned}
\overline f(x,t) &= \int_0^t S_{\varepsilon,R}(\kappa(t-s))\eta(x)\,\left(\widetilde Q(s) + \widetilde P(s)\right)\,ds + \qquad (4.6)\\
&\quad + \kappa \int_0^t S_{\varepsilon,R}(\kappa(t-s))\left[\eta(x) T_{\varepsilon,R}\left((M_{\lambda/2} - M_{\lambda/2,0})\tilde f\right)(s)\right]\,ds \\
&\quad + \kappa\, x_0^{\lambda/2} \int_0^t S_{\varepsilon,R}(\kappa(t-s)) \int_0^\infty \tilde f(x-y,s)\,(\eta(x)-\eta(x-y))\,\Phi(y,R,\varepsilon)\,dy\,ds \\
&= \overline f_1(x,t) + \overline f_2(x,t) + \overline f_3(x,t) \qquad (4.7)
\end{aligned}
$$

for all $x \in \mathbb{R}$. This follows from the fact that the unique solution f in the space $L^\infty((1/4,2)\times(0,1)) \cap L^2(0,1;H^{1/2}(1/4,2)) \cap H^1(0,1;L^2(1/4,2))$ of:

$$
\begin{aligned}
&\frac{\partial f}{\partial t} - x_0^{\lambda/2}\kappa T_{\varepsilon,R}(f) = G(x,t) \qquad (4.8)\\
&f(0,x) = 0,
\end{aligned}
$$

where $G \in L^\infty((0,1)\times(0,+\infty)$ and f and G compactly supported in $(1/4,2)\times(0,1)$, is given by Duhamel's formula. The uniqueness of f can be obtained by taking the difference of two such solutions and taking the scalar product of (4.8) with that difference in L^2. Such computations are possible by the regularity that is assumed on the solutions.

$$||\overline{f}_1||^2_{H^\sigma(\mathbb{R})} \le C\int_0^t ||\eta\,\widetilde{Q}||^2_{H^\sigma(\mathbb{R})} ds_1 + C\left|\left|\int_0^t S_{\varepsilon,R}(\kappa(t-s))\eta(x)\,\widetilde{P}(s)\,ds\right|\right|^2_{H^\sigma(\mathbb{R})} \tag{4.9}$$

Let us estimate the second term in the right hand side of (4.9). Formulas (3.1) and (3.4) imply:

$$\left|\left|\int_0^t S_{\varepsilon,R}(\kappa(t-s))\eta(x)\,\widetilde{P}(s)\,ds\right|\right|^2_{H^\sigma(\mathbb{R})} \le$$
$$\left|\left|\int_0^t e^{-\sqrt{2}\Gamma(1/2)\varepsilon\kappa\,|\xi|^{1/2}(t-s)}|\widehat{\eta\,\widetilde{P}}(\xi,s)|\,ds(1+|\xi|^\sigma)^2\right|\right|^2_{L^2_\xi(\mathbb{R})} =$$
$$= \left|\left|\int_0^t e^{-\varepsilon\,\kappa\,\Lambda(t-s)}M\left(\eta\,\widetilde{P}\right)ds\right|\right|^2_{H^\sigma(\mathbb{R})}.$$

Integration in time and (3.25) yields:

$$\int_0^1 ||\overline{f}_1(t)||^2_{H^\sigma(\mathbb{R})}dt \le C\int_0^1 ||\eta\,\widetilde{Q}(t)||^2_{H^\sigma(\mathbb{R})}dt + \frac{C}{\varepsilon^2\,\kappa^2}\int_0^1 ||\eta\,\widetilde{P}(t)||^2_{H^{\sigma-1/2}} \tag{4.10}$$

In order to estimate the term corresponding to $\overline{f}_2$ we first write

$$\eta(x)T_{\varepsilon,R}\left((M_{\lambda/2}-M_{\lambda/2,0})\tilde{f}\right) = T_{\varepsilon,R}\left(\eta(x)\,(M_{\lambda/2}-M_{\lambda/2,0})\tilde{f}\right) + [\eta, T_{\varepsilon,R}]\left((M_{\lambda/2}-M_{\lambda/2,0})\tilde{f}\right)$$

where $[\eta, T_{\varepsilon,R}]$ is the conmutator of $T_{\varepsilon,R}$ and the multiplication by η

$$[\eta, T_{\varepsilon,R}]\,(\varphi)(x) = \eta(x)\,T_{\varepsilon,R}(\varphi)(x) - T_{\varepsilon,R}\,(\eta\,\varphi)\,(x)$$

Therefore

$$\begin{aligned}\overline{f}_2 &= \kappa\int_0^t S_{\varepsilon,R}(\kappa(t-s))\left[T_{\varepsilon,R}\left((M_{\lambda/2}-M_{\lambda/2,0})\overline{f}\right)(s)\right]ds\\ &+ \kappa\int_0^t S_{\varepsilon,R}(\kappa(t-s))\,[\eta, T_{\varepsilon,R}]\left((M_{\lambda/2}-M_{\lambda/2,0})\tilde{f}\right)ds\\ &= \overline{f}_{2,1}+\overline{f}_{2,2}\end{aligned} \tag{4.11}$$

where we have used that $\eta\widetilde{f} = \overline{f}$. Let us denote

$$\Psi(x,s) = (M_{\lambda/2}-M_{\lambda/2,0})\overline{f}(s) \tag{4.12}$$

and define the operator M as $M(\varphi) = \dfrac{1}{\sqrt{2\,\pi}}\displaystyle\int_{\mathbb{R}} |\widehat{\varphi}(\xi)|\,e^{i\,x\,\xi}\,d\xi$. Then:

$$\begin{aligned}\left|\widehat{\overline{f}_{2,1}}(\xi)\right|^2 &\le C\,\kappa^2\int_0^t\int_0^t e^{x_0^{\lambda/2}\kappa(t-s_1)T_1(\xi)}e^{x_0^{\lambda/2}\kappa(t-s_1)T_2(\xi)}\widehat{T(\Psi)}(\xi,s_1)\times\\ &\quad\times e^{x_0^{\lambda/2}\kappa(t-s_2)T_1(\xi)}e^{-x_0^{\lambda/2}\kappa(t-s_2)T_2(\xi)}\overline{\left(\widehat{T(\Psi)}(\xi,s_2)\right)}\,ds_1\,ds_2\\ &\le C\left(\kappa\int_0^t e^{x_0^{\lambda/2}\kappa(t-s)T_1(\xi)}\left|\widehat{T(\Psi)}(\xi,s)\right|ds\right)^2\\ &\le C\left|\kappa\int_0^t e^{x_0^{\lambda/2}\kappa(t-s)T_1(\xi)}\,T_1(M(\Psi))(s)ds\right|^2.\end{aligned}$$

where we have used that $|W(\xi,\varepsilon,R)|\le C|ReW(\xi,\varepsilon,R)|=C\,T_1(\xi)$. Therefore using (3.24):

$$\begin{aligned}\int_0^1||\overline{f}_{2,1}||^2_{H^\sigma(\mathbb{R})}\,dt &\le C\int_0^1||\int_0^t e^{x_0^{\lambda/2}\kappa(t-s)T_1(\xi)}\,\kappa\,T_1(M(\Psi))ds||^2_{H^\sigma(\mathbb{R})}dt\\ &\le C\int_0^1||M(\Psi)||^2_{H^\sigma(\mathbb{R})}ds=C\int_0^1||\Psi||^2_{H^\sigma(\mathbb{R})}ds.\end{aligned}$$

The function Ψ may be written as $\Psi(x,t)=\alpha(x)\,\overline{f}(x,t)$ with $\alpha(x)=\widetilde{\eta}(x)\,(x^{\lambda/2}-x_0^{\lambda/2})$ where $\widetilde{\eta}$ is a cutoff supported in the interval $|x-x_0|\le\varepsilon_0$ and $\widetilde{\eta}(x)=1$ in $|x-x_0|\le 2\delta$ where δ is given in (4.4). Notice that α may be assumed to satisfy condition (3.36). Lemma 3.7 then implies:

$$||\Psi||_{H^\sigma_x(\mathbb{R})}\le K\,\varepsilon_0||\overline{f}||_{H^\sigma}+C||\overline{f}||_{L^\infty(\mathbb{R}\times(0,1))}\tag{4.13}$$

where the constant C here and until the end of the Proof of Theorem 3.1 may depend on ε_0 but K is independent on it. We have then obtained:

$$\int_0^1||\overline{f}_{2,1}||^2_{H^\sigma(\mathbb{R})}\,dt\le K\varepsilon_0\int_0^1||\overline{f}(s)||^2_{H^\sigma(\mathbb{R})}\,d\,s+C||\overline{f}||^2_\infty\tag{4.14}$$

We consider now $\overline{f}_{2,2}$. Using, (3.26) we have:

$$||\overline{f}_{2,2}||_{H^\sigma_x(\mathbb{R})}\;\le\;C\,\kappa^{1-\beta}\int_0^t(t-s)^{-\beta}||\left(M_{\lambda/2}-M_{0,\lambda/2}\right)\tilde{f}||_{H^{\sigma-\rho}_x(\mathbb{R})}\,ds$$

and by (4.13) with σ replaced by $\sigma-\rho$:

$$||\overline{f}_{2,2}||_{H^\sigma_x(\mathbb{R})}\;\le\;C\,\kappa^{1-\beta}\int_0^t(t-s)^{-\beta}||\tilde{f}||_{H^{\sigma-\rho}_x(\mathbb{R})}\,ds+C\,\kappa^{1-\beta}\,||\widetilde{f}||_{L^\infty(\mathbb{R})\times(0,1)}.\tag{4.15}$$

Squaring and integrating (4.15) and adding to the results for β small, we obtain

$$\begin{aligned}\int_0^1||\overline{f}_2(s)||^2_{H^\sigma_x(\mathbb{R})}ds &\le \varepsilon_0\int_0^1||\overline{f}(s)||^2_{H^\sigma(\mathbb{R})}\,d\,s+C\int_0^1||\tilde{f}(s)||^2_{H^{\sigma-\rho}(\mathbb{R})}\,d\,s\\ &\quad+C||\tilde{f}||^2_\infty\end{aligned}\tag{4.16}$$

The last term $\overline{f}_3$ is estimated as follows. Using Lemma 3.8 we obtain

$$\left|\left|\int_0^\infty\tilde{f}(x-y,s)\,(\eta(x)-\eta(x-y))\,\Phi(y,R,\varepsilon)\,dy\right|\right|_{H^\sigma_x(\mathbb{R})}\le C||\tilde{f}||_{H^{(\sigma-1/2)_+}(\mathbb{R})}.\tag{4.17}$$

Then, (3.23) in Lemma 3.5 and an interpolation argument yield:

$$||\overline{f}_3||_{H^\sigma_x(\mathbb{R})}\;\le\;C\,\kappa\int_0^t||\tilde{f}(s)||_{H^{(\sigma-1/2)_+}(\mathbb{R})}ds,$$

whence

$$\int_0^1||\overline{f}_3(s)||^2_{H^\sigma_x(\mathbb{R})}ds\;\le\;C\,\kappa^2\int_0^1||\tilde{f}(s)||^2_{H^{(\sigma-1/2)_+}}\,d\,s+C\,\kappa^2\,||\tilde{f}\,||^2_\infty.\tag{4.18}$$

Adding (4.10), (4.16) and (4.18) and using $\rho<1/2$, we deduce:

$$\int_0^1||\overline{f}(s)||^2_{H^\sigma(\mathbb{R})}\le\varepsilon_0\int_0^1||\overline{f}(s)||^2_{H^\sigma(\mathbb{R})}+C\int_0^t||\tilde{f}(s)||_{H^{\sigma-\rho}(\mathbb{R})}ds$$

$$+C\int_0^1||\eta\,\widetilde{Q}||^2_{H^\sigma(\mathbb{R})}ds+C||\widetilde{f}||^2_{L^\infty(\mathbb{R}\times(0,1))}+\frac{C}{\varepsilon^2\,\kappa^2}\int_0^1||\eta\,\widetilde{P}(t)||^2_{H^{\sigma-1/2}}.$$

Choosing ε_0 small enough:

$$\int_0^1 ||\overline{f}(s)||^2_{H^\sigma(\mathbb{R})} \le C\int_0^t ||\tilde{f}(s)||_{H^{\sigma-\rho}(\mathbb{R})} ds + C\int_0^1 ||\eta\,\widetilde{Q}||^2_{H^\sigma(\mathbb{R})} ds + C||\widetilde{f}||^2_{L^\infty(\mathbb{R}\times(0,1))}$$
$$+\frac{C}{\varepsilon^2\,\kappa^2}\int_0^1 ||\eta\,\widetilde{P}(t)||^2_{H^{\sigma-1/2}}.$$

Using a partition of the unity $(\eta_i)_{i\in\mathbb{N}}$ of the interval $(1/2,\,2)$, and adding the contributions of all the terms we obtain:

$$\int_0^1 ||\tilde{f}(s)||^2_{H^\sigma(\mathbb{R})} \le C\int_0^t ||\tilde{f}(s)||_{H^{\sigma-\rho}(\mathbb{R})} ds + C\int_0^1 ||\widetilde{Q}||^2_{H^\sigma(\mathbb{R})} ds + C||\widetilde{f}||^2_{L^\infty(\mathbb{R}\times(0,1))} +$$
$$+\frac{C}{\varepsilon^2\,\kappa^2}\int_0^1 ||\widetilde{P}(t)||^2_{H^{\sigma-1/2}} \tag{4.19}$$

where the constants C depend on δ. An interpolation argument then implies:

$$\int_0^1 ||\tilde{f}(s)||^2_{H^\sigma(\mathbb{R})} \le \varepsilon\int_0^t ||\tilde{f}(s)||_{H^{\sigma}(\mathbb{R})} ds + C\int_0^1 ||\widetilde{Q}||^2_{H^\sigma(\mathbb{R})} ds + C||\widetilde{f}||^2_{L^\infty(\mathbb{R}\times(0,1))}$$
$$+\frac{C}{\varepsilon^2\,\kappa^2}\int_0^1 ||\widetilde{P}(s)||^2_{H^{\sigma-1/2}}$$

whence part (i) of Theorem 3.1 follows.

Remark 4.1 *Notice that in (4.14), we estimate the H^σ norm of $\overline{f}_{2,1}$ in terms of the H^σ norm of $\overline{f}$, not of $\tilde{f}$.*

Remark 4.2 *In the estimates of $\overline{f}_j$, $j=1,2,3$ the term $\overline{f}_{2,1}$ is the only one where we are using the continuity in the freezing coefficients argument to obtain (4.14).*

Proof of (ii) of Theorem 3.1. In order to prove part (ii) we first notice that the equation satisfied by $\overline{f}$ is:

$$\frac{\partial\overline{f}}{\partial t} - x_0^{\lambda/2} T_{\varepsilon,R}\left(\overline{f}\right) + a(t)\,\overline{f} = \eta(x)T_{\varepsilon,R}\left((M_{\lambda/2}-M_{\lambda/2,0})\widetilde{f}\right) + \eta(x)\widetilde{Q} + \eta(x)\widetilde{P}$$
$$+x_0^{\lambda/2}\int_0^\infty \tilde{f}(x-y)\left(\eta(x)-\eta(x-y)\right)\Phi(y,R,\varepsilon)\,dy - \left(a(x,t)-a(t)\right)\overline{f} \tag{4.20}$$

where x_0 and η have been chosen as before and where $a(t)=a(x_0,t)$. Then:

$$\begin{aligned}
\overline{f}(x,t) &= \int_0^t \omega(t,s)S_{\varepsilon,R}(t-s)\left(\eta(x)\,\widetilde{Q}(s)+\eta(x)\,\widetilde{P}(s)\right)\,ds\,+\\
&+\int_0^t \omega(t,s)S_{\varepsilon,R}(t-s)\left[\eta(x)T_{\varepsilon,R}\left((M_{\lambda/2}-M_{\lambda/2,0})\widetilde{f}\right)\right]\,ds\,+\\
&+x_0^{\lambda/2}\int_0^t \omega(t,s)S_{\varepsilon,R}(t-s)\int_0^\infty \tilde{f}(x-y,s)\left(\eta(x)-\eta(x-y)\right)\Phi(y,R,\varepsilon)\,dy\,ds\\
&-\int_0^t \omega(t,s)S_{\varepsilon,R}(t-s)(a(x,t)-a(t))\,\overline{f}(s)\,ds\\
&= \overline{f}_1(x,t)+\overline{f}_2(x,t)+\overline{f}_3(x,t)+\overline{f}_4(x,t)
\end{aligned}$$

where we have defined:

$$\omega(t,s) = e^{-\int_s^t a(\lambda)\,d\lambda} \tag{4.21}$$

We estimate first the term with $\overline{f}_1$. If $T \leq 1$, then the same argument of the proof of point (i) shows that

$$\left(\int_T^{T+1} ||\overline{f}_1(t)||^2_{H^\sigma_x} dt\right)^{1/2} \leq \left(\int_0^2 ||\overline{f}_1(t)||^2_{H^\sigma_x} dt\right)^{1/2}$$
$$\leq C\left(\int_0^2 ||\widetilde{Q}(t)||^2_{H^\sigma_x} dt\right)^{1/2} + \frac{C}{\varepsilon^2}\left(\int_0^2 ||\widetilde{P}(t)||^2_{H^\sigma_x} dt\right)^{1/2}. \quad (4.22)$$

If $T > 1$, using the change of variables $t = (T-1)+\tau$ we write

$$\begin{aligned}
&\left(\int_T^{T+1} ||\int_0^t \omega(t,s) S_{\varepsilon,R}(t-s)\eta(x)\left(\widetilde{Q}(s)+\widetilde{P}(s)\right) ds||^2_{H^\sigma} dt\right)^{1/2}\\
\leq\ & \sum_{n=1}^{[T]}\left(\int_T^{T+1} ||\int_{n-1}^n \omega(t,s) S_{\varepsilon,R}(t-s)\eta(x)\left(\widetilde{Q}(s)+\widetilde{P}(s)\right) ds||^2_{H^\sigma} dt\right)^{1/2}\\
&+\left(\int_T^{T+1} ||\int_{[T]}^t \omega(t,s) S_{\varepsilon,R}(t-s)\eta(x)\left(\widetilde{Q}(s)+\widetilde{P}(s)\right) ds||^2_{H^\sigma} dt\right)^{1/2}\\
=\ & I_1 + I_2.
\end{aligned}$$

The estimate of the term I_2 follows as in the proof of point (i) of the Theorem and gives

$$I_2 \leq C\left(\int_T^{T+1} ||\widetilde{Q}(s)||^2_{H^\sigma}\right)^{1/2} + \frac{C}{\varepsilon}\left(\int_T^{T+1} ||\widetilde{P}(s)||^2_{H^{\sigma-1/2}}\right)^{1/2}.$$

To estimate I_1 we argue as follows. Changing the time variable t as $t = \tau + (T-n)$ and obtain:

$$\begin{aligned}
I_1 =& \sum_{n=1}^{[T]}\left(\int_n^{n+1} ||\int_{n-1}^n \omega(\tau+(T-n),s) S_{\varepsilon,R}((T-n)+\tau-s)\times\right.\\
&\qquad\left.\times\ \eta(x)\left(\widetilde{Q}(s)+\widetilde{P}(s)\right) ds||^2_{H^\sigma} d\tau\right)^{1/2}\\
\leq& \sum_{n=1}^{T}\left(\int_n^{n+1} ||\int_{n-1}^n \omega(\tau+(T-n),s) S_{\varepsilon,R}(\tau-s)\eta(x)\left(\widetilde{Q}(s)+\widetilde{P}(s)\right) ds||^2_{H^\sigma} d\tau\right)^{1/2}
\end{aligned}$$

since $||S_\varepsilon(T-n)\,h||_{H^\sigma} \leq ||h||_{H^\sigma}$ because $T-N \geq 0$. We use now that for each n

$$\begin{aligned}
&\int_n^{n+1} ||\int_{n-1}^n \omega(\tau+(T-n),s)\, S_{\varepsilon,R}(\tau-s)\eta(x)\left(\widetilde{Q}(s)+\widetilde{P}(s)\right) ds||^2_{H^\sigma} d\tau\\
&\leq \int_{n-1}^{n+1} ||\int_{n-1}^\tau \omega(\tau+(T-n),n)\,\omega(n,s) S_{\varepsilon,R}(\tau-s)\eta(x)\times\\
&\qquad\times \mathbf{1}_{(n-1,n)}(s)\left(\widetilde{Q}(s)+\widetilde{P}(s)\right) ds||^2_{H^\sigma} d\tau\\
&\leq Ce^{-2A(T-n)}\left(\int_{n-1}^{n+1} \mathbf{1}_{(n-1,n)}(s)\,\omega(n,s)||\widetilde{Q}||^2_{H^\sigma}\, ds + \right.\\
&\qquad\left.+\frac{1}{\varepsilon^2}\int_{n-1}^{n+1} \mathbf{1}_{(n-1,n)}(s)\,\omega(n,s)||\widetilde{P}||^2_{H^{\sigma-1/2}}\right)\\
&\leq Ce^{-2A(T-n)}\int_{n-1}^n ||\widetilde{Q}||^2_{H^\sigma}\, ds + \frac{Ce^{-2A(T-n)}}{\varepsilon^2}\int_{n-1}^n ||\widetilde{P}||^2_{H^{\sigma-1/2}}\, ds,
\end{aligned}$$

whence:

$$I_1 + I_2 \le C \sum_{n=1}^{[T]} e^{-A(T-n)} \left(\int_{n-1}^{n} ||\widetilde{Q}||^2_{H^\sigma} \, ds \right)^{1/2} + C \left(\int_{[T]}^{T+1} ||\widetilde{Q}(s)||^2_{H^\sigma} \right)^{1/2} +$$

$$+ \quad \frac{C}{\varepsilon} \sum_{n=1}^{[T]} e^{-A(T-n)} \left(\int_{n-1}^{n} ||\widetilde{P}||^2_{H^{\sigma-1/2}} \, ds \right)^{1/2} + \frac{C}{\varepsilon} \left(\int_{[T]}^{T+1} ||\widetilde{P}(s)||^2_{H^{\sigma-1/2}} \right)^{1/2}$$

$$\le \quad C \sup_{0 \le T \le T_{max}} \left(\int_{T}^{T+1} ||\widetilde{Q}||^2_{H^\sigma} \, ds \right)^{1/2} + \frac{C}{\varepsilon} \sup_{0 \le T \le T_{max}} \left(\int_{T}^{T+1} ||\widetilde{P}||^2_{H^{\sigma-1/2}} \, ds \right)^{1/2}$$

and

$$\sup_{0 \le T \le T_{max}} \left(\int_{T}^{\min(T+1, T_{max})} ||\overline{f}_1(s)||^2_{H^\sigma} ds \right)^{1/2} \le$$

$$\le C \sup_{0 \le T \le T_{max}} \left(\int_{T}^{\min(T+1, T_{max})} ||\widetilde{Q}||^2_{H^\sigma} \, ds \right)^{1/2} +$$

$$+ \frac{C}{\varepsilon} \sup_{0 \le T \le T_{max}} \left(\int_{T}^{\min(T+1, T_{max})} ||\widetilde{P}||^2_{H^{\sigma-1/2}} \, ds \right)^{1/2} \tag{4.23}$$

