ON THE FUNDAMENTAL SOLUTION OF A LINEARIZED UEHLING-UHLENBECK EQUATION.

M. Escobedo[1], S. Mischler[2], J. J. L. Velázquez[3]

Abstract. In this paper we describe the fundamental solution of the equation that is obtained linearizing the Uehling-Uhlenbeck equation around the steady state of Kolmogorov type $f(k) = k^{-7/6}$. Detailed estimates on its asymptotics are obtained.

Contents of the paper.

1. Introduction.
2. The linearized Problem: Carleman equation.
3. Solving Carleman equation using the Wiener-Hopf method.
4. The fundamental solution to the Cauchy problem.
6. Appendix A. Properties of the kernel.
7. Appendix B. The Fourier transform of the kernel.
8. Appendix C. Auxiliary results.

1 Introduction.

This paper is devoted to the analysis of several mathematical properties of the Uehling Uhlenbeck equation. This equation, introduced by L. W. Nordheim in [16] and by E. A. Uehling and G. E. Uhlenbeck in [21], describes the

[1]Departamento de Matemáticas, Universidad del Pais Vasco/EHU, Bilbao 48080, Spain.
[2]CEREMADE, Université Paris-Dauphine, Place du Marechal De Lattre De Tassigny, 75775 Paris Cedex 16 France
[3]Departamento de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense, Madrid 28040, Spain.

evolution in the momentum space of a weakly interacting gas of bosons. In the homogeneous case, this equation has the form:

$$\frac{\partial f}{\partial t}(t,p) = Q(f(t,.))(p) \tag{1.1}$$

where $f \equiv f(t,p)$ is the particle density in the momentum space at time t. The precise form of the collision kernel Q is given by:

$$Q(f)(p_1) = \int_{(\mathbb{R}^3)^3} W(p_1,p_2,p_3,p_4)\, q(f) dp_2 dp_3 dp_4 \tag{1.2}$$

$$q(f) = f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \tag{1.3}$$

$$W(p_1,p_2,p_3,p_4) = w(p_1,p_2,p_3,p_4)\delta(p_1+p_2-p_3-p_4)\cdot \delta(|p_1|^2+|p_2|^2-|p_3|^2-|p_4|^2) \tag{1.4}$$

From now on for convenience we write, $f_i \equiv f(\cdot,p_i)$, $i = 1\cdots 4$, where δ represents the Dirac measure and w is basically the differential cross section. The function w is determined by the specific kind of interaction under consideration. Since boson-boson is a short range interaction it can be assumed that w is constant (cf. [10] vol. 3), and therefore it can be chosen as $w = 1$ after rescaling the time.

If we assume that the gas is isotropic, we may write $f(t,p) = f(t,|p|) \equiv f(t,k)$ with $k = |p|^2$. The equation (1.1) reduces then to

$$\frac{\partial f}{\partial t}(k_1,t) = \int\int_{D(k_1)} W(k_1,k_2,k_3,k_4)\, q(f)\, dk_3 dk_4 \tag{1.5}$$

(see for instance [19]), where $q(f)$ is as in (1.3) and $D(k_1)$ is defined by means of

$$D(k_1) \equiv \{(k_3,k_4)) : k_3+k_4 \geq k_1\} \tag{1.6}$$

and finally

$$W(k_1,k_2,k_3,k_4) = \frac{\min\left(\sqrt{k_1},\sqrt{k_2},\sqrt{k_3},\sqrt{k_4}\right)}{\sqrt{k_1}} \tag{1.7}$$

$$k_2 = k_3+k_4-k_1. \tag{1.8}$$

There are several reasons for considering the equation (1.5) with singular initial data as $k \to 0$. More precisely, data behaving as $k^{-7/6}$. This particular type of behaviour arises in applications of the equation (1.1) in Bose Einstein condensation, as in references [6, 18, 19], and in weak turbulence, as in [4, 3]. The interpretation of solutions of equation (1.1) behaving as $|p|^{-7/3} \equiv k^{-7/6}$ when $k \to 0$ is the existence of a net flux or sink of particles at the origin $p = 0$ (cf. [4, 3, 6, 9]). In this paper we restrict our analysis to the study of isotropic solutions by simplicity. The study of the stability properties of the perturbation of the singular solution $|p|^{-7/3}$ of (1.1)-(1.4) is an interesting problem that, however, will not be considered in this paper.