The term $\overline{f}_2$ is written as $\overline{f}_2 = \overline{f}_{2,1} + \overline{f}_{2,2}$ where $\overline{f}_{2,1}$ and $\overline{f}_{2,2}$ are defined as in (4.11). We first estimate $\overline{f}_{2,1}$. Consider then

$$\left(\int_{T}^{T+1} || \int_{0}^{t} \omega(t,s) S_{\varepsilon,R}(t-s) \left[T_{\varepsilon,R} \Psi(x,s) \right] \, ds ||^2_{H^\sigma} dt \right)^{1/2}$$

$$\le \quad \sum_{n=1}^{[T]} \left(\int_{T}^{T+1} || \int_{n-1}^{n} \omega(t,s) S_{\varepsilon,R}(t-s) \left[T_{\varepsilon,R} \Psi(x,s) \right] \, ds ||^2_{H^\sigma} dt \right)^{1/2}$$

$$+ \left(\int_{T}^{T+1} || \int_{[T]}^{t} \omega(t,s) S_{\varepsilon,R}(t-s) \left[T_{\varepsilon,R} \Psi(x,s) \right] \, ds ||^2_{H^\sigma} \right)^{1/2}$$

$$= \quad I_1 + I_2.$$

Arguing as in the derivation of (4.14) we obtain that there exists a positive constant ε_0 that can be chosen arbitrarily small if δ is small enough, and such that:

$$I_2 \le \varepsilon_0 \left(\int_{T}^{\min(T+1, T_{max})} ||\overline{f}(s)||^2_{H^\sigma} \right)^{1/2} + C \, ||\overline{f}||^2_{L^\infty}. \tag{4.24}$$

In the first term I_1, we change the time variable t as $t = \tau + (T-n)$ and obtain:

$$I_1 \le \sum_{n=1}^{[T]} \left(\int_{n}^{n+1} || \int_{n-1}^{n} \omega(\tau + (T-n), s) S_{\varepsilon,R}(\tau + (T-n) - s) \left[T_{\varepsilon,R} \Psi(x,s) \right] \, ds ||^2_{H^\sigma} d\tau \right)^{1/2}$$

$$\le \sum_{n=1}^{[T]} \left(\int_{n}^{n+1} || \int_{n-1}^{n} \omega(\tau + (T-n), s) S_{\varepsilon,R}(\tau - s) \left[T_{\varepsilon,R} \Psi(x,s) \right] \, ds ||^2_{H^\sigma} d\tau \right)^{1/2}.$$

Arguing again as in the derivation of (4.14) we obtain, for ε_0 defined as above:

$$\int_n^{n+1} || \int_{n-1}^{n} \omega(\tau+(T-n),s) S_{\varepsilon,R}(\tau-s)\left[T_{\varepsilon,R}\Psi(x,s)\right] ds||^2_{H^\sigma} d\tau$$
$$= \int_{n-1}^{n+1} || \int_{n-1}^{\tau} \omega(\tau+(T-n),s) S_{\varepsilon,R}(\tau-s)\left[T_{\varepsilon,R}\left(\mathbf{1}_{(n-1,n)}(s)\Psi(x,s)\right)\right] ds||^2_{H^\sigma} d\tau$$
$$\le e^{-2A(T-n)}\left(\varepsilon_0 \int_{n-1}^{n} ||\overline{f}||^2_{H^\sigma}\, ds + C\,||\overline{f}||^2_{L^\infty}\right).$$

Therefore

$$I_1 \le \varepsilon_0 \sum_{n=1}^{[T]} e^{-A(T-n)}\left(\int_{n-1}^{n} ||\overline{f}||^2_{H^\sigma}\, ds\right)^{1/2} + C\sum_{n=1}^{[T]} e^{-A(T-n)}||\overline{f}||_{L^\infty}$$
$$\le \varepsilon_0 \sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})} ||\overline{f}||^2_{H^\sigma}\, ds\right)^{1/2} + C\,||\overline{f}||_{L^\infty},$$

whence

$$I_1+I_2 \le \varepsilon_0 \sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})} ||\overline{f}||^2_{H^\sigma}\, ds\right)^{1/2} + C\,||\overline{f}||_{L^\infty}.$$

We then obtain the estimate:

$$\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})} ||\overline{f}_{2,1}(s)||^2_{H^\sigma} ds\right)^{1/2}$$
$$\le \varepsilon_0 \sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})} ||\overline{f}||^2_{H^\sigma}\, ds\right)^{1/2} + C\,||\overline{f}||_{L^\infty}. \tag{4.25}$$

A similar argument using the contractivity of $S_{\varepsilon,R}$ in the spaces H^σ gives for $\overline{f}_{2,2}$ and $\overline{f}_3$:

$$\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})} ||\overline{f}_{2,2}(t)||^2_{H^\sigma_x} dt\right)^{1/2}$$
$$\le C \sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})} ||\widetilde{f}||^2_{H^{\sigma-\rho}}\, ds\right)^{1/2} + C||\widetilde{f}||_{L^\infty} \tag{4.26}$$

$$\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})} ||\overline{f}_{3}(t)||^2_{H^\sigma_x} dt\right)^{1/2}$$
$$\le C \sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})} ||\widetilde{f}||^2_{H^{\sigma-1/2}}\, ds\right)^{1/2} + C||\widetilde{f}||_{L^\infty}. \tag{4.27}$$

We now estimate $\overline{f}_4$:

$$\left(\int_T^{T+1} ||\overline{f}_4(s)||^2_{H^\sigma} dt\right)^{1/2}$$
$$\le \sum_{n=1}^{[T]}\left(\int_T^{T+1} || \int_{n-1}^{n} \omega(t,s) S_{\varepsilon,R}(t-s)(a(x,t)-a(t))\,\overline{f}(s) ds||^2_{H^\sigma} dt\right)^{1/2}$$
$$+\left(\int_T^{T+1} || \int_{[T]}^{t} \omega(t,s) S_{\varepsilon,R}(t-s)(a(x,t)-a(t))\,\overline{f}(s) ds||^2_{H^\sigma}\right)^{1/2} = I_1+I_2.$$

We use now the continuity of the semigroup $S_{\varepsilon,R}$ in H^σ, the fact that $a \in H^\sigma(\mathbb{R})$ and since $\sigma > 1/2$, the imbedding of H^σ into $\mathbf{C}(\mathbb{R})$ is continuous to obtain the existence of a positive constant ε_0, which can be made arbitrarily small if δ is sufficiently small, such that

$$||S_{\varepsilon,R}(t-s)(a(x,t)-a(t))\,\overline{f}(s)||_{H^\sigma} \le ||(a(x,t)-a(t))\,\overline{f}(s)||_{H^\sigma} \le \left(\varepsilon_0||\overline{f}||_{H^\sigma} + C||\overline{f}||_\infty\right)\,||a||_{H^\sigma_x}.$$

Arguing as in the derivation of (4.25) we obtain

$$\sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\overline{f}_4(s)||^2_{H^\sigma} ds\right)^{1/2} \le \varepsilon_0 \sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\overline{f}||^2_{H^\sigma}\, ds\right)^{1/2} + C\,||\overline{f}||_{L^\infty}. \tag{4.28}$$

Adding formulas (4.23), (4.25)–(4.28) we obtain:

$$\sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\overline{f}(t)||^2_{H^\sigma_x} dt\right)^{1/2} \le \varepsilon_0 \sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\overline{f}||^2_{H^\sigma}\, ds\right)^{1/2}$$
$$+C \sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{f}||^2_{H^{\sigma-\rho}}\, ds\right)^{1/2} + C||\widetilde{f}||_{L^\infty}$$
$$+C \sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{Q}||^2_{H^\sigma}\, ds\right)^{1/2} + \frac{C}{\varepsilon} \sup_{0\le T\le T_{max}} \left(\int_T^{T+1} ||\widetilde{P}||^2_{H^{\sigma-1/2}}\, ds\right)^{1/2}$$

where we have used that $||\overline{f}||_{L^\infty} \le ||\widetilde{f}||_{L^\infty}$ and $\rho \le 1/2$. Then,

$$\sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\overline{f}(t)||^2_{H^\sigma_x} dt\right)^{1/2} \le C \sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{f}||^2_{H^{\sigma-\rho}}\, ds\right)^{1/2}$$
$$+C||\widetilde{f}||_{L^\infty} + C \sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{Q}||^2_{H^s}\, ds\right)^{1/2}$$
$$+\frac{C}{\varepsilon} \sup_{0\le T\le T_{max}} \left(\int_T^{T+1} ||\widetilde{P}||^2_{H^{\sigma-1/2}}\, ds\right)^{1/2}. \tag{4.29}$$

Using a partition of unity as in the derivation of (4.19), we arrive at

$$\sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{f}(t)||^2_{H^\sigma_x} dt\right)^{1/2} \le C \sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{f}||^2_{H^{\sigma-\rho}}\, ds\right)^{1/2}$$
$$+C||\widetilde{f}||_{L^\infty} + C \sup_{0\le T\le T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{Q}||^2_{H^\sigma}\, ds\right)^{1/2}$$
$$+\frac{C}{\varepsilon} \sup_{0\le T\le T_{max}} \left(\int_T^{T+1} ||\widetilde{P}||^2_{H^{\sigma-1/2}}\, ds\right)^{1/2} \tag{4.30}$$

where the constants $C > 0$ depend on δ. An interpolation argument yields

$$\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})}||\widetilde{f}(t)||_{H^\sigma}^2\,dt\right)^{1/2}\le$$
$$C\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})}||\widetilde{Q}(t)||_{H^\sigma}^2\,dt\right)^{1/2}+C\,||\widetilde{f}||_{L^\infty}$$
$$+\frac{C}{\varepsilon}\sup_{0\le T\le T_{max}}\left(\int_T^{T+1}||\widetilde{P}||_{H^{\sigma-1/2}}^2\,ds\right)^{1/2} \tag{4.31}$$

Using that $\chi = 1$ in the interval $(5/8, 11/8)$ we have:

$$||f||_{H^\sigma(3/4,5/4)}\le||\widetilde{f}||_{H^\sigma}. \tag{4.32}$$

Part (ii) of Theorem 3.1 then follows combining (4.31) and (4.32).

Proof of part (iii) of Theorem 3.1. The equation satisfied by $\overline{f}$ is now (4.20) with $\widetilde{P} = 0$ and $\varepsilon = 0$. Then:

$$\begin{aligned}\overline{f}(x,t) &= \int_0^t\omega(t,s)S_{\varepsilon,R}(t-s)\eta(x)\,\widetilde{Q}(s)\,ds\\ &\quad+\int_0^t\omega(t,s)S_{\varepsilon,R}(t-s)\left[\eta(x)T_{\varepsilon,R}\left((M_{\lambda/2}-M_{\lambda/2,0})\widetilde{f}\right)\right]ds\\ &\quad+x_0^{\lambda/2}\int_0^t\omega(t,s)S_{\varepsilon,R}(t-s)\int_0^\infty\tilde{f}(x-y,s)\left(\eta(x)-\eta(x-y)\right)\Phi(y,R,\varepsilon)\,dy\,ds\\ &\quad-\int_0^t\omega(t,s)S_{\varepsilon,R}(t-s)(a(x,t)-a(t))\,\overline{f}(s)\,ds\\ &= \overline{f}_1(x,t)+\overline{f}_2(x,t)+\overline{f}_3(x,t)+\overline{f}_4(x,t)\end{aligned}$$

where ω is given by (4.21). The term $\overline{f}_1$ is estimated using (3.24) for $T\le 1$. Then,

$$\left(\int_T^{T+1}||T_1(\overline{f}_1)(t)||_{H_x^\sigma}^2dt\right)^{1/2}\le\left(\int_0^2||T_1(\overline{f}_1)(t)||_{H_x^\sigma}^2dt\right)^{1/2}\le C\left(\int_0^2||\widetilde{Q}(t)||_{H_x^\sigma}^2dt\right)^{1/2} \tag{4.33}$$

If $T > 1$, using the change of variables $t = (T-1)+\tau$ we write

$$\begin{aligned}&\left(\int_T^{T+1}||T_1\left(\int_0^t\omega(t,s)S_{\varepsilon,R}(t-s)\eta(x)\,\widetilde{Q}(s)\,ds\right)||_{H^\sigma}^2dt\right)^{1/2}\\ \le\ &\sum_{n=1}^{[T]}\left(\int_T^{T+1}||\int_{n-1}^n\omega(t,s)T_1\left(S_{\varepsilon,R}(t-s)\eta(x)\,\widetilde{Q}(s)\right)ds||_{H^\sigma}^2dt\right)^{1/2}\\ &+\left(\int_T^{T+1}||\int_{[T]}^t\omega(t,s)T_1\left(S_{\varepsilon,R}(t-s)\eta(x)\,\widetilde{Q}(s)\right)ds||_{H^s}^2dt\right)^{1/2}\\ =\ &I_1+I_2.\end{aligned}$$

The estimate of the term I_2 can be made as in (4.33) to obtain$I_2\le C\left(\int_T^{T+1}||\widetilde{Q}(s)||_{H^\sigma}^2\right)^{1/2}$. To estimate I_1 we argue as follows. We change the time variable t as $t = \tau + (T-n)$ and obtain:

$$
\begin{aligned}
I_1 &= \sum_{n=1}^{[T]} \left(\int_n^{n+1} || \int_{n-1}^{n} \omega(\tau + (T-n), s) T_1 \, S_{\varepsilon,R}((T-n) + \tau - s) \left(\eta(x) \, \widetilde{Q}(s) \right) \, ds ||_{H^\sigma}^2 d\tau \right)^{1/2} \\
&\leq \sum_{n=1}^{T} \left(\int_n^{n+1} || \int_{n-1}^{n} \omega(\tau + (T-n), s) T_1 \, S_{\varepsilon,R}(\tau - s) \left(\eta(x) \, \widetilde{Q}(s) \right) \, ds ||_{H^\sigma}^2 d\tau \right)^{1/2}
\end{aligned}
$$

where we have used $||S_\varepsilon(T-n)\, h||_{H^\sigma} \leq ||h||_{H^\sigma}$. We notice now that, for each n

$$
\begin{aligned}
&\int_n^{n+1} || \int_{n-1}^{n} \omega(\tau + (T-n), s) \, T_1 S_{\varepsilon,R}(\tau - s)(\eta(x) \, \widetilde{Q}(s)) \, ds ||_{H^\sigma}^2 d\tau \\
&\leq \int_{n-1}^{n+1} || \int_{n-1}^{\tau} \omega(\tau + (T-n), n) \, \omega(n,s) T_1 \, S_{\varepsilon,R}(\tau - s)(\eta(x) \, \mathbf{1}_{(n-1,n)}(s) \, \widetilde{Q}(s)) \, ds ||_{H^\sigma}^2 d\tau \\
&\leq Ce^{-2A(T-n)} \left(\int_{n-1}^{n+1} \mathbf{1}_{(n-1,n)}(s) \, \omega(n,s) ||\widetilde{Q}||_{H^\sigma}^2 \, ds \right) \leq Ce^{-2A(T-n)} \int_{n-1}^{n} ||\widetilde{Q}||_{H^\sigma}^2 \, ds,
\end{aligned}
$$

whence:

$$
\begin{aligned}
I_1 + I_2 \quad &\leq \quad C \sum_{n=1}^{[T]} e^{-A(T-n)} \left(\int_{n-1}^{n} ||\widetilde{Q}||_{H^\sigma}^2 \, ds \right)^{1/2} + C \left(\int_{[T]}^{T+1} ||\widetilde{Q}(s)||_{H^\sigma}^2 \right)^{1/2} \\
&\leq \quad C \sup_{0 \leq T \leq T_{max}} \left(\int_T^{T+1} ||\widetilde{Q}||_{H^\sigma}^2 \, ds \right)^{1/2}
\end{aligned}
$$

and

$$
\sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||T_1(\overline{f}_1)(s)||_{H^\sigma}^2 ds \right)^{1/2} \leq C \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||\widetilde{Q}||_{H^\sigma}^2 \, ds \right)^{1/2} \quad (4.34)
$$

The term $\overline{f}_2$ is written as $\overline{f}_2 = \overline{f}_{2,1} + \overline{f}_{2,2}$ where $\overline{f}_{2,1}$ and $\overline{f}_{2,2}$ are defined as in (4.11). We first estimate $\overline{f}_{2,1}$. Consider then

$$
\begin{aligned}
&\left(\int_T^{T+1} ||T_1 \int_0^t \omega(t,s) S_{\varepsilon,R}(t-s) \, [T_{\varepsilon,R} \Psi(x,s)] \, ds ||_{H^\sigma}^2 dt \right)^{1/2} \\
\leq \quad &\sum_{n=1}^{[T]} \left(\int_T^{T+1} || \int_{n-1}^{n} \omega(t,s) T_1 \, S_{\varepsilon,R}(t-s) \, [T_{\varepsilon,R} \Psi(x,s)] \, ds ||_{H^\sigma}^2 dt \right)^{1/2} \\
&+ \left(\int_T^{T+1} || \int_{[T]}^{t} \omega(t,s) T_1 \, S_{\varepsilon,R}(t-s) \, [T_{\varepsilon,R} \Psi(x,s)] \, ds ||_{H^\sigma}^2 \right)^{1/2} \\
= \quad &I_1 + I_2.
\end{aligned}
$$

Arguing as in the derivation of (4.14), but using (3.38) instead of (3.37) we obtain that there exists a positive constant ε_0 that can be chosen arbitrarily small such that:

$$
I_2 \leq \varepsilon_0 \left(\int_T^{\min(T+1, T_{max})} ||T_1 \, \overline{f}(s)||_{H^\sigma}^2 \right)^{1/2} + C \, ||\overline{f}||_{L^\infty}^2. \quad (4.35)
$$

In the first term I_1, we change the time variable t as $t = \tau + (T - n)$ and obtain:

$$I_1 \leq \sum_{n=1}^{[T]} \left(\int_n^{n+1} || \int_{n-1}^{n} \omega(\tau + (T-n), s) T_1 \, S_{\varepsilon,R}(\tau + (T-n) - s) \, [T_{\varepsilon,R} \Psi(x,s)] \; ds||_{H^\sigma}^2 d\tau \right)^{1/2}$$
$$\leq \sum_{n=1}^{[T]} \left(\int_n^{n+1} || \int_{n-1}^{n} \omega(\tau + (T-n), s) T_1 \, S_{\varepsilon,R}(\tau - s) \, [T_{\varepsilon,R} \Psi(x,s)] \; ds||_{H^\sigma}^2 d\tau \right)^{1/2}.$$

Arguing again as in the derivation of (4.14) we obtain, for ε_0 defined as above,

$$\int_n^{n+1} || \int_{n-1}^{n} \omega(\tau + (T-n), s) T_1 \, S_{\varepsilon,R}(\tau - s) \, [T_{\varepsilon,R} \Psi(x,s)] \; ds||_{H^\sigma}^2 d\tau$$
$$\leq e^{-2A(T-n)} \left(\varepsilon_0 \int_{n-1}^{n} ||T_1 \, \overline{f}||_{H^s}^2 \, ds + C \, ||\overline{f}||_{L^\infty}^2 \right).$$

Therefore

$$I_1 \leq \varepsilon_0 \sum_{n=1}^{[T]} e^{-A(T-n)} \left(\int_{n-1}^{n} ||T_1 \, \overline{f}||_{H^\sigma}^2 \, ds \right)^{1/2} + C \sum_{n=1}^{[T]} e^{-A(T-n)} ||\overline{f}||_{L^\infty}$$
$$\leq \varepsilon_0 \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||T_1 \, \overline{f}||_{H^\sigma}^2 \, ds \right)^{1/2} + C \, ||\overline{f}||_{L^\infty},$$

whence

$$I_1 + I_2 \leq \varepsilon_0 \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||T_1 \, \overline{f}||_{H^\sigma}^2 \, ds \right)^{1/2} + C \, ||\overline{f}||_{L^\infty}$$

and therefore

$$\sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||T_1 \, \overline{f}_{2,1}(s)||_{H^\sigma}^2 ds \right)^{1/2}$$
$$\leq \varepsilon_0 \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||T_1 \, \overline{f}||_{H^\sigma}^2 \, ds \right)^{1/2} + C \, ||\overline{f}||_{L^\infty}. \tag{4.36}$$