Our purpose is to develop a rigorous mathematical theory of well posedness for (1.5) whose solutions behave like $k^{-7/6}$ as $k \to 0$. In order to do that, our approach will consist in deriving a suitable semigroup theory for the problem which is obtained by linearizing (1.5) around $k^{-7/6}$. We use the semigroup obtained in this paper in order to study the nonlinear problem in a forthcoming paper. We remark that global solutions for an analogous equation, with a modified version of the kernel, has been studied by X. G. Lu in [13, 14].

In the analysis made in this paper we will assume that the density f is large and therefore we will neglect the quadratic terms in the collision integral. The reason for this assumption is that the values of f are very large near the origin for the singular stationary solution $k^{-7/6}$. In the companion paper [5] we will construct solutions of the whole nonlinear equation (1.5)-(1.8) behaving like the singular solution $k^{-7/6}$ near the origin. For such solutions the quadratic terms of the collision integral give a negligible contribution. We then approximate the Uehling Uhlenbeck equation by the following one, that has better homogeneity properties:

$$\frac{\partial f}{\partial t}(k_1, t) = \widetilde{Q}(f)(k_1) \equiv \int\int_{D(k_1)} W(k_1, k_2, k_3, k_4)\, \tilde{q}(f)\, dk_3 dk_4 \tag{1.9}$$

where

$$\tilde{q}(f) = f_3 f_4 (f_1 + f_2) - f_1 f_2 (f_3 + f_4) \tag{1.10}$$

and W and $D(k_1)$ are as in (1.6) and (1.7)-(1.8). This equation has been extensively studied in the context of weak turbulence (cf. [4] and references therein).

The main contribution of this paper is to study the fundamental solution corresponding to the linearization of the equation (1.9), (1.10) around the solution $k^{-7/6}$. To this end, we make an extensive use of the ideas in [4]. In that paper, Balk and Zakharov have developed a very general technique for the study of linear kinetic equations with homogeneous kernels. In this paper we have adapted those ideas to the specific problem considered here and, in particular we have obtained rather refined L^∞ estimates for the fundamental solution associated to the equation that will be needed for the study of the nonlinear problem (1.1)-(1.4). From the physical point of view the solutions of (1.1)-(1.4) behaving like $k^{-7/6}$ near the origin can be thought as solutions having a flux of particles towards the origin (cf. [3], [5]).

There are relatively few mathematical results about the equation (1.1) or related and only some of them are rigorous from the point of view of analysis. The formal derivation of that equation taking as a starting point the hamiltonian for a system of many interacting particles and using methods of statistical physics was first given in [21]. This derivation is by now a standard textbook result and can be found, for instance in [2]. The equation (1.1) shares many properties with the classical Boltzmann equation. In particular it has an increasing entropy given by

$$H(f) := \int_{\mathbb{R}^3} \{(1 + f(p)) \ln(1 + f(p)) - f(p) \ln f(p)\} \; dp \tag{1.11}$$

The stationary solutions of the simplified equation (1.9) (1.10) of the form $f(k) = k^\alpha$ have been studied very much in detail. In particular it was proved in [22] that the only solution to the equation $Q(f) = 0$ of this kind with enough integrability conditions to ensure that the integral term $Q(f)$ is well defined is $f(k) = k^{-7/6}$. The proof of this result is obtained in [4] using some interesting symmetry properties of the equation (1.9) (1.10) (see [3] for a different proof of the same result). Finally we remark that the numerical simulations of [9] and [18, 19] indicate that some solutions of (1.9) (1.10) can blow up in a finite time in a self similar form. A fully rigorous proof of such a blow up phenomena has not been obtained yet. This blowing up behavior is very different from the type of behaviors exhibited by the classical Boltzmann equation. Also the type of singular behaviour that we obtain in this paper and in [5] is rather different from the usual solutions that have been so far obtained in Boltzmann or other kinetic equations.

The basic idea of the paper is the following. First we linearized in the equation (1.9) around the singular stationary solution $k^{-\frac{7}{6}}$. More precisely we write $f = k^{-\frac{7}{6}} + F$. The correction F then satisfies a linear equation of the form

$$F_t(k,t) = \frac{1}{k^{\frac{1}{3}}} \int_0^\infty \frac{\mu\left(\frac{k}{k'}\right)}{k'} F(k',t)\, dk' \tag{1.12}$$

where μ is a measure in $[0,\infty)$.