A similar argument using the contractivity of $S_{\varepsilon,R}$ in the spaces H^σ and formula (3.27) gives

$$\sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||T_1 \, \overline{f}_{2,2}(t)||_{H_x^\sigma}^2 dt \right)^{1/2}$$
$$\leq C \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||T_1 \widetilde{f}||_{H^{\sigma-\rho}}^2 \, ds \right)^{1/2} + C||\widetilde{f}||_{L^\infty} \tag{4.37}$$

To estimate $\overline{f}_3$ we combine (3.24) and (4.17) to obtain

$$\sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||T_1 \overline{f}_3(t)||_{H_x^\sigma}^2 dt \right)^{1/2}$$
$$\leq C \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} ||\widetilde{f}||_{H^{\sigma-1/2}}^2 \, ds \right)^{1/2} + C||\widetilde{f}||_{L^\infty}. \tag{4.38}$$

We now estimate $\overline{f}_4$:

$$
\begin{aligned}
&\left(\int_T^{T+1} ||T_1 \overline{f}_4(s)||_{H^\sigma}^2 dt\right)^{1/2} \\
\leq\ & \sum_{n=1}^{[T]} \left(\int_T^{T+1} || \int_{n-1}^{n} \omega(t,s) T_1\, S_{\varepsilon,R}(t-s)(a(x,t)-a(t))\, \overline{f}(s) ds||_{H^\sigma}^2 dt\right)^{1/2} \\
&+ \left(\int_T^{T+1} || \int_{[T]}^{t} \omega(t,s) T_1\, S_{\varepsilon,R}(t-s)(a(x,t)-a(t))\, \overline{f}(s) ds||_{H^\sigma}^2\right)^{1/2} \\
=\ & I_1 + I_2.
\end{aligned}
$$

Using (3.24) we get $I_2 \leq \left(\int_T^{T+1} ||(a(x,t)-a(t))\overline{f}||_{H^\sigma(\mathbb{R})}^2\right)^{1/2}$. Since $a \in H^{\sigma+1} \subset C^{1,\alpha}$ for some $\alpha > 0$, we obtain,for δ sufficiently small

$$
I_2 \leq C \left(\int_T^{T+1} \left(\varepsilon_0\, ||\overline{f}(t)||_{H^\sigma(\mathbb{R})}^2 + C||\overline{f}(t)||_{L^\infty}^2\right) dt\right)^{1/2}
$$

The term I_1 can be estimated similarly using the exponential decay of $\omega(t,s)$ as in the previous cases. Then

$$
\begin{aligned}
&\sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||T_1\, \overline{f}_4(s)||_{H^\sigma}^2 ds\right)^{1/2} \\
&\qquad \leq \varepsilon_0 \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\overline{f}||_{H^\sigma}^2\, ds\right)^{1/2} + C\, ||\overline{f}||_{L^\infty}. \qquad (4.39)
\end{aligned}
$$

Adding formulas (4.23), (4.25)–(4.28) we obtain:

$$
\begin{aligned}
&\sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||T_1\, \overline{f}(t)||_{H_x^\sigma}^2 dt\right)^{1/2} \leq \varepsilon_0 \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||T_1\, \overline{f}||_{H^\sigma}^2\, ds\right)^{1/2} \\
&+C \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{f}||_{H^{\sigma-\rho}}^2\, ds\right)^{1/2} + C||\widetilde{f}||_{L^\infty} \\
&+C \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{Q}||_{H^\sigma}^2\, ds\right)^{1/2}
\end{aligned}
$$

where we have used that $||\overline{f}||_{L^\infty} \leq ||\widetilde{f}||_{L^\infty}$ and $\rho \leq 1/2$. Then,

$$
\begin{aligned}
&\sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||T_1\, \overline{f}(t)||_{H_x^\sigma}^2 dt\right)^{1/2} \leq C \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{f}||_{H^{\sigma-\rho}}^2\, ds\right)^{1/2} \\
&+C||\widetilde{f}||_{L^\infty} + C \sup_{0\leq T\leq T_{max}} \left(\int_T^{\min(T+1,T_{max})} ||\widetilde{Q}||_{H^s}^2\, ds\right)^{1/2}. \qquad (4.40)
\end{aligned}
$$

Using a partition of unity as for the derivation of (4.19), we arrive at

$$\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})}||T_1\widetilde{f}(t)||^2_{H^\sigma_x}dt\right)^{1/2}\le C\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})}||\widetilde{f}||^2_{H^{\sigma-\rho}}\,ds\right)^{1/2}$$
$$+C||\widetilde{f}||_{L^\infty}+C\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})}||\widetilde{Q}||^2_{H^s}\,ds\right)^{1/2} \tag{4.41}$$

where the constants $C>0$ depend on δ. An interpolation argument yields

$$\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})}||T_1\,\widetilde{f}(t)||^2_{H^\sigma}\,dt\right)^{1/2}\le C\sup_{0\le T\le T_{max}}\left(\int_T^{\min(T+1,T_{max})}||\widetilde{Q}(t)||^2_{H^\sigma}\,dt\right)^{1/2}+$$
$$+C\,||\widetilde{f}||_{L^\infty}$$

Using that $\chi=1$ in the interval $(5/8,11/8)$, we obtain

$$||T_1f||_{H^\sigma(3/4,5/4)}\le C||T_1\widetilde{f}||_{H^\sigma}. \tag{4.42}$$

On the other hand $|W(k,R,0)|\le C\min\{|k|,R\}$. Therefore estimate (3.10) holds. This concludes the proof of part (iii) of Theorem 3.1. □

5 Interior regularity estimates for the operator $\mathcal{L}$.

The main result in this Section is the following result, which is basically a reformulation of Theorem 3.1 in a way that is more convenient to be applied to the solutions of (2.4)

Theorem 5.1 *Suppose that* $\sigma\in(1/2,2)$, $\nu\in L^2_t(0,1;H^\sigma_x(1/4,4))$, $\varepsilon\in[0,1]$, $K\in L^\infty((1/4,4)\times(0,1))\cap L^2_t(0,1;H^\sigma_x(1/4,4))$, *and* $h\in L^\infty((1/8,4)\times(0,1))\cap L^2(0,1;H^{1/2}(1/4,2))\cap H^1(0,1;L^2(1/4,2))$, $W\in L^2_t(0,1;H_x^{\sigma-1/2}(1/4,4))$ *satisfies :*

$$\begin{aligned}\frac{\partial h}{\partial t}&=\varepsilon\int_0^{x/2}\frac{(x-y)^{\lambda/2}h(x-y)-x^{\lambda/2}h(x)}{y^{3/2}}+\\&\quad+(1-\varepsilon)R^{3/2}\int_0^{x/2}\left((x-y)^{\lambda/2}h(x-y)-x^{\lambda/2}h(x)\right)(Ry)^{\lambda/2}f_0(R\,y)\,dy+\\&\quad+K(x,t)\,h(x,t)+\nu(x,t)+W(x,t),\end{aligned}$$

for all $x\in(1/4,4)$ *and* $R>1$ *and* $h(x,0)=0$. *Then for any* $T\in[0,1]$*:*

$$||h||_{L^2_t(0,T;H^\sigma_x(7/8,9/8))}\le C\left(||\nu||_{L^2_t(0,1;H^\sigma_x(1/4,4))}+||h||_{L^\infty((1/8,4)\times(0,1))}+\frac{1}{\varepsilon}||W||_{L^2_t(0,1;H_x^{\sigma-1/2}(1/4,4))}\right)$$

where the constant C *is independent of* ε *and* R *but depends on* $||K||_{L^\infty((1/2,2)\times(0,1))}$ *and* $||K||_{L^2_t(0,1;H^\sigma_x(1/4,4))}$.

Proof of Theorem 5.1. Let χ be a C^∞ function such that

$$\chi(x)=\begin{cases}1 & \text{if } x\in(1/2,2)\ ,\\ 0 & \text{if } x\notin(1/4,4)\ .\end{cases} \tag{5.1}$$

We define $\tilde h(x,t) = \chi(x)\, h(x)$. Then, for all $x \in \mathbb{R}$ the function $\tilde h$ satisfies

$$\begin{aligned}
\frac{\partial \tilde h}{\partial t} &= \int_0^{+\infty} \Big((x-y)^{\lambda/2} \tilde h(x-y) - x^{\lambda/2} \tilde h(x) \Big)\, \Phi(y,R,\varepsilon) dy \\
&\quad - \int_{x/2}^{+\infty} \Big((x-y)^{\lambda/2} \tilde h(x-y) - x^{\lambda/2} \tilde h(x) \Big)\, \Phi(y,R,\varepsilon) dy \\
&\quad + \int_0^{x/2} (x-y)^{\lambda/2} h(x-y)\, (\chi(x) - \chi(x-y))\, \Phi(y,R,\varepsilon)\, dy \\
&\quad + K(x,t)\, \tilde h(x,t) + \chi(x)\, \nu + \chi(x)\, W. \qquad (5.2)
\end{aligned}$$

where the function $\Phi(y,R,\varepsilon)$ has been defined in (3.2).

We write the equation (5.2) in terms of the new function $\overline{h}$ defined as:

$$\tilde h(x,t) = e^{\left(\int_0^t K(x,s)\, ds + c_0(\varepsilon,R,x) t\right)}\, \overline{h}(x,t)$$

with $c_0(\varepsilon,R,x) = x^{\lambda/2} \displaystyle\int_{x/2}^{\infty} \Phi(y,R,\varepsilon)\, dy$:

$$\frac{\partial \overline{h}}{\partial t} = T_{\varepsilon,R}\left(M_{\lambda/2}\, \overline{h}\right) + Q_1 + Q_2 + Q_3 + Q_4$$

where $T_{\varepsilon,R}$ has been defined in (3.1) and

$$\begin{aligned}
Q_1 &= -\int_{x/2}^{+\infty} (x-y)^{\lambda/2} \overline{h}(x-y) \Phi(y,R,\varepsilon) dy \\
Q_2 &= e^{-\left(\int_0^t K(x,s)\, ds + 2\sqrt{2}\varepsilon t x^{(\lambda-1)/2}\right)} \int_0^{x/2} (x-y)^{\lambda/2} h(x-y)\, (\chi(x) - \chi(x-y))\, \Phi(y,R,\varepsilon)\, dy \\
Q_3 &= e^{-\left(\int_0^t K(x,s)\, ds + c_0(\varepsilon,R,x) t\right)}\, \chi(x)\, \nu\,, \quad Q_4 = e^{-\left(\int_0^t K(x,s)\, ds + c_0(\varepsilon,R,x) t\right)}\, \chi(x)\, W.
\end{aligned}$$

These terms are estimated as follows:

$$\begin{aligned}
||Q_1||_{L^\infty(0,1;W^{1,\infty}((1/4,4)))} + ||Q_2||_{L^\infty(0,1;H_x^{\tilde\sigma}((1/4,4)))} &\le C\, ||h||_{L^\infty((1/8,4)\times(0,1))} \\
||Q_3||_{L_t^2(0,1;H_x^\sigma(1/4,4))} &\le C\, ||\nu||_{L_t^2(0,1;H_x^\sigma(1/4,4))}, \\
||Q_4||_{L_t^2(0,1;H_x^{\sigma-1/2}(1/4,4))} &\le C\, ||W||_{L_t^2(0,1;H_x^{\sigma-1/2}(1/4,4))},
\end{aligned}$$

where $\tilde\sigma = \min\{\sigma, 1\}$.

If $1/2 < \sigma \le 1$ Theorem 3.1 immediately yields:

$$\begin{aligned}
||\overline{h}||_{L_t^2(0,1;H_x^\sigma(3/4,5/4))} &\le C\left(||\nu||_{L_t^2(0,1;H_x^\sigma(1/4,4))} + ||h||_{L^\infty((1/8,4)\times(0,1))} \right) + \\
&\quad + \frac{C}{\varepsilon} ||W||_{L_t^2(0,1;H_x^{\sigma-1/2}(1/4,4))}.
\end{aligned}$$

If $\sigma > 1$ we apply Theorem 3.1 with $\sigma = 1$ to obtain:

$$\begin{aligned}
||\overline{h}||_{L_t^2(0,1;H_x^1(3/4,5/4))} &\le C\left(||\nu||_{L_t^2(0,1;H_x^\sigma(1/4,4))} + ||h||_{L^\infty((1/8,4)\times(0,1))} \right) + \\
&\quad + \frac{C}{\varepsilon} ||W||_{L_t^2(0,1;H_x^{\sigma-1/2}(1/4,4))}. \qquad (5.3)
\end{aligned}$$

Since Q_1 and Q_2 involve integrals of the function $\overline{h}$, (5.3) provides better estimates on Q_1 and Q_2 although on the smaller interval $(3/4, 5/4)$:

$$\begin{aligned}
||Q_1||_{L^\infty(0,1;H^\sigma((3/4,5/4)))} &+ ||Q_2||_{L^\infty(0,1;H^\sigma((3/4,5/4)))} \le C\, ||\overline{h}||_{L_t^2(0,1;H_x^1(3/4,5/4))} \\
&\le C\left(||\nu||_{L_t^2(0,1;H_x^\sigma(1/4,4))} + ||h||_{L^\infty((1/8,4)\times(0,1))} \right).
\end{aligned}$$

Using again the Theorem 3.1 with $\sigma < 2$:

$$\begin{aligned}||\overline{h}||_{L^2_t(0,1;H^\sigma_x(7/8,9/8))} &\leq C\left(||\nu||_{L^2_t(0,1;H^s_x(1/4,4))} + ||h||_{L^\infty((1/8,4)\times(0,1))}\right) + \\ &+\frac{C}{\varepsilon}||W||_{L^2_t(0,1;H^{\sigma-1/2}_x(1/4,4))}.\end{aligned}$$

This ends the proof of Theorem 5.1. □

6 Estimating the difference between $\mathcal{L}$ and L.

In this Section we estimate the operator $\mathcal{L} - L$ which appear in the equation (1.7).

$$\begin{aligned}(\mathcal{L}-L)(\varphi)(x,t) &= A_1 + A_2, \\ A_1(x) &= \int_0^{x/2} \left(H(x-y) - H(x)\right) y^{\lambda/2}\varphi(y,t)dy \\ &\quad -H(x)\int_{x/2}^{\infty} y^{\lambda/2}\varphi(y,t)dy - x^{\lambda/2}\varphi(x,t)\left(\int_{x/2}^{\infty} H(y)\,dy\right) \quad (6.1) \\ A_2(x) &= \int_0^{x/2}\left((x-y)^{\lambda/2}\varphi(x-y,t) - x^{\lambda/2}\varphi(x,t)\right)H(y)dy \quad (6.2) \\ H(y) &= y^{\lambda/2}\, f_0(y) - y^{-3/2}. \quad (6.3)\end{aligned}$$

Since it will be needed in the Section 6, we shall actually estimate more general operators where the function A_2 has the more general form:

$$\begin{aligned}A_{2,\varepsilon}(x) &= \int_0^{x/2}\left((x-y)^{\lambda/2}\varphi(x-y,t) - x^{\lambda/2}\varphi(x,t)\right)H_\varepsilon(x,y)dy \quad (6.4) \\ H_\varepsilon(x,y) &= y^{\lambda/2}\, f_0(y) - \frac{1}{y^{3/2}+\varepsilon^{3/2}x^{3/2}}. \quad (6.5)\end{aligned}$$

Notice that $\varepsilon = 0$ corresponds to the functions A_2 and H defined in (6.2) and (6.3). We will need to assume in many of the following estimates that $\sigma > 1+\delta$. Notice that, by reducing the value of δ, it is possible to obtain estimates for all $\sigma > 1$.

In the two following Lemmas we estimate the two terms A_1 and $A_{2,\varepsilon}$ assuming some conditions of the function f_0.

Lemma 6.1 *Suppose that f_0 satisfies conditions (2.1), (2.2) and $|||\varphi|||_{3/2,(3+\lambda)/2} < \infty$. Then*

$$|||A_1|||_{3/2,\,2+\delta} \leq C\,|||\varphi|||_{3/2,\,(3+\lambda)/2}.$$

Proof of Lemma 6.1. The estimate on $A_1(x)$ for $0 < x < 1$ is immediate:

$$\begin{aligned}\left|\int_0^{x/2}(H(x-y)-H(x))y^{\lambda/2}\varphi(y)dy\right| &\leq |||\varphi|||_{3/2,(3+\lambda)/2}\int_0^{x/2}|H(x-y)+H(x)|\,y^{(-3+\lambda)/2}dy \\ &\leq C\,|||\varphi|||_{3/2,(3+\lambda)/2}\,x^{-3/2} \quad (6.6)\end{aligned}$$

$$\left|H(x)\int_{x/2}^{\infty} y^{\lambda/2}\varphi(y)dy\right| \leq C|||\varphi|||_{3/2,(3+\lambda)/2}\,x^{-3/2}, \quad (6.7)$$

$$\left|x^{\lambda/2}\left(\int_{x/2}^{\infty} H(y)\,dy\right)\varphi(x)\right| \leq C\,|||\varphi|||_{3/2,(3+\lambda)/2}\,x^{\lambda/2-2} \leq C\,|||\varphi|||_{3/2,(3+\lambda)/2}\,x^{-3/2}. \quad (6.8)$$

Let us consider the case when $x > 1$. In order to estimate the first term in the right hand side of (6.1) we write:

$$H(x-y) - H(x) = y \int_0^1 H'(x-\theta y) d\theta$$

where

$$H'(z) = \frac{\lambda}{2} z^{(\lambda-2)/2} (f_0(z) - z^{-(3+\lambda)/2}) + z^{\lambda/2} (f_0'(z) + \frac{3+\lambda}{2} z^{-(3+\lambda)/2-1})$$

By assumptions (2.1) (2.2), for all $z > 1$:

$$|H'(z)| \leq \left(1 + \frac{\lambda}{2}\right) z^{-5/2-\delta}$$

In particular, for all $y < x/2$ and $0 < \theta < 1$ we have $x - \theta y > x/2$ and so, if $x > 2$:

$$|H(x-y) - H(x)| = \left| y \int_0^1 H'(x-\theta y) d\theta \right| \leq C\, y\, x^{-5/2-\delta}$$

and

$$\left| \int_0^{x/2} (H(x-y) - H(x))\, y^{\lambda/2} \varphi(y) dy \right| \quad \leq \quad C|||\varphi|||_{3/2,\,(3+\lambda)/2}\, x^{-2-\delta} \tag{6.9}$$

In order to estimate the second term in (6.1) we use:

$$|H(x)| = x^{\lambda/2}\, |f_0(x) - G(x)| \leq C\, x^{\lambda/2} x^{-(3+\lambda)/2-\delta} \quad \text{for } x > 1$$

whence, for $x > 1$:

$$\left| H(x) \int_{x/2}^{\infty} y^{\lambda/2} \varphi(y) dy \right| \leq x^{-3/2-\delta} \int_{x/2}^{\infty} y^{\lambda/2} \varphi(y) dy \leq C\, |||\varphi|||_{3/2,\,(3+\lambda)/2}\, x^{-2-\delta}. \tag{6.10}$$

The third term of (6.1) is bounded by

$$x^{\lambda/2} |\varphi(x)| \int_{x/2}^{\infty} y^{-3/2-\delta}\, dy \quad = \quad C\, |||\varphi|||_{3/2,\,(3+\lambda)/2}\, x^{-2-\delta} \quad \text{for } x > 1. \tag{6.11}$$

Lemma 6.1 then follows combining (6.6)-(6.11). □

Lemma 6.2 *Let* $0 \leq T \leq 1$. *Then, there exists a constant* $C > 0$ *such that, for any* $\varphi \in \mathcal{E}_{T;\sigma}$ *and for all* $t_0 \in (0, T)$ *:*

$$R^{2+\delta}\, N_{2;\,\sigma}\, (A_1;\, R,\, t_0) \leq C\, |||\varphi|||, \;\; \forall R > 1,$$
$$R^{2-\lambda/2}\, M_{2;\,\sigma}\, (A_1;\, R) \leq C\, |||\varphi|||, \;\; \forall\;\; 0 < R < 1.$$

Proof of Lemma 6.2. For $R > 1$ we write $\varphi(x,t) = \sum_{n=0}^{\infty} \chi(x/2^n)\, \varphi(x,t)$ where $\chi \in C_0^{\infty}$, $\operatorname{supp} \chi \subset (1/2, 2)$. Let us consider $R = 2^{n_0}$, $x \in (R/2, 2R)$ and rescale $x = RX$, $y = RY$, $\tau = (t - t_0) R^{(\lambda-1)/2}$, $\varphi(x,t) = R^{-(3+\lambda)/2)} \psi(X,\tau)$, $\mathcal{A}_1(X,\tau) = A_1(x,t)$ to obtain:

$$\begin{aligned} R^{2+\delta}\, |\mathcal{A}_1(X,\tau)| \quad &= \quad \int_0^{X/2} [H_R(X-Y) - H_R(X)]\; Y^{\lambda/2}\, \psi(Y,\tau)\, dY - \\ &\quad -H_R(X) \int_{X/2}^{\infty} Y^{\lambda/2} \psi(Y,\tau)\, dY - X^{\lambda/2}\, \psi(X,\tau) \int_{X/2}^{\infty} H_R(Y)\, dY \end{aligned} \tag{6.12}$$

where the function H_R is defined as follows:

$$H_R(X) = \left(R^{(3+\lambda)/2} X^{\lambda/2}\, f_0(R\,X) - X^{-3/2}\right) R^{\delta}. \tag{6.13}$$

and satisfies $|H_R(x)| + |H_R'(x)| + |H_R''(x)| \leq C$ for some positive constant C and all $x \in (1/2, 1)$. Since $|||\varphi|||_{3/2,(3+\lambda)/2} < \infty$, we have the following bound on $\psi(X,\tau)$

$$|\psi(X,\tau)| \leq C \min\left\{\frac{R^{\lambda/2}}{X^{3/2}}, \frac{1}{X^{(3+\lambda)/2}}\right\} |||\varphi(t)|||_{3/2,(3+\lambda)/2} \tag{6.14}$$

for all $X \geq 0$ and $\tau \in (0, T\,R^{(\lambda-1)/2})$. Using this estimate it then follows that the integrals in the right hand side of (6.12) are convergent. Moreover, using conditions (2.1) and (2.2) we obtain:

$$R^{2+\delta}\left(\int_0^1 d\tau \int_{1/2}^2 |D_x^\sigma \mathcal{A}_1(X,\tau)|^2 dX d\tau\right)^{1/2} \leq C|||\varphi|||, \quad \forall R > 1. \tag{6.15}$$

For $R \in (0,1)$ we scale the variables $x \in (R/2, 2R)$ and φ as $x = RX$, $y = RY$, $\varphi(x,t) = R^{-3/2}\psi(X,t)$, $\mathcal{A}_1(X,t) = A_1(x,t)$ to obtain in this case:

$$\begin{aligned} R^{2-\lambda/2}\,|\mathcal{A}_1(X,t)| \;&=\; \int_0^{X/2} [H_R(X-Y) - H_R(X)]\; Y^{\lambda/2}\,\psi(Y,t)\,dY - \\ &-H_R(X)\int_{X/2}^\infty Y^{\lambda/2}\psi(Y,\tau)\,dY - X^{\lambda/2}\,\psi(X,t)\int_{X/2}^\infty H_R(Y)\,dY \end{aligned} \tag{6.16}$$

Using again (2.1), (2.2) and (6.14) we deduce

$$R^{2-\lambda/2}\left(\int_0^1 d\tau \int_{1/2}^2 |D_x^\sigma \mathcal{A}_1(X,\tau)|^2 dX d\tau\right)^{1/2} \leq C|||\varphi|||, \quad \forall R \in (0,1). \tag{6.17}$$

Lemma 6.2 follows from (6.15) and (6.17). □

The following two technical Lemmas will be needed in order to estimate $A_{2,\varepsilon}$.