The integral operator on the right of (1.12) is homogeneous of degree zero. Using this fact, it follows that, after taking Mellin's transform in the variable k and Laplace's transform in t, (1.12) becomes the following delay equation in the complex plane:

$$zG(z,\xi) = G\left(z, \xi - \frac{i}{3}\right) \Phi\left(\xi - \frac{i}{3}\right) + G_0(\xi) \tag{1.13}$$

where z is the Laplace variable, G is the Laplace-Mellin transform of F, G_0 is the Mellin transform of $F(\xi,0) = F_0(\xi)$ and Φ is an explicit meromorphic function that can be explicitly computed, but it has a complicated functional form in terms of hypergeometric functions.

The equation (1.12) makes sense only if suitable decay assumptions are made for F as $k \to 0^+,\ k \to \infty$. Such decay assumptions provide some analyticity conditions for $G(z,\cdot)$ with the form:

$$G(z,\cdot) \text{ analytic in } \quad \xi \in \left(\frac{4}{3}, \frac{11}{6}\right) \tag{1.14}$$

The problem (1.13), (1.14) can be explicitly solved using the classical Wiener-Hopf method. Using the resulting formula we will obtain a solution $F(k,t)$ of (1.12) with initial data $F(k,0) = \delta(k - k_0)$ using the inversion formula for the Laplace-Mellin transform. Most of this paper consists in deriving such explicit formula as well as "a priori" estimates for the derived solution.

Unfortunately, the computations required to implement the plan sketched above are cumbersome for several reasons. First, the measure $\mu(\cdot)$ above is complicated due to the cumbersome structure of the collision kernel $\widetilde{Q}(f)$ in (1.9). Second, the formulae obtained using the Wiener-Hopf method are complicated since they require the computation of several singular integrals. Finally, the last step requires to compute the asymptotics of the Laplace-Mellin inversion of the function $G(z,\xi)$ that contain some additional integrations. On the other hand, in many of the previous steps we need to know the position of the complex zeroes of an involved meromorphic function in a strip of the complex plane, something that we have made numerically.

The plan of the paper is the following. In Section 2, we linearise the equation (1.9), (1.10) around the steady state $f(k) = k^{-7/6}$ and obtain the Cauchy problem (2.9). We state in Theorem 2.2 our main result about the existence, uniqueness and behaviour of the solutions. In order to introduce and motivate the natural functional framework we perform a first change of variables and obtain the new formulation of the problem in (2.13). This one is then reduced to the Carleman type equation (2.30) and (2.31). In Section 3 we solve in detail the Carleman equation using the classical Wiener-Hopf method. In Section 4 we derive several estimates for the resulting fundamental solution of the linearized problem. Finally, in several Appendices at the end of the paper we have collected some technical results which are used in the arguments.

2 The Linearized problem: Carleman equation

We first proceed to linearise the equation (1.6)-(1.10) around the particular solution $k^{-7/6}$. To this end we write

$$f = k^{-7/6} + F.$$

Plugging this expression into (1.5)-(1.8) and keeping only the terms which are linear with respect to F we obtain the linearized Uehling Uhlenbeck equation

$$\frac{\partial F}{\partial t} = Q_L(F) \equiv \int_{D(k_1)} W(k_1, k_2, k_3, k_4)\, q_l(F)\, dk_3\, dk_4 \tag{2.1}$$

for $t > 0, k > 0$, where

$$\begin{aligned} q_l(F) = & \frac{1}{k_3^{7/6} k_4^{7/6}}(F_1 + F_2) + \frac{1}{k_4^{7/6}}\Big(\frac{1}{k_1^{7/6}} + \frac{1}{k_2^{7/6}}\Big)F_3 + \frac{1}{k_3^{7/6}}\Big(\frac{1}{k_1^{7/6}} + \frac{1}{k_2^{7/6}}\Big)F_4 \\ - & \frac{1}{k_1^{7/6}}\frac{1}{k_2^{7/6}}(F_3 + F_4) - \frac{1}{k_1^{7/6}}\Big(\frac{1}{k_3^{7/6}} + \frac{1}{k_4^{7/6}}\Big)F_2 - \frac{1}{k_2^{7/6}}\Big(\frac{1}{k_3^{7/6}} + \frac{1}{k_4^{7/6}}\Big)F_1 \end{aligned} \tag{2.2}$$

We express, in the following proposition the collision integral Q_L in a more suitable way for our next calculations.

Proposition 2.1 *Equation (2.1)-(2.2) might be written as*

$$\frac{\partial F}{\partial t} = -\frac{a}{k^{1/3}} F(k) + \frac{1}{k^{4/3}} \int_0^\infty K(\frac{r}{k})\, F(r)\, dr, \tag{2.3}$$

where $K \in \mathbf{C}^\infty\left((0,1) \cup (1,+\infty)\right)$ *satisfies:*

$$K(r) \sim a_1 r^{1/2} \quad as \quad r \to 0 \tag{2.4}$$

$$K(r) \sim a_2 r^{-7/6} \quad as \quad r \to +\infty \tag{2.5}$$

$$K(r) \sim a_3(1-r)^{-5/6} + a_4 + \mathcal{O}((1-r)^{1/6}) \quad as \quad r \to 1^- \tag{2.6}$$

$$K(r) \sim a_5(r-1)^{-5/6} + a_6 + \mathcal{O}((1-r)^{1/6}) \quad as \quad r \to 1^+, \tag{2.7}$$

and $a_i \in \mathbb{R}$, $i = 1, \cdots 6$. *The constants* $a > 0$, a_i *and the kernel* $K(r)$ *can be explicitely computed and they are given in the Appendix A.*

Proof. Using the symmetries of the right hand side of (2.1), the equation may be written as follows:

$$\begin{aligned} \frac{\partial F}{\partial t} &= F_1 \int_{D(k_1)} W\{\frac{1}{k_3^{7/6}\, k_4^{7/6}} - \frac{1}{k_2^{7/6}}(\frac{1}{k_3^{7/6}} + \frac{1}{k_4^{7/6}})\}\, dk_3\, dk_4 \\ &+ 2\int_{D(k_1)} F_3\, W\,\{\frac{1}{k_4^{7/6}}(\frac{1}{k_1^{7/6}} + \frac{1}{k_2^{7/6}}) - \frac{1}{k_1^{7/6}}\frac{1}{k_2^{7/6}}\}\, dk_3\, dk_4 \\ &+ \int_{D(k_1)} F_2\, W\,\{\frac{1}{k_3^{7/6}\, k_4^{7/6}} - \frac{1}{k_1^{7/6}}(\frac{1}{k_3^{7/6}} + \frac{1}{k_4^{7/6}})\}\, dk_3\, dk_4 \\ &\equiv I_3\, F_1 + I_1 + I_2, \end{aligned} \tag{2.8}$$

where W is as in (1.7).

Tedious but elementary computations, which are sketched in Appendix A1, show that $I_1 + I_2$ can be written as the integral term in (2.3) and I_3 is exactly $-ak^{-1/3}$. The asymptotics (2.5)-(2.7) can be obtained by means of explicit computations.

As we already said in the Introduction, the main goal of this work is to obtain a semi explicit expression of the solution to the Cauchy problem associated to equation (2.3). To this end we construct the fundamental solution $F(t, k, k_0)$ of the Uehling Uhlenbeck equation, which solves:

$$\begin{cases} F_t(t,k,k_0) = -\frac{a}{k^{1/3}} F(t,k,k_0) + \frac{1}{k^{4/3}} \int_0^\infty K(\frac{r}{k})\, F(t,r,k_0)\, dr, \ t > 0, k > 0, \\ F(t,k,k_0) = \delta(k - k_0), \quad k_0 > 0. \end{cases} \tag{2.9}$$

We transform now (2.9) to a more convenient form by means of the following change of variables:

$$r = e^y, \quad k = e^x; \tag{2.10}$$

$$F(t,k,1) = \mathcal{G}(t,x), \quad K(r/k) = K(e^{-(x-y)}) = e^{x-y}\mathcal{K}(x-y), \tag{2.11}$$

with

$$\mathcal{K}(x) = e^{-x}K(e^{-x}). \tag{2.12}$$

We are then lead to the following Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{G}(t,x) = e^{-x/3}\left(-a\mathcal{G}(t,x) + \displaystyle\int_{-\infty}^{\infty} \mathcal{K}(x-y)\,\mathcal{G}(t,y)\,dy\right), \\ \mathcal{G}(0,x) = \delta(x), \end{cases} \tag{2.13}$$

for $t > 0$ and $x \in \mathbb{R}$.