Lemma 6.3 *For any given function $h \in H^\sigma(\mathbb{R})$ supported in $(1/2, 2)$, with $\sigma > 1 + \delta$, and $\delta \in [0, \min(1/2, \sigma - 1))$ there holds:*

$$\int_0^{5/8} |h(X-Y) - h(X)|\, Y^{-3/2-\delta} dY \leq C||h||_{H^\sigma}$$

Proof of Lemma 6.3. Using

$$h(X) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{h}(\xi) e^{ix\xi} d\xi$$

we obtain

$$\int_0^{5/8} |h(X-Y) - h(X)|\, Y^{-3/2-\delta} dY \leq C \int_0^{1/2}\int_{\mathbb{R}} |\widehat{h}(\xi)|\left|e^{-i\xi Y} - 1\right| Y^{-3/2-\delta} d\xi dY$$

$$\leq C\left(\int_{\mathbb{R}} |\widehat{h}(\xi)|^2 (1+|\xi|^\sigma)^2 d\xi\right)^{1/2}\left(\int_{\mathbb{R}}\left|\int_0^{5/8}\left|e^{-i\xi Y} - 1\right| Y^{-3/2-\delta} dY\right|^2 \frac{d\xi}{(1+|\xi|^{2\sigma})}\right)^{1/2}.$$

Using the change of variables $\xi y = z$ we arrive at

$$\int_0^{5/8}\left|e^{-i\xi Y)} - 1\right| Y^{-3/2-\delta} dY \leq C\xi^{1/2+\delta}\int_0^{5\,\xi/8} |e^{-iz} - 1| \frac{dz}{z^{3/2+\delta}} \leq C\xi^{1/2+\delta},$$

and

$$\int_{\mathbb{R}} \left| \int_0^{5/8} \left| e^{-i\xi Y)} - 1 \right| Y^{-3/2-\delta} dY \right|^2 \frac{d\xi}{1+|\xi|^{2\sigma}} \leq \int_{\mathbb{R}} |\xi|^{1+2\delta} \frac{d\xi}{(1+|\xi|^{2\sigma})} < \infty$$

since $\sigma > 1+\delta$. □

Lemma 6.4 *Consider the operators $\mathcal{W}_R$ and $\mathcal{W}_\infty$ defined as:*

$$\mathcal{W}_R(h) = R^{(3+\lambda)/2} \int_0^{x/2} \Big((x-y)^{\lambda/2} h(x-y) - x^{\lambda/2} h(x) \Big) y^{\lambda/2} f_0(Ry) dy \tag{6.18}$$

$$\mathcal{W}_\infty(h) = \int_0^{x/2} \Big((x-y)^{\lambda/2} h(x-y) - x^{\lambda/2} h(x) \Big) y^{-3/2} dy \tag{6.19}$$

$$\mathcal{W}_{\infty,\varepsilon}(h) = \int_0^{x/2} \Big((x-y)^{\lambda/2} h(x-y) - x^{\lambda/2} h(x) \Big) \frac{dy}{y^{3/2} + \varepsilon^{3/2} x^{3/2}} \tag{6.20}$$

Then, for any $\eta \in \mathbf{C}^\infty(\mathbb{R})$ of compact support contained in $(1/2, 3/2)$ such that $\eta = 1$ on $(3/4, 5/4)$, and for all $\sigma \geq 1/2$, there exists a positive constant C, depending only on the function η and its derivatives, such that for all $\psi \in H^\sigma(\mathbb{R})$:

$$||\mathcal{W}_\infty(\eta\,\psi)||_{H^{\sigma-1/2}(\mathbb{R})} + ||\mathcal{W}_R(\eta\,\psi)||_{H^{\sigma-1/2}(\mathbb{R})} + ||\mathcal{W}_{\infty,\varepsilon}(\eta\,\psi)||_{H^{\sigma-1/2}(\mathbb{R})} \leq C||\eta\,\psi||_{H^\sigma(\mathbb{R})}. \tag{6.21}$$

Moreover, for all $h \in H^\sigma(\mathbb{R})$ fixed:

$$\lim_{\varepsilon\to 0} ||\eta \cdot (\mathcal{W}_{\infty,\varepsilon} - \mathcal{W}_\infty)(\eta\, h)||_{H^{\sigma-1/2}(\mathbb{R})} = 0. \tag{6.22}$$

Proof of Lemma 6.4. The function $\mathcal{W}_R(\eta\,\psi)$ can be written as follows

$$R^{(3+\lambda)/2} \int_0^{x/2} \Big((x-y)^{\lambda/2} \eta(x-y)\, \psi(x-y) - x^{\lambda/2} \eta(x) \psi(x) \Big) y^{\lambda/2} f_0(Ry) dy$$
$$= T_{0,R} \circ M_{\lambda/2}(\eta\,\psi) + \mathcal{Z}$$

with $||\mathcal{Z}||_{H^\sigma} \leq C||\eta\,\psi||_{H^\sigma}$.
Using now the fact that the operator $T_{0,R}$ is the multiplier by a function bounded by $|\xi|^{1/2}$ and $M_{\lambda/2}\, h$ is the product of h by $x^{\lambda/2}$ which is a smooth function in the interval $(1/2, /2)$ the result follows. The same argument yields the estimate for $\mathcal{W}_\infty$:

$$||\mathcal{W}_\infty(\eta\,\psi)||_{H^{\sigma-1/2}(\mathbb{R})} \leq C||\eta\,\psi||_{H^\sigma}. \tag{6.23}$$

The third operator $\mathcal{W}_{\infty,\varepsilon}$ may be written as a pseudo differential operator with symbol

$$P_\varepsilon(x,k) = \int_0^\infty \frac{(e^{-iky} - 1)}{y^{3/2} + \varepsilon^{3/2} x^{3/2}} dy. \tag{6.24}$$

Therefore,

$$||\eta \cdot (\mathcal{W}_{\infty,\varepsilon} - \mathcal{W}_\infty)(\eta\, h)||^2_{H^{\sigma-1/2}(\mathbb{R})} = \int_{\mathbb{R}} d\tilde{k} (1+|\tilde{k}|^2)^{\sigma-1/2} \int_{\mathbb{R}} dk_1\, \overline{\widehat{\psi}(k_1)} \times$$
$$\times \int_{\mathbb{R}} dk_2\, \widehat{\psi}(k_2)\, \mathcal{Z}_\varepsilon(k_1, k_2, \tilde{k})$$

where

$$\mathcal{Z}_\varepsilon(k_1,k_2,\tilde{k}) = \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dx_2 \overline{[P_\varepsilon(x_1,k_1) - P_0(x_1,k_1)]} \times$$
$$\times [P_\varepsilon(x_2,k_2) - P_0(x_2,k_2)]\, e^{-i\,(k_1-\tilde{k})\,x_1} e^{-i\,(k_2-\tilde{k})\,x_2} \eta(x_1)\eta(x_2)$$

We now show:

$$\left|\mathcal{Z}_\varepsilon(k_1,k_2,\tilde{k})\right| \le C_m \frac{|k_1|^{1/2}|k_2|^{1/2}}{(1+|\tilde{k}-k_1|^m)(1+|\tilde{k}-k_1|^m)}. \tag{6.25}$$

To this end we notice that we may write:

$$P_\varepsilon(x,k) - P_0(x,k) \;=\; \int_0^\infty dy\left(e^{-iky}-1\right)\frac{\varepsilon^{3/2}x^{3/2}}{y^{3/2}(y^{3/2}+\varepsilon^{3/2}x^{3/2})}$$

whence,

$$\begin{aligned}\int_{\mathbb{R}} e^{-i(k-\tilde{k})x}\left(P_\varepsilon(x,k)-P_0(x,k)\right)\eta(x)dx &= \int_{\mathbb{R}} e^{-i(k-\tilde{k})x}\int_0^\infty \frac{e^{-iky}-1}{y^{3/2}} R\left(\frac{y}{\varepsilon x}\right)\eta(x)dx\,dy \\ R(\xi) &= \frac{1}{\xi^{3/2}+1}\end{aligned} \tag{6.26}$$

For $|k-\tilde{k}| \le 1$ we immediately obtain from (6.26) that, for some positive constant C independent of ε:

$$\left|\int_{\mathbb{R}} e^{-i(k-\tilde{k})x}\left(P_\varepsilon(x,k)-P_0(x,k)\right)\eta(x)dx\right| \le C \tag{6.27}$$

On the other hand, using $e^{-i(k-\tilde{k})x} = \frac{i}{k-\tilde{k}}\frac{\partial}{\partial x}\left(e^{-i(k-\tilde{k})x}\right)$ and integrating by parts m times in the right hand side of (6.26) we obtain that for any $m \in \mathbb{N}$ there exists a positive constant C_m such that

$$\left|\int_0^\infty e^{-i(k-\tilde{k})x} R\left(\frac{y}{\varepsilon x}\right)\eta(x)dx\right| \le \frac{C_m}{1+|k-\tilde{k}|^m}. \tag{6.28}$$

In the derivation of (6.28) we have used $\dfrac{\partial}{\partial x}R\left(\dfrac{y}{\varepsilon x}\right) = -\dfrac{1}{x}\,\xi\, R'\left(\xi\right)$ with $\xi = \left(\dfrac{y}{\varepsilon x}\right)$ and the fact that the function $\xi\, R'\left(\xi\right)$ has the same structure than $R(\xi)$. Namely, it is a rational function of $\xi^{3/2}$ decreasing as $\xi \to \infty$ like $\xi^{-3/2}$. This is also true for all the derivatives of higher order. Moreover, since $\mathrm{supp}(\eta) \subset (1/2, 2)$, the term $\eta(x)/x$ is uniformly bounded in $\mathbb{R}$.

Define now the function

$$M(y,k-\tilde{k}) = \frac{1}{y^{3/2}}\int_0^\infty e^{-i(k-\tilde{k})x} R\left(\frac{y}{\varepsilon x}\right)\eta(x)dx$$

An integration by parts yields:

$$\int_0^\infty (e^{-iky}-1)M(y,k-\tilde{k})dy = -ik\int_0^\infty e^{-iky}\int_y^\infty M(\sigma,k-\tilde{k})\,d\sigma dy. \tag{6.29}$$

This identity still holds in the straight lines Γ of the complex plane defined by

$$|Im(y) = \varepsilon_0|Re(y)|, \quad sign(Im(y)) = -sign(k)$$

Using then (6.28) we obtain:

$$\left|ik\int_\Gamma e^{-iky}\int_y^\infty M(\sigma,k-\tilde{k})\,d\sigma dy\right|\le\frac{C_m|k|}{1+|k-\tilde{k}|^m}\int_\Gamma\frac{e^{-ky}}{|y|^{1/2}}dy\le\frac{C'_m|k|^{1/2}}{1+|k-\tilde{k}|^m}. \tag{6.30}$$

Using (6.30) twice, estimate (6.25) follows. Therefore,

$$\begin{aligned}&\left|\int_{\mathbb{R}}d\tilde{k}(1+|\tilde{k}|^2)^{\sigma-1/2}\int_{\mathbb{R}}dk_1\,\overline{\widehat{\psi}(k_1)}\int_{\mathbb{R}}dk_2\,\widehat{\psi}(k_2)\,\mathcal{Z}_\varepsilon(k_1,k_2,\tilde{k})\right| \qquad (6.31)\\ &\le\int_{\mathbb{R}}d\tilde{k}(1+|\tilde{k}|^2)^{\sigma-1/2}\int_{\mathbb{R}}dk_1\,\overline{\widehat{\psi}(k_1)}\int_{\mathbb{R}}dk_2\,\widehat{\psi}(k_2)\,\frac{|k_1|^{1/2}|k_2|^{1/2}}{(1+|k_1-\tilde{k}|)^m\,(1+|k_2-\tilde{k}|)^m}.\end{aligned}$$

Using that $|\tilde{k}|\le|k_1|+|\tilde{k}-k_1|$ we have:

$$\int_{\mathbb{R}}\frac{(1+|\tilde{k}|^2)^{\sigma-1/2}d\tilde{k}}{(1+|k_1-\tilde{k}|)^m\,(1+|k_2-\tilde{k}|)^m}\le C\frac{(1+|k_1|)^{2\sigma-1}}{(1+|k_2-k_1|)^{m'}} \tag{6.32}$$

for some $m'<m$. Using (6.32) in (6.31) and Cauchy-Schwartz's inequality we obtain:

$$\begin{aligned}&\left|\int_{\mathbb{R}}d\tilde{k}(1+|\tilde{k}|^2)^{\sigma-1/2}\int_{\mathbb{R}}dk_1\,\overline{\widehat{\psi}(k_1)}\int_{\mathbb{R}}dk_2\,\widehat{\psi}(k_2)\,\mathcal{Z}_\varepsilon(k_1,k_2,\tilde{k})\right|\le\\ &\le||\psi||_{H^\sigma(\mathbb{R})}\int_{\mathbb{R}}dk_1\left|\int_{\mathbb{R}}dk_2\frac{|k_2|^{1/2}(1+|k_1|)^{\sigma-1/2}|\widehat{\psi}(k_2)|}{(1+|k_1-k_2|)^{m'}}\right|^2\\ &\le||\psi||_{H^\sigma(\mathbb{R})}\int_{\mathbb{R}}dk_1\left|\int_{\mathbb{R}}dk_2\frac{(1+|k_2|)^\sigma|\widehat{\psi}(k_2)|}{(1+|k_1-k_2|)^{m''}}\right|^2 \qquad (6.33)\end{aligned}$$

for some $m''<m'$. Young's inequality then implies:

$$\left|\int_{\mathbb{R}}d\tilde{k}(1+|\tilde{k}|^2)^{\sigma-1/2}\int_{\mathbb{R}}dk_1\,\overline{\widehat{\psi}(k_1)}\int_{\mathbb{R}}dk_2\,\widehat{\psi}(k_2)\,\mathcal{Z}_\varepsilon(k_1,k_2,\tilde{k})\right|\le C||\psi||^2_{H^\sigma(\mathbb{R})}. \tag{6.34}$$

and therefore

$$||\eta\cdot(\mathcal{W}_{\infty,\varepsilon}-\mathcal{W}_\infty)\,(\eta\,h)||_{H^{\sigma-1/2}(\mathbb{R})}\le C||\psi||_{H^\sigma(\mathbb{R})}. \tag{6.35}$$

Combining (6.23) and (6.35) we obtain the estimate for $\mathcal{W}_{\infty,\varepsilon}(\eta\,h)$ in (6.21).

It remains to prove that (6.22) holds true. By the estimate (6.25) in $\mathcal{Z}_\varepsilon(k_1,k_2,\tilde{k})$ this is reduced to prove that for any k_1, k_2 and $\tilde{k}$, $\mathcal{Z}_\varepsilon(k_1,k_2,\tilde{k})\to 0$ as $\varepsilon\to 0$. This follows from the fact that the support of η is compact and that $P_\varepsilon(x,k)\to P_0(x,k)$ as $\varepsilon\to 0$ as it follows from the explicit expressions (6.24) .

If $y=\varepsilon\,t$, we obtain,

$$P_\varepsilon(x,k)=\frac{1}{\sqrt{\varepsilon}}\int_0^\infty\frac{(e^{-ik\varepsilon t}-1)}{t^{3/2}+x^{3/2}}dt.$$

Therefore it follows that, for some positive constant C independent of $\varepsilon>0$ and $x>0$:

$$|P_\varepsilon(x,k)|\le C|k|^{1/2},\ \ \forall\varepsilon>0,\ \forall x>0$$

and (6.22) follows. □

We have the following estimate for $A_{2,\varepsilon}$ in (6.4).

Lemma 6.5 *Suppose that f_0 satisfies conditions (2.1), (2.2) and $\sigma > 1+\delta$, then*

$$\sup_{t_0\in(0,T)}\sup_{R>1} R^{2+\delta} N_\infty(A_{2,\varepsilon};\, t_0, R) \leq C|||\varphi|||, \tag{6.36}$$

$$\sup_{t_0\in(0,T)}\sup_{R>1} R^2\, N_{2;\,\sigma-\frac{1}{2}}(A_{2,\varepsilon};\, t_0, R) \leq C|||\varphi|||, \tag{6.37}$$

$$\sup_{0<R<1} R^{2-\lambda/2} M_\infty(A_{2,\varepsilon};\, R) \leq C|||\varphi|||, \tag{6.38}$$

$$\sup_{0<R<1} R^{2-\lambda/2} M_{2;\,\sigma-\frac{1}{2}}(A_{2,\varepsilon};\, R) \leq C|||\varphi|||, \tag{6.39}$$

where the functions $N_\infty(\cdot\,;\, t_0, R)$, $N_{2;\,\sigma}(\cdot\,;\, t_0, R)$, $M_\infty(\cdot\,;\, R)$ and $M_{2;\sigma}(\cdot\,;\, R)$ are defined in (2.9) - (2.12).

Proof of Lemma 6.5. For $R > 1$ we write

$$\varphi(x,t) = \sum_{n=0}^{\infty} \chi(x/2^n)\, \varphi(x,t)$$

where $\chi \in C_0^\infty$, supp$\chi \subset (1/2, 2)$

$$|A_{2,\,\varepsilon}(x,t)| \leq \sum_{n=n_0-2}^{n_0+1} \int_0^{x/2} \left|(x-y)^{\lambda/2}\varphi(x-y,t)\chi\left(\frac{x-y}{2^n}\right) - x^{\lambda/2}\varphi(x,t)\chi\left(\frac{x}{2^n}\right)\right| y^{-3/2-\delta} dy.$$

Let us consider $R = 2^{n_0}$, $x \in (R/2, 2R)$ and rescale $x = RX$, $y = RY$, $\tau = (t-t_0)R^{(\lambda-1)/2}$, $\varphi(x,t) = R^{-(3+\lambda)/2)}\psi(X,\tau)$, $\mathcal{A}_{2,\varepsilon}(X,\tau) = A_{2,\varepsilon}(x,t)$ to obtain:

$$|\mathcal{A}_{2,\varepsilon}(X,\tau)| \leq R^{-2-\delta} \times$$
$$\times \sum_{\ell=-2}^{1} \int_0^{X/2} \left|(X-Y)^{\lambda/2}\psi(X-Y,\tau)\chi\left(\frac{X-Y}{2^\ell}\right) - X^{\lambda/2}\psi(X,\tau)\chi\left(\frac{X}{2^\ell}\right)\right| Y^{-3/2-\delta} dY.$$

Using Lemma 6.3, we deduce, for $X \in (3/4, 5/4)$:

$$|\mathcal{A}_{2,\varepsilon}(X,\tau)| \leq R^{-(2+\delta)}\, C\, \left(||\psi(\tau)||_{L^\infty(1/8,8)} + ||\psi(\tau)||_{H^\sigma(1/8,8)}\right)$$

whence:

$$\begin{aligned} &R^{2+\delta}\left(\int_0^{\min(1,\, R^{(\lambda-1)/2}(T-t_0))} ||\mathcal{A}_{2,\varepsilon}(t)||^2_{L^\infty(3/4,5/4)} d\tau\right)^{1/2} \\ &\leq C \sup_{0\leq\tau\leq\min(1,\, R^{(\lambda-1)/2}(T-t_0))} ||\psi(\tau)||_{L^\infty(1/8,8)} + \\ &+C\left(\int_0^{\min(1,\, R^{(\lambda-1)/2}(T-t_0))} ||\psi(\tau)||^2_{H^\sigma(1/8,8)} d\tau\right)^{1/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} R^{2+\delta} N_\infty(A_{2,\varepsilon}; t_0, R) \;\;\leq\;\; & C\, R^{(3+\lambda)/2}\left[\sup_{t_0\leq t\leq\min(t_0+R^{-(\lambda-1)/2},T)} ||\varphi(t)||_{L^\infty(R/8,8R)} + \right. \\ & \left. + \sum_{\ell=-3}^{3} N_{2;\sigma}(\varphi; t_0, 2^\ell R)\right] \leq C|||\varphi|||, \end{aligned}$$

and (6.36) follows.