Analysing the solution of (2.13), is crucial to understand the analyticity properties of the function $\Phi(\xi) = -a + \widehat{\mathcal{K}}(\xi)$, where $\widehat{\mathcal{K}}$ is the Fourier transform of $\mathcal{K}$. It turns out that this function is meromorphic with explicit poles in the imaginary axes (cf. Appendix B). On the other hand, the positions of the zeros of $\Phi(\xi)$ determine the asymptotics of $\mathcal{G}(t,x)$. The zeros of Φ that are closer to the line $\mathcal{I}m\xi = 4/3$ are placed at $\xi = 7i/6$, $\xi = 13i/6$ and there are two zeros placed symmetrically with respect to the imaginary axes at the points

$$\xi = \pm u_0 + v_0 i \tag{2.14}$$

where the values of u_0 and v_0 are computed numerically and are given by

$$u_0 = 0.331..., \; v_0 = 1.84020....$$

2.1 Functional framework.

As a first step we make precise in which class of functions it is natural to solve (2.13).

Due to (2.4) and (2.5), the behaviour of the kernel $\mathcal{K}$ is

$$\mathcal{K}(x) \sim a_1 e^{\frac{x}{6}} \quad \text{as} \quad x \to -\infty \tag{2.15}$$

$$\mathcal{K}(x) \sim a_2 e^{-\frac{3}{2}x} \quad \text{as} \quad x \to +\infty. \tag{2.16}$$

Therefore, in order for the integral term in (2.13) to be convergent it is natural to assume :

$$|\mathcal{G}(t,x)| \leq Ce^{-Mx} \quad \text{for} \quad x<0, \qquad |\mathcal{G}(t,x)| \leq Ce^{-mx} \quad \text{for} \quad x>0 \quad (2.17)$$

for some $m > -1/6$ and $M < 3/2$, and where from now on, C is a generic constant that might change from line to line.
Suppose now that $\mathcal{G}$ is a function satisfying (2.17). Then, by (2.15) and (2.16)

$$|\int_{-\infty}^{\infty} \mathcal{K}(x-y)\mathcal{G}(y)dy| \leq C\left(e^{-\frac{3}{2}x} + e^{-mx}\right) \tag{2.18}$$

for $x > 0$. We deduce from (2.18) that the right hand side of (2.13) may be estimated as:

$$e^{-x/3}\left|-a\mathcal{G}(x) + \int_{-\infty}^{\infty} K(x-y)\,\mathcal{G}(y)\,dy\right| \leq C\left(e^{-(m+\frac{1}{3})x} + e^{-\frac{11}{6}x}\right), \tag{2.19}$$

for $x > 0$. Therefore, for any initial data of (2.13), say compactly supported, it would follow from the equation that (2.17) for $x > 0$ holds for some $m \in (1/6, 11/6]$. Iterating the argument, we deduce (2.17) for $x > 0$ with $M < 3/2$ and $m = 11/6$.

Notice that, since we are interested in solving the problem (2.13) whose initial data is a Dirac mass, one would have an additional term $e^{-3x/2}$ in the right hand side of (2.18). We define then:

$$\mathcal{U}(M) = \{H \in L^\infty_{loc}(\mathbb{R});\ H \text{ satisfies (2.17) with } m = 11/6\},$$
$$\mathcal{V}(M,x_0) = \{\mathcal{G};\ \mathcal{G}(x) = \alpha\,\delta(x-x_0) + H(x), \quad \alpha \in \mathbb{R},\ H \in \mathcal{U}(M)\},\ x_0 \in \mathbb{R}. \tag{2.20}$$

These spaces have a natural translation in the k variable by means of (2.10), (2.11), namely:

$$\tilde{\mathcal{U}}(M) = \{h \in L^\infty_{loc}(0,+\infty);\ |h(k)| \leq Ck^{M-1},\ \text{for } k \leq 1, \text{ and } |h(k)| \leq Ck^{5/6},\ \text{for } k > 1\}$$

$$\tilde{\mathcal{V}}(M,k_0) = \{F;\ F(k) = \alpha\,\delta(k-k_0) + h(k),\ \alpha \in \mathbb{R},\ h \in \tilde{\mathcal{U}}(M)\}, \quad k_0 > 0. \tag{2.21}$$

We finally remark that the Fourier transform of the elements of $\mathcal{U}(M)$, $\mathcal{V}(M, x_0)$ are analytic in suitable strips of the complex plane. More precisely, let us define,

$$\widehat{\mathcal{G}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} \mathcal{G}(x)\, dx. \tag{2.22}$$

It is easily checked that for any $\mathcal{G} \in \mathcal{V}(M, x_0)$, the function $\widehat{\mathcal{G}}$ is analytic in the strip

$$S_M = \{\xi;\ \xi = u + iv,\ M < v < 11/6,\, u \in \mathbb{R}\}\,. \tag{2.23}$$

Moreover, notice that for $\mathcal{G} \in \mathcal{V}(M, x_0)$, the corresponding Fourier transform $\widehat{\mathcal{G}}$ is a uniformly bounded function in such strips.

2.2 The main result.

The scaling properties of (2.3) suggest the following functional dependence:

$$F(t, k, k_0) = \frac{1}{k_0} F\left(\frac{t}{k_0^{1/3}}, \frac{k}{k_0}, 1\right). \tag{2.24}$$

Therefore, it is enough to study (2.9) for $k_0 = 1$. Our main result is the following.

Theorem 2.2 *Assume that $M \in (7/6, 3/2)$. Then, for all $k_0 > 0$, there exists a unique solution $F(t, \cdot, k_0)$ of (2.9) in the class of functions $\tilde{\mathcal{V}}(M, k_0)$. Moreover, $F(t, \cdot, k_0) \in \tilde{\mathcal{V}}(7/6, k_0)$, has the form given in (2.24). For $k \in (0, 2)$ the function $F(t, k, 1)$ can be written as:*

$$F(t, k, 1) = e^{-a\,t} \delta(k-1) + \sigma(t)\, k^{-7/6} + \mathcal{R}_1(t, k) + \mathcal{R}_2(t, k),$$

where $\sigma \in \mathbf{C}[0, +\infty)$ satisfies:

$$\sigma(t) = \begin{cases} A\, t^4 + \mathcal{O}(t^{4+\varepsilon}) & \text{as } t \to 0^+, \\ \mathcal{O}(t^{-(3v_0 - 5/2)}) & \text{as } t \to +\infty \end{cases} \tag{2.25}$$

$\mathcal{R}_1$ satisfies,

$$\mathcal{R}_1(t, k) \equiv 0 \quad \text{for } |k-1| \geq \frac{1}{2}, \tag{2.26}$$

$$|\mathcal{R}_1(t, k)| \leq C \frac{e^{-(a-\varepsilon)t}}{|k-1|^{5/6}} \quad \text{for } |k-1| \leq \frac{1}{2}, \tag{2.27}$$

and $\mathcal{R}_2$ satisfies:

$$\mathcal{R}_2(t,k) \leq \begin{cases} \dfrac{C}{t^{5/2+\varepsilon}}\left(\dfrac{t^3}{k}\right)^{\tilde{b}} & \text{for } 0 \leq t \leq 1 \\ \dfrac{C}{t^{3v_0-\varepsilon}}\left(\dfrac{t^3}{k}\right)^{\tilde{b}} & \text{for } t > 1. \end{cases} \tag{2.28}$$

On the other hand, for $k > 2$,

$$F(t,k,1) \leq \begin{cases} \dfrac{C}{t^{\frac{9}{2}+\varepsilon}}\left(\dfrac{t^3}{k}\right)^{\frac{11}{6}} & \text{for } 0 \leq t \leq 1 \\ \dfrac{C}{t^{1+3v_0-\varepsilon}}\left(\dfrac{t^3}{k}\right)^{\frac{11}{6}} & \text{for } t > 1, \end{cases} \tag{2.29}$$

In these formulae, A is an explicit numerical constant, $\varepsilon > 0$ is an arbitrarily small number, $\tilde{b}$ is an arbitrary number in the interval $(1, 7/6)$, and v_0 as in (2.14). The constant C depends on ε and $\tilde{b}$ but is independent on t.

2.3 Carleman equation.

In order to solve (2.13) we transform this equation into a Carleman equation taking the Fourier transform in the x variable and the Laplace transform in the t variable. We define the Fourier transform with respect to the x variable, $\widehat{\mathcal{G}}(t,\xi)$, as in (2.22) and the Laplace transform of this last function with respect to t by means of

$$G(z,\xi) = \int_0^\infty e^{-zt}\hat{\mathcal{G}}(t,\xi)dt.$$

In this manner, (2.13) becomes:

$$zG(z,\xi) = G(z,\xi - \frac{i}{3})\Phi(\xi - \frac{i}{3}) + \frac{1}{\sqrt{2\pi}}, \tag{2.30}$$

where $\Phi(\xi) = -a + \widehat{\mathcal{K}}(\xi)$ and $\widehat{\mathcal{K}}$ is the Fourier transform of $\mathcal{K}$.