We now prove (6.37). To this end notice that:

$$\mathcal{A}_{2,\varepsilon}(X,\tau) = R^{-2} \sum_{\ell=-2}^{1} \int_0^{X/2}$$

$$\left((X-Y)^{\lambda/2} \psi(X-Y,\tau)\chi\left(\frac{X-Y}{2^\ell}\right) - X^{\lambda/2}\psi(X,\tau)\chi\left(\frac{X}{2^\ell}\right)\right) \mathcal{H}_\varepsilon(X,Y)dY$$

where $\mathcal{H}_\varepsilon(X,Y) = R^{3/2} H_\varepsilon(x,y)$. The integrals in the right hand side of this formula can be written in term of the operators $\mathcal{W}_R,\ \mathcal{W}_{\infty,\varepsilon}$ in (6.18), (6.20). Using Lemma 6.4 we obtain:

$$R^2 \left(\int_0^{\min(1,\, R^{(\lambda-1)/2}(T-t_0))} ||\mathcal{A}_{2,\varepsilon}(\tau)||^2_{H^{\sigma-1/2}(3/4,5/4)} d\tau \right)^{1/2} \leq$$

$$\leq C \left(\int_0^{\min(1,\, R^{(\lambda-1)/2}(T-t_0))} ||\psi(\tau)||^2_{H^\sigma(1/8,8)} d\tau \right)^{1/2} +$$

$$+C \sup_{0\leq\tau\leq\min(1,\, R^{(\lambda-1)/2}(T-t_0))} ||\psi(\tau)||_{L^\infty(1/8,8)}.$$

Therefore

$$R^{\sigma+1} N_{2;\,\sigma-\frac{1}{2}}(\mathcal{A}_{2,\varepsilon};t_0,R) \quad \leq \quad C\, R^{(3+\lambda)/2} \left[\sum_{\ell=-3}^{3} N_{2;\sigma}(\varphi;t_0,2^\ell R) + \right.$$

$$\left. + \sup_{t_0\leq t\leq \min(t_0+R^{-(\lambda-1)/2},T)} ||\varphi(t)||_{L^\infty(R/8,8R)} \right].$$

whence (6.37) follows.

We consider now the case where $0<R\leq 1$. The arguments are very similar to those used in the previous case. In order to prove (6.37) we write $\varphi(x,t) = \sum_{n=0}^{\infty} \chi(2^n\, x)\, \varphi(x,t)$ where $\chi \in C_0^\infty$ and $\text{supp}\chi \subset (1/2,2)$. Then

$$|\mathcal{A}_{2,\varepsilon}(x,t)| \quad \leq \quad C \sum_{n=n_0-2}^{n_0+1} \int_0^{x/2} \left| (x-y)^{\lambda/2}\varphi(x-y,t)\chi\left(2^n(x-y)\right) - x^{\lambda/2}\varphi(x,t)\chi\left(2^n\, x\right) \right| y^{-3/2} dy.$$

Let us consider $R=2^{n_0}$, $x\in(R/2,2R)$ and rescale $x=RX$, $y=RY$, $\varphi(x,t) = R^{-3/2}\psi(X,t)$, $\mathcal{A}_{2,\varepsilon}(X,t) = A_{2,\varepsilon}(x,t)$ to obtain:

$$|\mathcal{A}_{2,\varepsilon}(X,t)| \quad \leq \quad R^{\lambda/2-2} \sum_{\ell=-2}^{1} \int_0^{X/2}$$

$$\left| (X-Y)^{\lambda/2}\psi(X-Y,t)\chi\left(2^\ell(X-Y)\right) - X^{\lambda/2}\psi(X,t)\chi\left(2^\ell\, X\right) \right| Y^{-3/2} dY.$$

Using Lemma 6.3 we deduce that for $X\in(3/4,5/4)$:

$$|\mathcal{A}_{2,\varepsilon}(X,t)| \leq R^{-2+\frac{\lambda}{2}}\, C\, \left(||\psi(t)||_{L^\infty(1/8,8)} + ||\psi(t)||_{H^\sigma(1/8,8)} \right)$$

whence,

$$R^{2-\lambda/2} \left(\int_0^T ||\mathcal{A}_{2,\,\varepsilon}(t)||^2_{L^\infty(3/4,5/4)} dt \right)^{1/2} \leq C \sup_{0\leq t\leq T} ||\psi(t)||_{L^\infty(1/8,8)} +$$

$$+C \left(\int_0^T ||\psi(t)||^2_{H^\sigma(1/8,8)} dt \right)^{1/2}.$$

Therefore

$$R^{2-\lambda/2} M_\infty(\mathcal{A}_{2,\varepsilon}, R) \le C\, R^{3/2} \left[\sup_{0\le t\le T} ||\varphi(t)||_{L^\infty(R/8,8R)} + \sum_{\ell=-3}^{3} M_{2;\,\sigma}(\varphi, 2^\ell\, R) \right] \le C|||\varphi|||$$

and (6.38) follows.

We now prove (6.39). Since:

$$\mathcal{A}_{2,\varepsilon}(X,t) = R^{-2+\lambda/2} \sum_{\ell=-2}^{1} \int_0^{X/2}$$
$$\Big((X-Y)^{\lambda/2} \psi(X-Y,t) \chi\left(2^\ell (X-Y)\right) - X^{\lambda/2} \psi(X,t) \chi\left(2^\ell\, X\right) \Big) \mathcal{H}_\varepsilon(X,Y) dY$$

where $\mathcal{H}_\varepsilon(X,Y) = R^{3/2} H_\varepsilon(x,y)$. Rewriting the integrals in the right hand side of this formula in term of the operators $\mathcal{W}_R,\ \mathcal{W}_{\infty,\varepsilon}$ and using Lemma 6.4 we obtain:

$$\begin{aligned} R^{2-\lambda/2} \left(\int_0^T ||\mathcal{A}_{2,\varepsilon}(t)||^2_{H^{\sigma-1/2}(3/4,5/4)} dt \right)^{1/2} &\le C \left(\int_0^T ||\psi(t)||^2_{H^\sigma(1/8,8)} dt \right)^{1/2} \\ &\quad + C \sup_{0\le t\le T} ||\psi(t)||_{L^\infty(1/8,8)}. \end{aligned}$$

Therefore

$$R^{2-\lambda/2} M_{2;\,\sigma-\frac{1}{2}}(\mathcal{A}_{2,\varepsilon}; R) \le C\, R^{3/2} \left[\sum_{\ell=-3}^{3} M_{2;\,\sigma}(\varphi; 2^\ell\, R) + \sup_{0\le t\le T} ||\varphi(t)||_{L^\infty(R/8,8R)} \right].$$

whence (6.39) follows. □

7 Regularity for some auxiliary evolution equations

The following result has been proved in [4]:

Proposition 7.1 *The fundamental solution $g(t,x,x_0)$ of the operator L defined in (1.3) such that $g(0,x,x_0) = \delta(x-x_0)$ satisfies:*

$$g(t,x,x_0) = \frac{1}{x_0} g\left(t\, x_0^{(\lambda-1)/2}, \frac{x}{x_0}, 1 \right) \tag{7.1}$$
$$|g(t,x,1)| \le C\, t\, x^{-3/2}, \quad \textit{for all } 0 \le t \le 1,\ 0 < x \le 1/2, \tag{7.2}$$
$$|g(t,x,1)| \le C\, t\, x^{-(3+\lambda)/2}, \quad \textit{for all } 0 \le t \le 1,\ x \ge 3/2, \tag{7.3}$$
$$|g(t,x,1)| \le C\, t^{-2}\, \Phi\left(\frac{x-1}{t^2} \right), \quad \textit{for all } 0 \le t \le 1,\ 1/2 \le x \le 3/2, \tag{7.4}$$

where,

$$\Phi(\xi) = \frac{1}{1+|\xi|^{3/2-\delta}}. \tag{7.5}$$

Moreover

$$g(t,x,x_0) \le C t^{2/(\lambda-1)}\, \sigma^{-3/2}, \quad \textit{for all } t \ge 1,\ 0 < \sigma \le 1, \tag{7.6}$$
$$|g(t,x,1)| \le C\, t^{2/(\lambda-1)}\, \sigma^{-(3+\lambda)/2}, \quad \textit{for all } t \ge 1,\ \sigma \ge 1, \tag{7.7}$$

with

$$\sigma = t^{2/(\lambda-1)}\, x. \tag{7.8}$$

Lemma 7.2 *For $T \in (0,1]$ there is a constant $C > 0$ such that, for all $||\nu||_{X_{3/2,\,2+\delta}(T)} < \infty$:*

$$\sup_{0\le t\le T} \left|\left|\left| \int_0^t G(t-s)\,\nu(s)ds \right|\right|\right|_{3/2,\,\frac{3+\lambda}{2}} \le CT^{\beta}||\nu||_{X_{3/2,\,2+\delta}(T)}$$

where

$$\beta = \min\left(1, \frac{2\delta}{\lambda-1}\right). \tag{7.9}$$

Proof of Lemma 7.2. We assume first that

$$R^{-(\lambda-1)/2} \le t. \tag{7.10}$$

Let us suppose that

$$x \in \left(\frac{3R}{4}, \frac{5R}{4}\right). \tag{7.11}$$

Using Proposition 7.1:

$$\begin{aligned}
&\int_0^t G(t-s)\nu(s,y)\,ds = \int_0^t ds \int_0^\infty dy\nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} \\
&\le \int_{t-R^{-\frac{\lambda-1}{2}}}^t ds \int_{|x-y|\le R/2} \nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} \\
&+ \int_0^{t-R^{-\frac{\lambda-1}{2}}} ds \int_{|x-y|\le R/2} \nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} \\
&+ \int_0^t ds \int_{|y|\le R/2} \nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} \\
&+ \int_0^t ds \int_{|y|\ge 2R} \nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} \\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.
\end{aligned} \tag{7.12}$$

To estimate $\mathcal{I}_1$ we use the fact that (7.4) implies:

$$g(s,z) \le \frac{C}{s^2}\Phi\left(\frac{z-1}{s^2}\right) \tag{7.13}$$

for $0 \le s \le 1$, $z \in (1/2, 3/2)$.

$$\begin{aligned}
|\mathcal{I}_1| &\le C \int_{t-R^{-\frac{\lambda-1}{2}}}^t ds \int_{|x-y|\le R/2} \frac{\nu(s,y)}{((t-s)\,y^{\frac{\lambda-1}{2}})^2}\Phi\left(\frac{\frac{x}{y}-1}{(t-s)^2\,y^{\lambda-1}}\right)\frac{d\,y}{y} \\
&\le C \int_{t-R^{-\frac{\lambda-1}{2}}}^t ||\nu(s)||_{L^\infty(R/2,\,2R)} ds.
\end{aligned}$$

where we have used (7.10) in the last inequality. Using Hölder's inequality we deduce:

$$|\mathcal{I}_1| \le C\,R^{-(\lambda-1)/2} N_\infty(\nu; t_0, R)$$

whence:

$$|\mathcal{I}_1| \le C\,R^{-(3+\lambda)/2}\,R^{-\delta}\left[R^{2+\delta}\,N_\infty(\nu;t_0,R)\right] \le C\,R^{-(3+\lambda)/2}\,t^{2\delta/(\lambda-1)}|||\nu|||_{X_{3/2,\,2+\delta}}. \tag{7.14}$$

We consider now the term $\mathcal{I}_2$:

$$\mathcal{I}_2 = \int_0^{t-R^{-\frac{\lambda-1}{2}}} ds \int_{|x-y|\leq R/2} \nu(y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y}.$$

In the region of integration we have $(t-s)y^{\frac{\lambda-1}{2}} \geq 1$. Using then (7.7) we deduce

$$\left|g\left((t-s)y^{(\lambda-1)/2}, \frac{x}{y}\right)\right| \leq C\,(t-s)^{-\frac{\lambda+1}{\lambda-1}} \frac{y}{x^{(3+\lambda)/2}} \tag{7.15}$$

for $s \geq 1$ and $1/7 \leq |z| \leq 7$. Therefore:

$$\begin{aligned}
|\mathcal{I}_2| &\leq R^{-(3+\lambda)/2} \int_{R^{-\frac{\lambda-1}{2}}}^{t} ||\nu(t-s)||_{L^\infty(R/2,2R)}\, s^{-\frac{\lambda+1}{\lambda-1}} ds \\
&\leq CR^{-(3+\lambda)/2} \sum_{n=1}^{\left[t\,R^{(\lambda-1)/2}\right]} R^{(\lambda+1)/2} n^{-(\lambda+1)/(\lambda-1)} R^{-(\lambda-1)/2} N_\infty(\nu; nR^{-(\lambda-1)/2}, R) \\
&\leq CR^{-(3+\lambda)/2}\, R^{-1-\delta} |||\nu|||_{X_{3/2,\,2+\delta}} \leq CR^{-(3+\lambda)/2}\, t^{2(1+\delta)/(\lambda-1)} |||\nu|||_{X_{3/2,\,2+\delta}}, \qquad (7.16)
\end{aligned}$$

where we have used (7.10) in the last step.

We next consider the term $\mathcal{I}_3$.

$$\begin{aligned}
\mathcal{I}_3 &= \int_0^t ds \int_{|y|\leq R/2} \nu(y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} = \qquad (7.17)\\
&= \int_0^t ds \int_0^{t^{-2/(\lambda-1)}} \nu(y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} + \\
&\qquad + \int_0^t ds \int_{t^{-2/(\lambda-1)}}^{R/2} \nu(y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} = \mathcal{I}_{3,1} + \mathcal{I}_{3,2}. \qquad (7.18)
\end{aligned}$$

We can use (7.3) in the region of integration of $\mathcal{I}_{3,1}$. Therefore:

$$\left|g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\right| \leq C(t-s)\, x^{-(3+\lambda)/2}\, y^{\lambda+1}. \tag{7.19}$$

Then:

$$\begin{aligned}
|\mathcal{I}_{3,1}| &\leq Cx^{-(3+\lambda)/2} \int_0^t ds(t-s) \int_0^{t^{-2/(\lambda-1)}} |\nu(y)| y^\lambda dy \\
&= C\,x^{-(3+\lambda)/2} \int_0^t ds(t-s) \left(\int_0^1 |\nu(y)|y^\lambda dy + \int_1^{t^{-2/(\lambda-1)}} |\nu(y)| y^\lambda dy\right) \\
&= \mathcal{I}_{3,1,1} + \mathcal{I}_{3,1,2}. \qquad (7.20)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{3,1,1} &\leq C\,x^{-(3+\lambda)/2} \sum_{n=0}^\infty 2^{-n(\lambda+1)} \int_0^t ds(t-s) ||\nu(s)||_{L^\infty(2^{-(n+1)},\,2^{-n})} \\
&\leq Cx^{-(3+\lambda)/2} t^{3/2} \sum_{n=0}^\infty 2^{-n(\lambda+1)} M_\infty(\nu; 2^{-n}) \\
&\leq Cx^{-(3+\lambda)/2} t^{3/2} |||\nu|||_{X_{3/2,\,2+\delta}} \sum_{n=0}^\infty 2^{-n(\lambda-1/2)} \qquad (7.21)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{3,1,2} &\leq C\, x^{-(3+\lambda)/2}\, t \sum_{0\leq 2^n\leq t^{-2/(\lambda-1)}} \sum_{\ell=1}^{\left[t\,(2^n)^{(\lambda-1)/2}\right]} \int_{2^{-n(\lambda-1)/2}\,\ell}^{\min\{2^{-n(\lambda-1)/2}\,(\ell+1),t\}} ||\nu(s)||_{L^\infty(2^n,\,2^{n+1})} ds \\
&\qquad \times \int_{2^n}^{2^{n+1}} y^\lambda\, dy \\
&\leq C\, x^{-(3+\lambda)/2}\, t \sum_{0\leq 2^n\leq t^{-2/(\lambda-1)}} \sum_{\ell=0}^{\left[t\,(2^n)^{(\lambda-1)/2}\right]} \int_{2^{-n(\lambda-1)/2}\,\ell}^{\min\{2^{-n(\lambda-1)/2}\,(\ell+1),t\}} ||\nu(s)||_{L^\infty(2^n,\,2^{n+1})} ds \times \\
&\qquad \times 2^{n(\lambda+1)} \\
&\leq C\, x^{-(3+\lambda)/2}\, t \sum_{0\leq 2^n\leq t^{-2/(\lambda-1)}} \sum_{\ell=0}^{\left[t\,(2^n)^{(\lambda-1)/2}\right]} 2^{-n(\lambda-1)/2} \times N_\infty(\nu; 2^{-n(\lambda-1)/2}\ell, 2^n) ds\, 2^{n(\lambda+1)} \\
&\leq C\, x^{-(3+\lambda)/2}\, t^2\, |||\nu|||_{X_{3/2,2+\delta}} (t^{-2/(\lambda-1)})^{\lambda-1-\delta} = C\, \frac{t^{2\delta/(\lambda-1)}}{x^{(3+\lambda)/2}}\, |||\nu|||_{X_{3/2,2+\delta}}. \qquad (7.22)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\mathcal{I}_{3,2} &= \int_0^t ds \int_{t^{-2/(\lambda-1)}}^{R/2} \nu(y,s) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right) \frac{dy}{y} \\
&= \int_{t^{-2/(\lambda-1)}}^{R/2} \frac{dy}{y} \int_0^{t-y^{-(\lambda-1)/2}} \nu(y,s) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right) ds + \\
&+ \int_{t^{-2/(\lambda-1)}}^{R/2} \frac{dy}{y} \int_{t-y^{-(\lambda-1)/2}}^{t} \nu(y,s) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right) ds \\
&= \mathcal{I}_{3,2,1} + \mathcal{I}_{3,2,2}. \qquad (7.23)
\end{aligned}$$

In the term $\mathcal{I}_{3,2,1}$ we use (7.15) that gives

$$\begin{aligned}
|\mathcal{I}_{3,2,1}| &\leq x^{-(3+\lambda)/2} \int_{t^{-2/(\lambda-1)}}^{R/2} dy \int_0^{t-y^{-(\lambda-1)/2}} (t-s)^{-(\lambda+1)/(\lambda-1)} |\nu(y,s)| ds \\
&\leq C\, x^{-(3+\lambda)/2} \sum_{0\leq 2^n\leq t^{-2/(\lambda-1)}} \sum_{\ell=1}^{\left[t\,(2^n)^{(\lambda-1)/2}\right]} \int_{2^n}^{2^{n+1}} dy \int_{2^{-n(\lambda-1)/2}\,\ell}^{2^{-n(\lambda-1)/2}\,(\ell+1)} ds \\
&\qquad s^{-(\lambda+1)/(\lambda-1)} |\nu(y,t-s)| \\
&\leq C\, x^{-(3+\lambda)/2}\, |||\nu|||_{X_{3/2,2+\delta}} \sum_{0\leq 2^n\leq t^{-2/(\lambda-1)}} (2^n)^{-\delta} \sum_{\ell=1}^{\left[t\,(2^n)^{(\lambda-1)/2}\right]} \ell^{-(\lambda+1)/(\lambda-1)} \\
&= C\, x^{-(3+\lambda)/2}\, |||\nu|||_{X_{3/2,2+\delta}} t^{2\delta/(\lambda-1)} \qquad (7.24)
\end{aligned}$$

In the term $\mathcal{I}_{3,2,2}$, we use (7.19) which gives:

$$\begin{aligned}
|\mathcal{I}_{3,2,2}| &\leq x^{-(3+\lambda)/2}\int_{t^{-2/(\lambda-1)}}^{R/2} y^{\lambda+1}\frac{dy}{y}\int_{t-y^{-(\lambda-1)/2}}^{t}(t-s)|\nu(y,s)|ds \\
&\leq Cx^{-(3+\lambda)/2}\sum_{t^{-2/(\lambda-1)}\leq 2^n\leq R/2}\int_{2^n}^{2^{n+1}} y^{\lambda+1}\frac{dy}{y}\int_{t-2^{-n(\lambda-1)/2}}^{t}(t-s)|\nu(y,s)|ds \\
&\leq Cx^{-(3+\lambda)/2}|||\nu|||_{X_{3/2,2+\delta}}\sum_{t^{-2/(\lambda-1)}\leq 2^n\leq R/2} 2^{-n\delta} \\
&\leq Cx^{-(3+\lambda)/2}|||\nu|||_{X_{3/2,2+\delta}}t^{2\delta/(\lambda-1)}. \qquad (7.25)
\end{aligned}$$

Estimates (7.24) and (7.25) yield

$$|\mathcal{I}_{3,2}| \leq Cx^{-(3+\lambda)/2}\, t^{\frac{2\delta}{\lambda-1}}\, |||\nu|||_{X_{3/2,\,2+\delta}}. \qquad (7.26)$$

Then, using also (7.18) and (7.20), we deduce that

$$|\mathcal{I}_{3}| \leq Cx^{-(3+\lambda)/2}\, t^{\frac{2\delta}{\lambda-1}}\, |||\nu|||_{X_{3/2,\,2+\delta}}. \qquad (7.27)$$

We estimate now the term $\mathcal{I}_4$. To this end we have:

$$\begin{aligned}
\int_0^t ds\int_{y\geq 2R}\nu(y)g\left((t-s)y^{\frac{\lambda-1}{2}},\frac{x}{y}\right)\frac{dy}{y} &\leq \int_{y\geq 2R}\frac{dy}{y}\int_0^{t-y^{-(\lambda-1)/2}}[\cdots]ds + \\
&+\int_{y\geq 2R}\frac{dy}{y}\int_{t-y^{-(\lambda-1)/2}}^{t}[\cdots]ds = \mathcal{I}_{4,1}+\mathcal{I}_{4,2}.
\end{aligned}$$

We split $\mathcal{I}_{4,1}$ in two pieces as follows:

$$\begin{aligned}
\mathcal{I}_{4,1} &= \int_{y\geq 2R}\frac{dy}{y}\int_0^{t-x^{-(\lambda-1)/2}}[\cdots]ds + \int_{y\geq 2R}\frac{dy}{y}\int_{t-x^{-(\lambda-1)/2}}^{t-y^{-(\lambda-1)/2}}[\cdots]ds \\
&= \mathcal{I}_{4,1,1}+\mathcal{I}_{4,1,2}. \qquad (7.28)
\end{aligned}$$

In the term $\mathcal{I}_{4,1,1}$ we are in the region where (7.7) holds. Then, we use (7.15) to obtain:

$$\begin{aligned}
|\mathcal{I}_{4,1,1}| &\leq Cx^{-(3+\lambda)/2}\int_{y\geq 2R}dy\int_{x^{-(\lambda-1)/2}}^{t} s^{-(\lambda+1)/(\lambda-1)}|\nu(y,(t-s))|ds \\
&\leq Cx^{-(3+\lambda)/2}\sum_{2^n\geq 2R}2^{n(\lambda+1)/2}2^n\sum_{x^{-(\lambda-1)/2}\leq 2^{-n(\lambda-1)/2}\ell\leq t}\ell^{-(\lambda+1)/(\lambda-1)}\times \\
&\qquad\times 2^{-n(\lambda-1)/2}N_\infty(\nu;t-2^{-n(\lambda-1)/2}\ell,2^n) \\
&\leq Cx^{-(3+\lambda)/2}|||\nu|||_{X_{3/2,2+\delta}}\sum_{2^n\geq 2R}2^{-n\delta} \\
&\leq Cx^{-(3+\lambda)/2}|||\nu|||_{X_{3/2,2+\delta}}R^{-2\delta}\leq Cx^{-(3+\lambda)/2}|||\nu|||_{X_{3/2,2+\delta}}t^{2\delta/(\lambda-1)} \qquad (7.29)
\end{aligned}$$

In the integral $\mathcal{I}_{4,1,2}$ we use (7.6) to obtain:

$$|g\left((t-s)y^{\frac{\lambda-1}{2}},\frac{x}{y}\right)| \leq C(t-s)^{-1/(\lambda-1)}\,y\,x^{-3/2}. \qquad (7.30)$$

This yields,

$$
\begin{aligned}
|\mathcal{I}_{4,1,2}| &\leq Cx^{-3/2}\int_{y\geq 2R} dy \int_{y^{-(\lambda-1)/2}}^{x^{-(\lambda-1)/2}} s^{-1/(\lambda-1)}|\nu(y,(t-s))|ds \\
&\leq Cx^{-3/2}\sum_{2^n\geq 2R}\int_{2^n}^{2^{n+1}} dy \times \\
&\qquad\qquad \times \sum_{\substack{1\leq \ell\leq 2^{-n(\lambda-1)/2}\\ \ell\leq x^{-(\lambda-1)/2}}} \int_{2^{-n(\lambda-1)/2}\ell}^{2^{-n(\lambda-1)/2}(\ell+1)} s^{-1/(\lambda-1)}||\nu(t-s)||_{L^\infty(2^n,2^{n+1})}ds \\
&\leq Cx^{-3/2}\sum_{2^n\geq 2R} 2^{2n}2^{-n\lambda/2}N_\infty(\nu, t-2^{-n(\lambda-1)/2}\ell, 2^n) \\
&\leq Cx^{-3/2}R^{-\lambda/2}|||\nu|||_{X_{3/2,2+\delta}}\sum_{2^n\geq 2R} 2^{-n\delta} \leq Cx^{-(3+\lambda)/2}|||\nu|||_{X_{3/2,2+\delta}}t^{2\delta/(\lambda-1)} \qquad (7.31)
\end{aligned}
$$

using (7.10) in the last step.

In the term $\mathcal{I}_{4,2}$ we use (7.2) to obtain,

$$
|g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)| \leq C(t-s)\, y^{(\lambda+2)/2}\, x^{-3/2} \qquad (7.32)
$$

and then

$$
\begin{aligned}
|I_{4,2}| &\leq Cx^{-3/2}\int_{y\geq 2R} y^{(\lambda+2)/2}\frac{dy}{y}\int_{t-y^{-(\lambda-1)/2}}^{t}(t-s)|\nu(y,s)|ds \\
&\leq Cx^{-3/2}\sum_{2^n\geq 2R}\int_{2^n}^{2^{n+1}} y^{\lambda/2}\,dy\int_{t-y^{-(\lambda-1)/2}}^{t}(t-s)|\nu(y,s)|ds \\
&\leq Cx^{-3/2}\sum_{2^n\geq 2R}2^{-n(\lambda-1)}2^{n\lambda/2}2^n N_\infty(\nu, t-2^{-n(\lambda-1)/2}, 2^n) \\
&\leq Cx^{-(3+\lambda)/2}\, t^{2\delta/(\lambda-1)}\, |||\nu|||_{X_{3/2,\,2+\delta}}. \qquad (7.33)
\end{aligned}
$$

where we have used (7.10) in the last inequality.

Estimates (7.29), (7.31) and (7.33) give

$$
|I_4| \leq CR^{-(3+\lambda)/2}\, t^{2\delta/(\lambda-1)}\, |||\nu|||_{X_{3/2,\,2+\delta}}, \qquad (7.34)
$$

which, combined with (7.14), (7.16) and (7.27) yields,

$$
R^{(3+\lambda)/2}\left|\left|\int_0^t G(t-s)\,\nu(s)\,ds\right|\right|_{L^\infty(R/2,\,2R)} \leq Ct^{2\delta/(\lambda-1)}|||\nu|||_{X_{3/2,\,2+\delta}} \qquad (7.35)
$$

for $R\geq t^{-2/(\lambda-1)}$.

We assume now:

$$
1\leq R\leq t^{-2/(\lambda-1)} \qquad (7.36)
$$

Then,

$$\int_0^t G(t-s)\nu(s,y)\,ds = \int_0^t ds \int_0^\infty dy\nu(s,y)g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y}$$
$$\leq \int_0^t ds \int_{|x-y|\leq R/2} \nu(s,y)g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y}$$
$$+ \int_0^t ds \int_{y\leq 1} \nu(s,y)g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y}$$
$$+ \int_0^t ds \int_{1\leq y\leq 5R/4} \nu(s,y)g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y}$$
$$+ \int_0^t ds \int_{|y|\geq 5R/4} \nu(s,y)g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y}$$
$$= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4. \tag{7.37}$$

In the term $\mathcal{J}_1$ we use again (7.13) to obtain

$$\begin{aligned}
|\mathcal{J}_1| &\leq C \int_0^t ds \int_{|x-y|\leq R/2} \frac{\nu(s,y)}{((t-s)\,y^{\frac{\lambda-1}{2}})^2}\Phi\left(\frac{\frac{x}{y}-1}{(t-s)^2\,y^{\lambda-1}}\right)\frac{d\,y}{y} \\
&\leq C \int_0^t ds||\nu(s)||_{L^\infty(R/2,2R)} \leq C\sqrt{t}\left(\int_0^t ds||\nu(s)||^2_{L^\infty(R/2,2R)}\right)^{1/2} \\
&\leq CR^{-(3+\lambda)/2}\left[\sqrt{t}R^{(\lambda-1)/4}R^{-\delta}\right]|||\nu|||_{X_{3/2,2+\delta}} \\
&\leq CR^{-(3+\lambda)/2}t^{2\delta(\lambda-1)}|||\nu|||_{X_{3/2,2+\delta}}.
\end{aligned} \tag{7.38}$$

We estimate $\mathcal{J}_2$ using (7.19) and therefore:

$$\begin{aligned}
|\mathcal{J}_2| &\leq C\,x^{-(3+\lambda)/2}\int_0^t ds \sum_{n=0}^\infty ||\nu(s)||_{L^\infty(2^{-(n+1)},2^{-n})}(t-s)\int_{2^{-(n+1)}}^{2^{-n}} y^\lambda dy \\
&\leq C\,x^{-(3+\lambda)/2}t\sum_{n=0}^\infty 2^{-n(\lambda+1)}\sqrt{t}M_\infty(\nu;0,2^{-n}), \\
&\leq C\,x^{-(3+\lambda)/2}t^{3/2}|||\nu|||_{X_{3/2,2+\delta}}\sum_{n=0}^\infty 2^{-n(\lambda-1/2)}.
\end{aligned} \tag{7.39}$$

We use also (7.19) in $\mathcal{J}_3$. Then:

$$\begin{aligned}
|\mathcal{J}_3| &\leq C\,x^{-(3+\lambda)/2}\int_0^t ds(t-s)\int_1^{5R/4} y^\lambda|\nu(y,s)|dy \\
&\leq C\,x^{-(3+\lambda)/2}\sum_{1\leq 2^n\leq 5R/4}\int_0^t ds(t-s)||\nu(s)||_{L^\infty(2^n,2^{n+1})}\int_{2^n}^{2^{n+1}} y^\lambda dy \\
&\leq C\,x^{-(3+\lambda)/2}t^{3/2}\sum_{1\leq 2^n\leq 5R/4} 2^{n(\lambda+1)}2^{-n(\lambda-1)/4}N_\infty(\nu,0,2^n) \\
&\leq C\,x^{-(3+\lambda)/2}t^{3/2}|||\nu|||_{X_{3/2,2+\delta}}R^{\frac{3}{4}(\lambda-1)-\delta} \leq C\,x^{-(3+\lambda)/2}|||\nu|||_{X_{3/2,2+\delta}}t^{\frac{2\delta}{\lambda-1}}.
\end{aligned} \tag{7.40}$$

In the term $\mathcal{J}_4$,

$$\mathcal{J}_4 = \int_{|y|\geq 5R/4}\frac{dy}{y}\int_0^{\left(t-y^{-(\lambda-1)/2}\right)_+} ds\,\nu(s,y)g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right) +$$
$$\int_{|y|\geq 5R/4}\frac{dy}{y}\int_{\left(t-y^{-(\lambda-1)/2}\right)_+}^t ds\,\nu(s,y)g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right) = \mathcal{J}_{4,1} + \mathcal{J}_{4,2}. \tag{7.41}$$

In the first term at the right hand side of (7.41) we are in the region where (7.6) holds and then we have (7.30) to obtain:

$$|\mathcal{J}_{4,1}| \le C\, x^{-3/2} \int_{|y|\ge 5R/4} dy \int_0^{\left(t-y^{-(\lambda-1)/2}\right)_+} (t-s)^{-1/(\lambda-1)} |\nu(y,s)| ds.$$

Notice that this integral is nonzero if and only if $y \ge t^{-2/(\lambda-1)}$. In that case:

$$\begin{aligned}
|\mathcal{J}_{4,1}| &\le C\, x^{-3/2} \sum_{2^n \ge t^{-2/(\lambda-1)} \ge R} \int_{2^n}^{2^{n+1}} dy \int_{y^{-(\lambda-1)/2}}^{t} s^{-1/(\lambda-1)} ||\nu(t-s)||_{L^\infty(2^n,2^{n+1})} ds \\
&\le C\, x^{-3/2} \sum_{2^n \ge t^{-2/(\lambda-1)} \ge R} \sum_{\substack{1 \le \ell \le 2^{-n(\lambda-1)/2} \\ \ell \le t}} 2^n 2^{-n(\lambda-1)} \ell^{-1/(\lambda-1)} 2^{n/2} \times \\
&\quad \times N_\infty(\nu; t - 2^{-n(\lambda-1)/2}\ell, 2^n) \qquad (7.42)
\end{aligned}$$

$$\le C\, x^{-3/2} |||\nu|||_{X_{3/2,2+\delta}} t^{2\delta/(\lambda-1)} R^{-\lambda/2} \le C\, |||\nu|||_{X_{3/2,2+\delta}} t^{2\delta/(\lambda-1)} R^{-(3+\lambda)/2}. \qquad (7.43)$$

In the integral $\mathcal{J}_{4,2}$, we are in a region where (7.2) holds true. Then we may use (7.32) to get:

$$|\mathcal{J}_{4,2}| \le C x^{-3/2} \int_{|y|\ge 5R/4} y^{\lambda/2} dy \int_{\left(t-y^{-(\lambda-1)/2}\right)_+}^{t} ds\, (t-s) |\nu(y,s)|.$$

The last integral is bounded as follows:

$$\begin{aligned}
|\mathcal{J}_{4,2}| &\le Cx^{-3/2} \int_{5R/4}^{t^{-2/(\lambda-1)}} y^{\lambda/2} dy \int_0^t ds\, (t-s)|\nu(y,s)| + \\
&\quad + Cx^{-3/2} \int_{t^{-2/(\lambda-1)}}^{\infty} y^{\lambda/2} dy \int_{\left(t-y^{-(\lambda-1)/2}\right)_+}^{t} ds\, (t-s)|\nu(y,s)| \\
&\le Cx^{-3/2} t^{3/2} \sum_{R \le 2^n \le t^{-2/(\lambda-1)}} (2^n)^{\lambda/2+1} 2^{-n(\lambda-1)/4} N_\infty(\nu; 0, 2^n) + \\
&\quad + Cx^{-3/2} \sum_{2^n \ge t^{-2/(\lambda-1)}} 2^{2n} 2^{-n\lambda/2} N_\infty(\nu; t - 2^{-n(\lambda-1)/2}, 2^n) \\
&\le Cx^{-3/2} R^{-\lambda/2} t^{3/2} |||\nu|||_{X_{3/2,2+\delta}} t^{-\frac{2}{\lambda-1}\frac{3(\lambda-1)}{4}} + \\
&+ \; Cx^{-3/2} |||\nu|||_{X_{3/2,2+\delta}} R^{-\lambda/2} \sum_{2^n \ge t^{-2/(\lambda-1)}} 2^{-n\delta} = CR^{-(3+\lambda)/2} |||\nu|||_{X_{3/2,2+\delta}} t^{2\delta/(\lambda-1)}.
\end{aligned}$$

This yields,

$$|\mathcal{J}_{4,2}| \le C\, R^{-(3+\lambda)/2} t^{2\delta/(\lambda-1)} |||\nu|||_{X_{3/2,\,2+\delta}}, \qquad (7.44)$$

which, combined with (7.44) gives

$$|\mathcal{J}_4| \le C\, R^{-(3+\lambda)/2} t^{2\delta/(\lambda-1)} |||\nu|||_{X_{3/2,\,2+\delta}}. \qquad (7.45)$$

Adding (7.38), (7.39), (7.40) and (7.45):

$$R^{(3+\lambda)/2}\, ||||_{L^\infty(R/2, 2\,R)} \le C t^{2\delta/(\lambda-1)} |||\nu|||_{X_{3/2,\,2+\delta}} \qquad (7.46)$$

for all $R \ge t^{-2/(\lambda-1)}$. Adding (7.35) and (7.46) yields, for all $R > 1$:

$$R^{(3+\lambda)/2} \left|\left| \int_0^t G(t-s)\, \nu(s)\, ds \right|\right|_{L^\infty(R/2,\, 2R)} \le C t^{2\delta/(\lambda-1)} |||\nu|||_{X_{3/2,\,2+\delta}}. \qquad (7.47)$$

We now consider the region where $0 < R < 1$. Then, for $|x - R| \leq R/8$:

$$
\begin{aligned}
&\int_0^t G(t-s)\nu(s,y)\,ds \leq \int_0^t ds \int_{y \leq 3R/4} |\nu(s,y)| g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} \\
&+ \int_0^t ds \int_{y \geq 5R/4} |\nu(s,y)| g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} \\
&+ \int_0^t ds \int_{|x-y| \leq R/2} \nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right)\frac{dy}{y} \\
&= \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 \qquad (7.48)
\end{aligned}
$$

The last integral in the right hand side of (7.48) is estimated as follows. Since in that term (7.4) holds we still have (7.13) and then

$$
\begin{aligned}
|\mathcal{K}_3| &\leq C \int_0^t ||\nu(s)||_{L^\infty(R/2,2R)} ds \int_{|x-y| \leq R/2} \frac{1}{(t-s)^2\, y^\lambda} \Phi\left(\frac{x-y}{(t-s)^2\, y^\lambda}\right) dy \\
&\leq C \int_0^t ||\nu(s)||_{L^\infty(R/2,2R)} \leq C\sqrt{t}\, M_\infty(\nu; R) \leq C\sqrt{t}\, R^{-3/2}\, |||\nu|||_{X_{3/2,2+\delta}}. \qquad (7.49)
\end{aligned}
$$

Using (7.3) we deduce that, in the integral $\mathcal{K}_1$ the following estimate holds:

$$
g\left((t-s)y^{(\lambda-1)/2}, \frac{x}{y}\right) \leq C(t-s)y^{\lambda+1}x^{-(3+\lambda)/2}.
$$

Using this estimate we deduce:

$$
\begin{aligned}
|\mathcal{K}_1| &\leq C\, x^{-(3+\lambda)/2} \sum_{2^{-n} \leq R} \int_0^t ds \int_{2^{-(n+1)}}^{2^{-n}} y^\lambda ||\nu(s)||_{L^\infty(2^{-(n+1)},2^{-n})}(t-s)dy \\
&\leq C\, x^{-(3+\lambda)/2} t \sum_{2^{-n} \leq R} 2^{-n(\lambda+1)}\sqrt{t}\; M_\infty(\nu, 2^{-n}) \\
&\leq C\, x^{-(3+\lambda)/2} t^{3/2} |||\nu|||_{X_{3/2,2+\delta}}\, R^{\lambda-1/2} \leq C\, x^{-3/2} t^{3/2} |||\nu|||_{X_{3/2,2+\delta}} \qquad (7.50)
\end{aligned}
$$

for $x \in (R/2, 2R)$. We are then left with the term $|\mathcal{K}_2|$.

$$
\begin{aligned}
\mathcal{K}_2 &= \int_{5R/4 \leq y \leq 2} \frac{dy}{y} \int_0^t ds\, \nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right) \\
&+ \int_{y \geq 2} \frac{dy}{y} \int_0^{(t-y^{-(\lambda-1)/2})_+} ds\, \nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right) \\
&+ \int_{y \geq 2} \frac{dy}{y} \int_{(t-y^{-(\lambda-1)/2})_+}^t ds\, \nu(s,y) g\left((t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y}\right) \\
&= \mathcal{K}_{2,1} + \mathcal{K}_{2,2} + \mathcal{K}_{2,3}. \qquad (7.51)
\end{aligned}
$$

In the term $\mathcal{K}_{2,1}$, we may use (7.32) to obtain:

$$
\begin{aligned}
|\mathcal{K}_{2,1}| &\leq C\, x^{-3/2} \int_{5R/4 \leq y \leq 2} dy y^{\lambda/2} \int_0^t ds |\nu(s,y)|(t-s) \\
&\leq C\, x^{-3/2} t^{3/2} \sum_{n=0,\, R \leq 2^{-n}} 2^{-n(1+\lambda/2)}\, M_\infty(\nu, 2^{-n}) \\
&\leq C\, x^{-3/2} t^{3/2} |||\nu|||_{X_{3/2,2+\delta}}. \qquad (7.52)
\end{aligned}
$$

In $\mathcal{K}_{2,2}$, (7.30) holds and then,

$$|\mathcal{K}_{2,2}| \quad \leq \quad C\,x^{-3/2}\int_{y\geq 2}dy\int_0^{(t-y^{-(\lambda-1)/2})_+}ds\,|\nu(s,y)|(t-s)^{-1/(\lambda-1)} \tag{7.53}$$

We notice also here that the last integral in the right hand side of (7.53) is nonzero only if $y \geq t^{-2/(\lambda-1)}$. Therefore

$$\begin{aligned}
|\mathcal{K}_{2,2}| &\leq C\,x^{-3/2}\sum_{n=1,\,2^n\geq t^{-2/(\lambda-1)}}2^n\int_{y^{-(\lambda-1)/2}}^t ds\,s^{-1/(\lambda-1)}||\nu(s)||_{L^\infty(2^n,2^{n+1})}\\
&\leq C\,x^{-3/2}\sum_{n=1,\,2^n\geq t^{-2/(\lambda-1)}}2^{3n/2}\sum_{\ell=1,\,2^{-n(\lambda-1)/2}\,\ell\leq t}\ell^{-1/(\lambda-1)}2^{-n(\lambda-1)/2}\times\\
&\qquad\qquad\times N_\infty(\nu;t-2^{-n(\lambda-1)/2}\,\ell,2^n)\\
&\leq C\,x^{-3/2}|||\nu|||_{X_{3/2,2+\delta}}\sum_{n=1,\,2^n\geq t^{-2/(\lambda-1)}}2^{-n(\lambda/2+\delta)}\\
&\leq C\,x^{-3/2}|||\nu|||_{X_{3/2,2+\delta}}t^{\frac{2}{\lambda-1}(\lambda/2+\delta)}.
\end{aligned} \tag{7.54}$$

In $\mathcal{K}_{2,3}$, we may use (7.32), whence

$$\begin{aligned}
|\mathcal{K}_{2,3}| &\leq C\,x^{-3/2}\int_{y\geq 2}y^{\lambda/2}dy\int_{(t-y^{-(\lambda-1)/2})_+}^t|\nu(y,s)|(t-s)ds\\
&\leq C\,x^{-3/2}t\sum_{n=1}^\infty 2^{n(\lambda/2+1)}2^{-n(\lambda-1)/2}N_\infty(\nu;t-2^{-n(\lambda-1)/2},2^n)\\
&\leq C\,x^{-3/2}t|||\nu|||_{X_{3/2,2+\delta}}\sum_{n=1}^\infty 2^{-n(1/2+\delta)}\leq C\,x^{-3/2}t|||\nu|||_{X_{3/2,2+\delta}}.
\end{aligned} \tag{7.55}$$

By (7.51), (7.52), (7.54) and (7.55) we have

$$|\mathcal{K}_2| \quad \leq \quad C\,x^{-3/2}\left(t+t^{(2\delta+\lambda)/(\lambda-1)}\right)|||\nu|||_{X_{3/2,2+\delta}}. \tag{7.56}$$

Adding (7.49), (7.50) and (7.56) we obtain the following estimate for $0<R\leq 1$:

$$R^{3/2}\left|\left|\int_0^t G(t-s)\,\nu(s)\,ds\right|\right|_{L^\infty(R/2,\,2R)}\leq Ct\,|||\nu|||_{X_{3/2,2+\delta}}. \tag{7.57}$$

The Lemma follows combining (7.47) and (7.57). □

Lemma 7.3 *For all* $\varphi\in Y^\sigma_{3/2,(3+\lambda)/2}(T)$ *with* $\sigma>1+\delta$, $\varepsilon\geq 0$ *and* $0<T<1$*:*

$$\sup_{0\leq t\leq T}\left|\left|\left|\int_0^t G(t-s)(\mathcal{L}-L_\varepsilon)\varphi(s)ds\right|\right|\right|_{3/2,(3+\lambda)/2}\leq CT^\beta|||\varphi|||$$

for some constant $C>0$ *independent of* T*, of* ε *and* φ*.*

Proof of Lemma 7.3. By Lemma 7.2:

$$\sup_{0\leq t\leq T}||\int_0^t G(t-s)A_1(s)ds||_{3/2,(3+\lambda)/2}\leq CT^\beta||A_1||_{X_{3/2,(3+\lambda)/2}(T)}.$$

Moreover, for all $h(t,x), q, p$ $||h||_{X_{q,p}(T)} \leq C \sup_{0\leq t\leq T} |||h(t)|||_{q,p}$.
Using Lemma 6.1 $T^{\beta}||A_1||_{X_{3/2,(3+\lambda)/2}(T)} \leq C\,T^{\beta}|||\varphi|||_{3/2,(3+\lambda)/2} \leq C|||\varphi|||$. Whence

$$\sup_{0\leq t\leq T} || \int_0^t G(t-s)A_1(s)ds||_{3/2,(3+\lambda)/2} \leq C\,T^{\beta}\,|||\varphi|||. \tag{7.58}$$

A similar argument is used for the term $A_{2,\,\varepsilon}$. First, by Lemma 7.2,

$$\sup_{0\leq t\leq T} || \int_0^t G(t-s)A_{2,\,\varepsilon}(s)ds||_{3/2,(3+\lambda)/2} \leq CT^{\beta}||A_{2,\,\varepsilon}||_{X_{3/2,(3+\lambda)/2}(T)}.$$

Then, by Lemma 6.5,$||A_{2,\,\varepsilon}||_{X_{3/2,(3+\lambda)/2}(T)} \leq C|||\varphi|||$ and

$$\sup_{0\leq t\leq T} || \int_0^t G(t-s)A_{2,\,\varepsilon}(s)ds||_{3/2,(3+\lambda)/2} \leq CT^{\beta}|||\varphi||| \tag{7.59}$$

and Lemma 7.3 follows from (7.58) and (7.59). □

Lemma 7.4 *Suppose that $\sigma > 1+\delta$. There exists a positive constant C such that, for all $0 < T^* < 1$, for all $\theta \in [0,1]$, for all $\nu \in Y^{\sigma}_{3/2,\,2+\delta}(T)$ and all φ satisfying $|||\varphi||| < +\infty$ and solving:*

$$\frac{\partial\varphi}{\partial t} = L(\varphi) + \theta\,(\mathcal{L}-L)\,(\varphi) + \nu,\ \ x>0,\ \ t\in(0,T^*) \tag{7.60}$$

we have:

$$|||\varphi||| \leq C\,||\nu||_{Y^{\sigma}_{3/2,\,2+\delta}(T^*)}.$$

Remark 7.5 *The result of Lemma 7.4 remains true if the space $Y^{\sigma}_{3/2,\,2+\delta}(T^*)$ is replaced by $Y^{\sigma}_{3/2,\,2}(T^*)$. However, a solution of (7.60) satisfying $|||\varphi||| < +\infty$ does not exists in general if $\nu \in Y^{\sigma}_{3/2,\,2}(T^*)$.*

Proof of Lemma 7.4. We first rewrite the equation (7.60) as follows:

$$\frac{\partial\varphi}{\partial t} = (1-\theta)L(\varphi) + \theta\,\mathcal{L}(\varphi) + \nu$$

Then, for $x \in (3R/4, 5R/5)$ and $R > 1$ we define the new variables $x = XR$, $y = YR$, $t = (\tau/R^{(\lambda-1)/2})$ and $\varphi(x,t) = R^{-(3+\lambda)/2}\,\Psi(X,\tau)$. Since $t \in (0,T_*)$, $\tau \in (0, T_*\,R^{(\lambda-1)/2})$.

$$\begin{aligned}
\frac{\partial\Psi}{\partial\tau} &= (1-\theta)\,L(\Psi) + \theta\left[R^{3/2}\int_0^{X/2}\Big((X-Y)^{\lambda/2}\Psi(X-Y) - X^{\lambda/2}\Psi(X)\Big)\,(Ry)^{\lambda/2}f_0(R\,y)\,dY\right]\\
&\quad -\theta\,X^{\lambda/2}\Psi(X)\int_{X/2}^{\infty} Y^{\lambda/2}f_0(RY)dY + \tilde{\nu}_1\\
\tilde{\nu}_1 &= R^2\,\nu(R\,X, \tau\,R^{-(\lambda-1)/2}) + \theta R^{(3+\lambda)/2}\,\times\\
&\quad \times\int_0^{X/2}\Big((X-Y)^{\lambda/2}f_0(R(X-Y)) - X^{\lambda/2}f_0(RX)\Big)\,Y^{\lambda/2}\Psi(Y)dY\\
&\quad -\theta\,X^{\lambda/2}R^{(3+\lambda)/2}f_0(RX)\int_{X/2}^{\infty}Y^{\lambda/2}\Psi(Y)dY
\end{aligned} \tag{7.61}$$

Using the expression of the operator L given in (1.3)

$$\begin{aligned}
\frac{\partial\Psi}{\partial\tau} &= (1-\theta)\int_0^{X/2}\Big((X-Y)^{\lambda/2}\Psi(X-Y)-X^{\lambda/2}\Psi(x)\Big)Y^{\lambda/2}Y^{-3/2}dy \\
&+[R^{(3+\lambda)/2}]\left[R^{3/2}\int_0^{X/2}\Big((X-Y)^{\lambda/2}\Psi(X-Y)-X^{\lambda/2}\Psi(X)\Big)(Ry)^{\lambda/2}f_0(R\,y)\,dY\right] \\
&-2(1-\theta)\sqrt{2}X^{(\lambda-1)/2}\Psi(X)-\theta\,R^{(3+\lambda)/2}\,X^{\lambda/2}\Psi(X)\int_{X/2}^{\infty}Y^{\lambda/2}f_0(RY)dY \\
&+\tilde{\nu}_1+\tilde{\nu}_2 \\
\tilde{\nu}_2 &= (1-\theta)\int_0^{X/2}\Big((X-Y)^{-3/2}-X^{-3/2}\Big)Y^{\lambda/2}\Psi(Y)dY- \\
&\qquad -(1-\theta)X^{-3/2}\int_{X/2}^{\infty}Y^{\lambda/2}\Psi(Y)dY
\end{aligned}\tag{7.62}$$

We can rewrite the equation as

$$\frac{\partial\Psi}{\partial\tau}=T_{1-\theta,\,R}\left(M_{\lambda/2}\Psi\right)-a(X,t)\Psi+Q \tag{7.63}$$

$$a(X,t)=2(1-\theta)\sqrt{2}X^{(\lambda-1)/2}+\theta\,R^{(3+\lambda)/2}\,X^{\lambda/2}\int_{X/2}^{\infty}Y^{\lambda/2}f_0(RY)dY \tag{7.64}$$

$$Q=\tilde{\nu}_1+\tilde{\nu}_2. \tag{7.65}$$

Since $|||\varphi|||_{3/2,(3+\lambda)/2}<\infty$, we can combine (3.7) in Theorem 3.1 with (6.14) to obtain:

$$\begin{aligned}
\sup_{0\le T\le R^{(\lambda-1)/2}}\left(\int_T^{\min(T+1,T^*\,R^{(\lambda-1)/2})}||\Psi(s)||^2_{H^\sigma_x(3/4,5/4)}ds\right)^{1/2} &\le C\sup_{0\le t\le T^*}|||\varphi(t)|||_{3/2,(3+\lambda)/2} \\
&+\sup_{0\le T\le R^{(\lambda-1)/2}}\left(\int_T^{\min(T+1,T^*\,R^{(\lambda-1)/2})}||Q(s)||^2_{H^\sigma_x(1/2,2)}ds\right)^{1/2}
\end{aligned}$$

Moreover, in order to estimate the norm of $Q(s)$ we first notice that, using (6.14):

$$||\theta X^{-3/2}\int_{X/2}^{\infty}Y^{\lambda/2}\Psi(Y,s)dY||_{H^\sigma_x(1/2,2)}\le C|||\varphi(t)|||_{3/2,(3+\lambda)/2}+C||\Psi(s)||_{H^{(\sigma-1)_+}(1/2,2)}.$$

The same estimate holds trivially for the term $\theta\,X^{\lambda/2}R^{(3+\lambda)/2}f_0(RX)\int_{X/2}^{\infty}Y^{\lambda/2}\Psi(Y)dY$ in $\widetilde{\nu}_1$. We are then left with the term $\int_0^{X/2}\left((X-Y)^{-3/2}-X^{-3/2}\right)Y^{\lambda/2}\Psi(Y)dY$. Using that $|\Psi(Y)|\le Y^{-(3+\lambda)/2}$ we deduce:

$$\begin{aligned}
||\int_0^{X/2}\Big((X-Y)^{-3/2}-X^{-3/2}\Big)Y^{\lambda/2}\Psi(Y)dY||_{H^\sigma(1/2,2)} &\le C|||\varphi(t)|||_{3/2,(3+\lambda)/2} \\
&+C||\Psi(s)||_{H^{(\sigma-1)_+}(1/2,2)}.
\end{aligned}$$

This gives

$$\begin{aligned}
\sup_{0\le T\le T^*\,R^{(\lambda-1)/2}}\left(\int_T^{\min(T+1,T_*\,R^{(\lambda-1)/2})}||\Psi(s)||_{H^\sigma_X(3/4,5/4)}ds\right)^{1/2} &\le C\sup_{0\le t\le T^*}|||\varphi|||_{3/2,(3+\lambda)/2} \\
+C\sup_{0\le T\le T^*\,R^{(\lambda-1)/2}}\left(\int_T^{\min(T+1,T_*\,R^{(\lambda-1)/2})}||\Psi(s)||_{H_X^{(\sigma-1)_+}(1/2,2)}ds\right)^{1/2} &+C||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T)}
\end{aligned}$$

A bootstrap argument then yields:

$$\sup_{0\le T\le T_*\,R^{(\lambda-1)/2}} \left(\int_T^{\min(T+1, T_*\,R^{(\lambda-1)/2})} ||\Psi(s)||_{H^\sigma_X(3/4,5/4)} ds \right)^{1/2} \le$$
$$\le C \sup_{0\le t\le T^*} |||\varphi|||_{3/2,(3+\lambda)/2} + C||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T^*)} \tag{7.66}$$

(actually in an interval slightly smaller than $(3/4, 5/4)$, for example $(7/8, 9/8)$).
We deduce,

$$\sup_{0\le t_0\le T^*} \sup_{R>1} \left(R^{(3+\lambda)/2} N_{2;\,\sigma}(\varphi; R, t_0) \right) \le C \sup_{0\le t\le T^*} |||\varphi(t)|||_{3/2,(3+\lambda)/2} + C\,||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T^*)} \tag{7.67}$$

We consider now the case where $0 < R \le 1$. We rescale the equation for $x \in (3R/4, 5R/5)$ and $R < 1$. The new variables are now $x = XR$, $y = YR$, and $\varphi(x,t) = R^{-3/2}\,\Psi(X,t)$. Arguing as above, the function Ψ satisfies now:

$$\frac{\partial \Psi}{\partial \tau} = R^{\frac{\lambda-1}{2}} T_{1-\theta,\,R}\left(M_{\lambda/2}\Psi \right) + Q$$

$$Q = R^{3/2}\,\nu(R\,X,\tau) + R^{(\lambda-1)/2} \left((1-\theta) \int_0^{X/2} \left((X-Y)^{-3/2} - X^{-3/2} \right) Y^{\lambda/2}\Psi(Y)dY - \right.$$
$$-(1-\theta)X^{-3/2} \int_{X/2}^\infty Y^{\lambda/2}\Psi(Y)dY - 2(1-\theta)\sqrt{2}X^{(\lambda-1)/2}\Psi(X) -$$
$$\left. -(1-\theta)X^{-3/2} \int_{X/2}^\infty Y^{\lambda/2}\Psi(Y)dY \right)$$
$$+\theta\,R^{\lambda+1} \int_0^{X/2} \left((X-Y)^{\lambda/2}\Psi(X-Y) - X^{\lambda/2}\Psi(X) \right) y^{\lambda/2} f_0(R\,y)\,dY$$
$$+\theta R^{\lambda+1} \int_0^{X/2} \left((X-Y)^{\lambda/2} f_0(R(X-Y)) - X^{\lambda/2} f_0(RX) \right) Y^{\lambda/2}\Psi(Y)dY$$
$$-\theta\,R^{\lambda/2}\,X^{\lambda/2}\Psi(X) \int_0^\infty y^{\lambda/2} f_0(y)dy + \theta\,R^{\lambda/2}\,X^{\lambda/2}\Psi(X) \int_0^{X/2} Y^{\lambda/2}\,f_0(R\,Y)\,dY$$
$$-\theta\,R^{\lambda/2}\,f_0(R\,X)\,X^{\lambda/2} \int_0^\infty y^{\lambda/2}\,\varphi(y,t)\,dy - (1-\theta)\,X^{-3/2} \int_0^\infty \varphi(y,t) y^{\lambda/2} dy$$
$$+R^{\frac{\lambda-1}{2}} X^{-3/2} \int_0^{X/2} \Psi(Y) Y^{\lambda/2} dY. \tag{7.68}$$

Where we have used that:

$$x^{-3/2} \int_{x/2}^\infty g(y) y^{\lambda/2} dy = x^{-3/2} \int_0^\infty g(y) y^{\lambda/2} dy - x^{-3/2} \int_0^{x/2} g(y) y^{\lambda/2} dy$$
$$= a(t) x^{-3/2} - x^{-3/2} \int_0^{x/2} g(y) y^{\lambda/2} dy.$$

By Theorem 3.1 with $\kappa = R^{(\lambda-1)/2}$

$$||\Psi||_{L^2_t(0,T_*;H^\sigma(3/4,5/4))} \le C\,||Q||_{L^2_t(0,T_*;H^\sigma_x(1/2,2))}$$

We now have:

$$||Q||_{H^\sigma(3/4,5/4)} \le C||\Psi||_{H^{(\sigma-1)_+}(1/2,2)} + \sup_{0<t<T_*} |||\varphi(t)|||_{3/2,(3+\lambda/2)} + C\,||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T)}$$

As before, a bootstrap argument as in the case $R > 1$ gives

$$||\Psi||_{L^2_t(0,T_*;H^\sigma(3/4,5/4))} \leq C \sup_{0\leq t\leq T^*} |||\varphi(t)|||_{3/2,(3+\lambda)/2} + C\,||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T)}$$

and then, rewriting this estimate in the original variables

$$R^{3/2}\,M_{2;\,\sigma}(\varphi, R) \leq C\left(\sup_{0<t<T_*} |||\varphi(t)|||_{3/2,(3+\lambda)/2} + ||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T)}\right). \tag{7.69}$$

Combined with (7.67) we deduce

$$\begin{aligned}\sup_{0\leq t_0\leq T^*}\sup_{R>1}\left(R^{(3+\lambda)/2}N_{2;\,\sigma}(\varphi; R, t_0)\right) + \sup_{0<R\leq 1} R^{3/2}\,M_{2;\,\sigma}(\varphi, R)&\\ \leq C\left(\sup_{0<t<T_*} |||\varphi(t)|||_{3/2,(3+\lambda)/2} + ||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T)}\right),&\end{aligned} \tag{7.70}$$

and then

$$|||\varphi||| \leq C \sup_{0<t<T_*} |||\varphi(t)|||_{3/2,(3+\lambda/2)} + C||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T^*)}.$$

We use now

$$\varphi(t) = \theta\int_0^t G(t-s)\ (\mathcal{L}-L)\,(\varphi)(s)\,ds + \int_0^t G(t-s)\,\nu(s)\,ds$$

which yields

$$\begin{aligned}|||\varphi(t)|||_{3/2,(3+\lambda)/2} \leq C\int_0^t |||G(t-s)\ (\mathcal{L}-L)\,(\varphi)(s)|||_{3/2,(3+\lambda)/2}\,ds\,+&\\ +\int_0^t |||G(t-s)\,\nu(s)|||_{3/2,(3+\lambda)/2}\,ds.&\end{aligned}$$

By Lemma 7.2 and Lemma 7.3:

$$|||\varphi(t)|||_{3/2,(3+\lambda)/2} \leq C\,{T^*}^\beta|||\varphi||| + (T^* + {T^*}^{2\delta(\lambda-1)})^\beta||\nu||_{X_{3/2,\,2+\delta}(T^*)}$$

and the result follows taking T^* small enough. □

8 Proof of Theorem 2.1.

We introduce the auxiliary operators L_ε, for $\varepsilon > 0$, defined as follows:

$$\begin{aligned}L_\varepsilon(g) &= \int_0^{x/2}\left((x-y)^{-3/2} - x^{-3/2}\right)y^{\lambda/2}g(y)dy\\ &+ \int_0^{x/2}\left((x-y)^{\lambda/2}g(x-y) - x^{\lambda/2}g(x)\right)\frac{dy}{y^{3/2}+\varepsilon^{3/2}\,x^{3/2}}\\ &- x^{-3/2}\int_{x/2}^\infty y^{\lambda/2}g(y)dy - 2\sqrt{2}x^{(\lambda-1)/2}g(x).\end{aligned} \tag{8.1}$$

For all $\varepsilon > 0$ the operator L_ε is more regular than L. Notice in particular that g and $L_\varepsilon g$ have the same regularity.

Lemma 8.1 *Let $0 \le T \le 1$. Then, there exists a constant $C > 0$ such that, for any $\varphi \in \mathcal{E}_{T;\sigma}$, for all $t_0 \in (0,T)$ and $\varepsilon \ge 0$:*

$$R^2 \, N_{2;\,\sigma-1/2}\left((\mathcal{L} - L_\varepsilon)(\varphi);\, R,\, t_0\right) \le C \, |||\varphi|||, \ \ \forall R > 1, \tag{8.2}$$

$$R^{2-\lambda/2} \, M_{2;\,\sigma-1/2}\left((\mathcal{L} - L_\varepsilon)(\varphi);\, R\right) \le C \, |||\varphi|||, \ \ \forall \ 0 < R < 1. \tag{8.3}$$

Proof of Lemma 8.1. Notice that the operator $(\mathcal{L} - L_\varepsilon)$ may be written as $A_1 + A_{2,\varepsilon}$ where A_1 and $A_{2,\varepsilon}$ are defined in (6.1) and (6.4)-(6.5). Lemma 8.1 then follows using Lemma 6.2 and Lemma 6.5. □

Lemma 8.2 *(i) There exists a constant $C > 0$ such that, for all $\varepsilon \in (0,1]$, $\theta \in [0,1)$, $\varphi \in \mathcal{E}_{T;\sigma}$ and $u \in \mathcal{E}_{T;\sigma}$ satisfying:*

$$\partial_t \varphi = (1-\theta) \, L(\varphi) + \theta \, \mathcal{L}(\varphi) + (\mathcal{L} - L_\varepsilon)\,(u)$$

there holds:

$$|||\varphi||| \le C \sup_{0 \le t \le T^*} |||\varphi|||_{3/2,(3+\lambda)/2} + \frac{C}{1-\theta} |||u|||.$$

Proof of Lemma 8.2 The proof of this Lemma is similar to that of Lemma 7.4. The difference comes from the fact that we must use the regularising effect of the operator $T_{1-\theta,R}$ of Theorem 3.1. We then start by scaling the variables.

In the case $R > 1$ and for $x \in (3R/4, 5R/5)$ we define the new variables: $x = XR$, $y = YR$, $t = (\tau / R^{(\lambda-1)/2})$ and $\varphi(x,t) = R^{-(3+\lambda)/2} \, \Psi(X, \tau)$. Since $t \in (0, T_*)$, $\tau \in (0, T_* \, R^{(\lambda-1)/2})$. The function $\psi(X,\tau)$ satisfies equations (7.63)-(7.65) with $\widetilde{\nu}_1$ and $\widetilde{\nu}_2$ are defined as in (7.61), (7.62) but where ν is now given by

$$\nu = (\mathcal{L} - L_\varepsilon)\,(u). \tag{8.4}$$

Using Lemma 8.1 and Theorem 3.1 with $\varepsilon = 1 - \theta$, we obtain , arguing as in the proof of (7.66),

$$\sup_{0 \le T \le T_* \, R^{(\lambda-1)/2}} \left(\int_T^{\min(T+1, T_* \, R^{(\lambda-1)/2})} ||\Psi(s)||^2_{H^\sigma_X(3/4,5/4)} ds \right)^{1/2} \le$$

$$\le C \sup_{0 \le t \le T^*} |||\varphi|||_{3/2,(3+\lambda)/2} + \frac{C}{1-\theta} |||u||| \tag{8.5}$$

Notice that the only difference between the proof of (8.5) and that of (7.66) comes from the control of the term ν defined in (8.4). However that term is estimated as the term P in (3.5) with $\kappa = 1$, and $\varepsilon = 1 - \theta$ combined with (8.2).

We consider now the range $R \in (0,1)$ and rescale the equation for $x \in (3R/4, 5R/5)$. The new variables are now $x = XR$, $y = YR$, $\varphi(x,t) = R^{-3/2} \, \Psi(X,t)$ and $u(x,t) = R^{-3/2} U(X,t)$. Arguing as above, the function Ψ satisfies now the same equation (7.68) where the term Q is defined in (7.68) where here again ν is given by (8.4). The term $R^{3/2}\nu(R,X,t)$ in (7.68) is rewritten using (8.1) as follows:

$$R^{3/2}\nu(R,X,t) \;=\; R^{(\lambda-1)/2}\,(\mathcal{L} - L_\varepsilon)\,(U) = \mathcal{Q}_0(X,t) + \mathcal{Q}_1(X,t)$$

where,

$$\mathcal{Q}_0(X,t) \;=\; R^{(\lambda-1)/2} \int_0^{X/2} \Big((X-Y)^{\lambda/2} U(X-Y) - X^{\lambda/2} U(X) \Big) \frac{dY}{Y^{3/2} + \varepsilon^{3/2} \, X^{3/2}}$$

$$\mathcal{Q}_1(X,t) \;=\; R^{(\lambda-1)/2} \int_0^{X/2} \Big((X-Y)^{-3/2} - X^{-3/2} \Big) Y^{\lambda/2} U(Y,t) dY$$

$$- \;\; X^{-3/2} \int_{R\,X/2}^{\infty} y^{\lambda/2} u(y,t) dy - 2\sqrt{2} R^{(\lambda-1)/2} X^{(\lambda-1)/2} U(X).$$

and $\mathcal{Q}_1(X,t)$ satisfies,$|M_{2,\sigma}(\mathcal{Q}_1;1)| \leq C\,|||u|||$. Using now Theorem 3.1 with $\varepsilon = 1-\theta$ and $\kappa = R^{(\lambda-1)/2}$ and estimating all the remaining terms as in the proof of (7.69) we obtain

$$R^{3/2} M_{2,\sigma}(\varphi;R) \leq C \sup_{0\leq t\leq T^*} |||\varphi|||_{3/2,(3+\lambda)/2} + \frac{C}{1-\theta}|||u|||. \tag{8.6}$$

Combining (8.5) and (8.6) the Lemma follows. □

Lemma 8.3 *Let $0 \leq T \leq 1$. Then for any $\varphi \in \mathcal{E}_{T;\sigma}$, for all $t_0 \in (0,T)$:*

$$\begin{aligned}
\lim_{\varepsilon\to 0} N_{2;\,\sigma-1/2}\left((L-L_\varepsilon)(\varphi);\,R,\,t_0\right) &= 0, \;\; \forall R>1,\\
\lim_{\varepsilon\to 0} M_{2;\,\sigma-1/2}\left((L-L_\varepsilon)(\varphi);\,R\right) &= 0, \;\; \forall\; 0<R<1.
\end{aligned}$$

Proof of Lemma 8.3. After rescaling the variables $x = R\,X$, $t = t_0 + R^{-(\lambda-1)/2}\tau$ and $\varphi(x,t) = \psi(X,\tau)$, the two identities reduce to:

$$\lim_{\varepsilon\to 0}\int_0^{\tau^*} ||(L-L_\varepsilon)(\psi)(\tau)||^2_{H^\sigma_X(1/2,2}\, d\tau = 0 \tag{8.7}$$

with $0<\tau^*<1$. Using (1.3) and (8.1) we have

$$(L-L_\varepsilon)(\psi) = (\mathcal{W}_\infty - \mathcal{W}_{\infty,\,\varepsilon})(\psi)$$

where $\mathcal{W}_\infty$ and $\mathcal{W}_{\infty,\,\varepsilon}$ are defined in (6.19) and (6.20). Therefore the Lemma follows combining (6.22) and the Lebesgue convergence Theorem. □

End of the proof of Theorem 2.1.

Our goal is to solve (1.7) for $\theta = 1$. To this end we use a continuation argument starting at $\theta = 0$.

For $\theta = 0$ equation (1.7) has a solution $\varphi \in \mathcal{E}_{T;\sigma}$. This is a consequence of the results of [4] and of Lemma 7.4 in Section 3 with $\theta = 0$.

Then, we define:

$$\theta^* = \sup\Big\{\theta\geq 0;\text{ for all }\nu\in Y^\sigma_{3/2,2+\delta}(T),\text{there exists }\varphi\in\mathcal{E}_{T;\sigma}\text{ solution of (1.7) }\Big\} \tag{8.8}$$

The Lemmas 7.3 and 7.4 show that there exists a constant $C>0$ such that, for any $\theta<\theta^*$ and for all $\nu \in Y^\sigma_{3/2,2+\delta}(T)$ there exists a function $\varphi\in\mathcal{E}_{T;\sigma}$ such that

$$|||\varphi||| \leq C||\nu||_{Y^\sigma_{3/2,\,2+\delta}(T)}.$$

Suppose that $\theta^*<1$. We will show that for all $\theta>\theta^*$ with $\theta-\theta^*$ sufficiently small and all $\nu \in Y^\sigma_{3/2,2+\delta}(T)$ there exists a function $\varphi\in\mathcal{E}_{T;\sigma}$ and solving (1.7). This would give a contradiction.

To this end we use a fixed point argument.

Given $\widetilde{\varphi}\in\mathcal{E}_{T;\sigma}$ and $\nu\in Y^\sigma_{3/2,2+\delta}(T)$ we define $\varphi_{\varepsilon,n}\in\mathcal{E}_{T;\sigma}$ as the solution of

$$\partial_t\varphi_{\varepsilon,n} = (1-\theta_n)\,L(\varphi_{\varepsilon,n}) + \theta_n\,\mathcal{L}(\varphi_{\varepsilon,n}) + (\theta-\theta_n)\,(\mathcal{L}-L_\varepsilon)\,(\widetilde{\varphi}) + \nu \tag{8.9}$$

where θ_n is a sequence such that $\theta_n<\theta^*$, $\theta_n\to\theta^*$ as $n\to+\infty$. The functions $\varphi_{\varepsilon,n}$ are well defined since $\theta_n<\theta_*$ and $(\mathcal{L}-L_\varepsilon)\,(\widetilde{\varphi})\in Y^\sigma_{3/2,2+\delta}(T)$. Combining Lemma 7.4 and Lemma 8.2 we obtain:

$$|||\varphi_{\varepsilon,n}||| \leq C(\theta-\theta_n)\sup_{0\leq t\leq T^*}|||\varphi|||_{3/2,(3+\lambda)/2} + C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + ||\nu||_{Y^\sigma_{3/2,2+\delta}}. \tag{8.10}$$

Since $\varphi_{\varepsilon,n}$ satisfies equation (8.9) we have:

$$\varphi_{\varepsilon,n} = \int_0^t G(t-s)\left[\theta_n(\mathcal{L}-L)(\varphi_{\varepsilon,n}) + (\theta-\theta_n)(\mathcal{L}-L_\varepsilon)(\widetilde{\varphi}) + \nu\right]ds. \tag{8.11}$$

Using now Lemma 7.2 and Lemma 7.3 we obtain:

$$\sup_{0\le t\le T} ||\varphi_{\varepsilon,n}||_{3/2,(3+\lambda)/2} \le CT^{\beta}\left(|||\varphi_{\varepsilon,n}||| + (\theta-\theta_n)|||\widetilde{\varphi}||| + ||\nu||_{X_{3/2,2+\delta}}\right) \tag{8.12}$$

Therefore, using (8.10) and (8.12) for T small we obtain

$$|||\varphi_{\varepsilon,n}||| \le C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + ||\nu||_{Y^{\sigma}_{3/2,2+\delta}}. \tag{8.13}$$

Moreover, given $\widetilde{\varphi}\in\mathcal{E}_{T;\sigma}$, $\widetilde{\varphi}'\in\mathcal{E}_{T;\sigma}$ and denoting the corresponding solutions as $\varphi_{\varepsilon,n}$ and $\varphi'_{\varepsilon,n}$ a similar argument yields:

$$|||\varphi_{\varepsilon,n}-\varphi'_{\varepsilon,n}||| \le C\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}-\widetilde{\varphi}'||| \tag{8.14}$$

By Lemma 8.3 we deduce that

$$\begin{aligned}\lim_{\varepsilon,\ \varepsilon'\to 0} N_{2;\,\sigma-1/2}\left((L_{\varepsilon'}-L_{\varepsilon})(\widetilde{\varphi});\, R,\, t_0\right) &= 0,\ \forall R>1,\ \forall t_0\in(0,T),\\ \lim_{\varepsilon,\ \varepsilon'\to 0} M_{2;\,\sigma-1/2}\left((L_{\varepsilon'}-L_{\varepsilon})(\widetilde{\varphi});\, R\right) &= 0,\ \forall\ 0<R<1,\ \forall t_0\in(0,T).\end{aligned}$$

We use now the regularising effects obtained in Theorem 3.1 combined with the rescaling argument that have already been used in the proof of Lemma 7.4, to obtain:

$$\begin{aligned}\lim_{\varepsilon,\ \varepsilon'\to 0} N_{2;\,\sigma}\left(\varphi_{\epsilon,n}-\varphi_{\varepsilon',n};\, R,\, t_0\right) &= 0,\ \forall R>1,,\ \forall t_0\in(0,T),\\ \lim_{\varepsilon,\ \varepsilon'\to 0} M_{2;\,\sigma}\left(\varphi_{\epsilon,n}-\varphi_{\varepsilon',n};\, R\right) &= 0,\ \forall\ 0<R<1,\ \forall t_0\in(0,T)\\ \lim_{\varepsilon,\ \varepsilon'\to 0} ||\varphi_{\epsilon,n}-\varphi_{\varepsilon',n}||_{L^{\infty}([0,T]\times[R/2,2R])} &= 0\ \forall R>0\ \forall t\in(0,T).\end{aligned}$$

There exists then a function φ_n defined in $\mathbb{R}^+\times[0,T]$ such that

$$\lim_{\varepsilon,\ \varepsilon'\to 0} N_{2;\,\sigma}\left(\varphi_{\epsilon,n}-\varphi_n;\, R,\, t_0\right) = 0,\ \forall R>1,,\ \forall t_0\in(0,T), \tag{8.15}$$

$$\lim_{\varepsilon,\ \varepsilon'\to 0} M_{2;\,\sigma}\left(\varphi_{\epsilon,n}-\varphi_n;\, R\right) = 0,\ \forall\ 0<R<1,\ \forall t_0\in(0,T), \tag{8.16}$$

$$\lim_{\varepsilon,\ \varepsilon'\to 0} ||\varphi_{\epsilon,n}-\varphi_n||_{L^{\infty}([0,T]\times[R/2,2R])} = 0\ \forall R>0,\ \forall t\in(0,T). \tag{8.17}$$

By (8.13) we have:

$$\begin{aligned}R^{(3+\lambda)/2} N_{2;\,\sigma}\left(\varphi_{\epsilon,n};\, R,\, t_0\right) &\le C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + C||\nu||_{Y^{\sigma}_{3/2,2+\delta}},\ \forall R>1,,\ \forall t_0\in(0,T),\\ R^{3/2} M_{2;\,\sigma}\left(\varphi_{\epsilon,n};\, R,\, t_0\right) &\le C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + C||\nu||_{Y^{\sigma}_{3/2,2+\delta}},\ \forall\ 0<R<1,\ \forall t_0\in(0,T)\\ \max\left\{x^{3/2}, x^{(3+\lambda)/2}\right\}|\varphi_{\epsilon,n}(x,t)| &\le C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + C||\nu||_{Y^{\sigma}_{3/2,2+\delta}},\ \forall x\in(R/2,2R),\\ &\qquad \forall R>0\ \forall t\in(0,T).\end{aligned}$$

Taking limits as $\varepsilon\to 0$:

$$\begin{aligned}R^{(3+\lambda)/2} N_{2;\,\sigma}\left(\varphi_n;\, R,\, t_0\right) &\le C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + C||\nu||_{Y^{\sigma}_{3/2,2+\delta}},\ \forall R>1,,\ \forall t_0\in(0,T),\\ R^{3/2} M_{2;\,\sigma}\left(\varphi_n;\, R,\, t_0\right) &\le C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + C||\nu||_{Y^{\sigma}_{3/2,2+\delta}},\ \forall\ 0<R<1,\ \forall t_0\in(0,T),\\ \max\left\{x^{3/2}, x^{(3+\lambda)/2}\right\}|\varphi_n(x,t)| &\le C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + C||\nu||_{Y^{\sigma}_{3/2,2+\delta}},\ \forall x\in(R/2,2R),\\ &\qquad \forall R>0\ \forall t\in(0,T).\end{aligned}$$

whence $\varphi_n \in \mathcal{E}_{T;\sigma}$ and,

$$|||\varphi_n||| \leq C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}||| + ||\nu||_{Y^\sigma_{3/2,2+\delta}}.$$

A similar argument yields

$$|||\varphi_n - \varphi_n'||| \leq C\,\frac{\theta-\theta_n}{1-\theta_n}|||\widetilde{\varphi}-\widetilde{\varphi}'|||.$$

Notice that $\varphi_n \in L^2(0,T;H^\sigma_{loc}(\mathbb{R}^+))$. Moreover, passing to the weak limit in the equation (8.9) as $\varepsilon \to 0$ we obtain that φ_n solves

$$\partial_t\varphi_n = (1-\theta_n)\,L(\varphi_n) + \theta_n\,\mathcal{L}(\varphi_n) + (\theta-\theta_n)\,(\mathcal{L}-L)\,(\widetilde{\varphi}) + \nu \tag{8.18}$$

in the sense of distributions. Then, $\varphi_n \in H^1(0,T;H^\sigma_{loc}(\mathbb{R}^+))$.

Formula (8.18) implies that the application $\widetilde{\varphi} \mapsto \varphi_n$ has a fixed point for any $\nu \in Y^\sigma_{3/2,2+\delta}$, n sufficiently large and $\theta-\theta^* > 0$ sufficiently small. Let us denote by φ such a fixed point that satisfies:

$$\partial_t\varphi = (1-\theta_n)\,L(\varphi) + \theta_n\,\mathcal{L}(\varphi) + (\theta-\theta_n)\,(\mathcal{L}-L)\,(\varphi) + \nu \tag{8.19}$$

whence,

$$\partial_t\varphi = (1-\theta)\,L(\varphi) + \theta\,\mathcal{L}(\varphi) + \nu \tag{8.20}$$

and since $\theta > \theta^*$ this yields a contradiction. It then follows that $\theta^* = 1$. We prove now the solvability of the equation for $\theta = 1$.
To this end consider a sequence $\theta_k \to 1$ and the corresponding sequence of solutions $\varphi_k \in \mathcal{E}_{T;\sigma}$ to :

$$\partial_t\varphi_k = (1-\theta_k)\,L(\varphi_k) + \theta_k\,\mathcal{L}(\varphi_k) + \nu. \tag{8.21}$$

By Lemma 7.4 we have

$$|||\varphi_k||| \leq C||\nu||_{Y^\sigma_{3/2,2+\delta}}. \tag{8.22}$$

Therefore,

$$R^{(3+\lambda)/2}\,N_{2;\,\sigma}\,(\varphi_k;\,R,\,t_0) \leq C||\nu||_{Y^\sigma_{3/2,2+\delta}},\ \ \forall R>1,,\ \forall t_0\in(0,T), \tag{8.23}$$

$$R^{3/2}\,M_{2;\,\sigma}\,(\varphi_k;\,R,\,t_0) \leq C||\nu||_{Y^\sigma_{3/2,2+\delta}},\ \ \forall\, 0<R<1,\ \forall t_0\in(0,T) \tag{8.24}$$

$$\max\left\{x^{3/2},x^{(3+\lambda)/2}\right\}|\varphi_k(x,t)| \leq C||\nu||_{Y^\sigma_{3/2,2+\delta}},\ \forall x\in(R/2,2R),\quad \forall R>0\ \forall t\in(0,T). \tag{8.25}$$

The sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is then weakly compact in $L^2(t_0,t_0+R^{-(\lambda-1)/2,T};H^\sigma(R/2,2R))$ for all $R>0$ and $t_0\in(0,T]$. Therefore, using a diagonal procedure, there exists a sub sequence, still denoted $\{\varphi_k\}_{k\in\mathbb{N}}$, and a function φ defined in all $\mathbb{R}^+\times(0,T]$ such that φ_k converges to φ weakly in $L^2((0,T);H^\sigma(R_1,R_2))$ for all $R_2>R_1>0$. Since the left hand sides in the inequalities (8.23)-(8.25) are all of them convex functions of φ_k, these inequalities are preserved under weak limits. Therefore

$$R^{(3+\lambda)/2}\,N_{2;\,\sigma}\,(\varphi;\,R,\,t_0) \leq C||\nu||_{Y^\sigma_{3/2,2+\delta}},\ \ \forall R>1,\ \text{for a. e. }\ t_0\in(0,T), \tag{8.26}$$

$$R^{3/2}\,M_{2;\,\sigma}\,(\varphi;\,R,\,t_0) \leq C||\nu||_{Y^\sigma_{3/2,2+\delta}},\ \ \forall\, 0<R<1,\ \text{for a. e. }\ t_0\in(0,T) \tag{8.27}$$

$$\max\left\{x^{3/2},x^{(3+\lambda)/2}\right\}|\varphi(x,t)| \leq C||\nu||_{Y^\sigma_{3/2,2+\delta}},\ \forall x\in(R/2,2R),\quad \forall R>0,\ \text{for a. e. }\ t\in(0,T) \tag{8.28}$$

whence $\varphi \in \mathcal{E}_{T;\sigma}$.

On the other hand, it is possible to pass to the limit in the equation (8.21) in the weak sense of $L^2(0,T;H^\sigma(R_1,R_2))$ for any $R_2 > R_1 > 0$ to obtain that $\varphi \in L^2(0,T;H^\sigma_{loc}(\mathbb{R}^+)) \cap H^1(0,T;H^{\sigma-1/2}_{loc}(\mathbb{R}^+))$ is a solution of

$$\partial_t \varphi = \mathcal{L}(\varphi) + \nu. \tag{8.29}$$

in the sense of distributions.

Finally, in order to prove uniqueness let us assume that φ_1 and φ_2 are two solutions of (8.29). Then, the function $\psi = \varphi_1 - \varphi_2$ satisfies,

$$\partial_t \psi = \mathcal{L}(\psi), \quad \psi(x,0) = 0.$$

and Lemma 7.4 for $\theta = 1$ and $\nu = 0$ shows that $\psi = 0$ and uniqueness holds. □

Remark 8.4 *In all this argument we have needed the condition $\sigma > 1$ in order to estimate the term A_2 (cf. for example Lemma 6.5 and Lemma 7.3). On the other hand, the main property of the spaces H^σ that is actually used is that they are an algebra, and this only requires $\sigma > 1/2$. An alternative approach, in order to avoid the more stringent condition $\sigma > 1$ could be to estimate the terms A_2 in Section 6 using σ Sobolev derivatives, with $\sigma \in (1/2, 1)$, and an almost half derivative as does the norm $[\,\cdot\,]$ defined in (2.21) (2.20). On the other hand, the condition $\sigma < 2$ has been imposed only in order to avoid further technicalities.*

9 Proof of Theorem 2.2

Consider the function $F_{R,t_0}(X,\tau)$ defined in (2.20). The function $\Psi(X,\tau) = R^{(3+\lambda)/2} F_{R,t_0}(X,\tau)$ satisfies equation (7.64) with $\theta = 1$. Then, using (3.10) we obtain

$$R^{(3+\lambda)/2} \left(\int_{t_0}^{\min(t_0 + R^{-(\lambda-1)/2}, T)} \int_{\mathbb{R}} |\widehat{F}_{R,t_0}(k,\tau)|^2 \, |k|^{2\,\sigma} \min\{|k|,\, R\} \, dk, dt \right)^{1/2} \\ \leq C \left(|||\varphi||| + ||\nu||_{Y_{3/2,2+\delta}} \right)$$

whence Theorem 2.2 follows. □

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