Since we are looking for solutions of (2.13) in the class $\widetilde{\mathcal{V}}(M,0)$ with M as in the statement of Theorem 2.2, it follows that $G(z,\cdot)$ is analytic in a

strip S_M (cf. (2.23)) whose width is larger that 1/3.
Therefore, the basic problem that we need to solve is the following:

For any $z \in \mathbb{C}$, $\mathcal{R}ez > 0$, find a function $G(z, \cdot)$ analytic in the strip $S_M = \{\xi;\ \xi = u + iv,\ M < v < 11/6,\ u \in \mathbb{R}\}$ for some $M < 3/2$ (2.31) and satisfying (2.30) on S_M.

Problem (2.31) may be solved by means of suitable Wiener-Hopf types arguments as introduced in this context in [4]. The analyticity properties of the function $\Phi(\xi)$ play a crucial role solving (2.31). Therefore, we summarize the relevant properties of Φ in the Appendix B. Concerning the analyticity of G with respect to the z variable, it turns out that it is possible to extend G analytically to $z \in \mathbb{C} \setminus \mathbb{R}^-$ as it will be explained in the next section.

3 Solving Carleman equation using Wiener-Hopf method.

3.1 Reformulating the Carleman equation.

Problem (2.31) might be solved in a more convenient manner after transforming the strip S_M in the exterior of a line by means of a conformal mapping. We will prove Theorem 2.2, assuming by definiteness that $M = 4/3$. This is useful inorder to discharge the notation at several points, but the same arguments could be made for any value of $M \in (7/6, 3/2)$. Let us introduce the following new set ofvariables:

$$\zeta = T(\xi) \equiv e^{6\pi(\xi - \frac{4}{3}i)} \tag{3.1}$$

$$g(z, \zeta) = G(z, \xi) \tag{3.2}$$

$$\widetilde{\varphi}(\zeta) = \Phi(\xi). \tag{3.3}$$

Notice that the transformation T transforms the complex plane $\mathbb{C}$ in the Riemann surface associated to the logarithmic function that we will denote from now on as $\mathcal{S}$. The function $\tilde{\varphi}$ is meromorphic in this Riemann surface. We can characterize uniquely each of the sheets of this Riemann surface by means of the argument θ of ζ, where $\zeta = re^{i\theta}$. Notice in particular that the strip $S_{4/3}$, where by assumption the function $G(z, \xi)$ is analytic, is transformed by means of (3.1), into the portion of $\mathcal{S}$ such that $\theta \in (0, 3\pi)$.

Therefore, the function $g(z,\cdot)$ is analytic in that region. Let us denote as D the following portion of the Riemann surface $\mathcal{S}$:

$$D = \{\zeta \in \mathcal{S};\ \zeta = re^{i\theta},\ r > 0,\ 0 < \theta < 2\pi\}. \tag{3.4}$$

The definition of the function $g(z,\zeta)$ as well as (2.30) imply that g solves the following problem:

$$zg(z, x - i0) = \varphi(x)\, g(z, x + i0) + \frac{1}{\sqrt{2\pi}} \quad \text{for all } x \in \mathbb{R}^+ \tag{3.5}$$

$$g \text{ is analytic and bounded in } \ D, \tag{3.6}$$

where, for any $x \in \mathbb{R}^+$:

$$g(z, x + i0) = \lim_{\varepsilon\to 0} g(z, xe^{i\varepsilon}),\ \ g(z, x - i0) = \lim_{\varepsilon\to 0} g(z, xe^{i(2\pi-\varepsilon)}) \tag{3.7}$$

$$\varphi(x) = \lim_{\varepsilon\to 0} \widetilde{\varphi}(xe^{i\varepsilon}). \tag{3.8}$$

Problem (3.5)-(3.8) is explicitly solvable using the Wiener Hopf method. The result is the following:

Theorem 3.1 *For any $z \in \mathbb{C} \setminus \mathbb{R}^-$, there exists a unique bounded function $g = g(z,\cdot)$, solving (3.5)-(3.8) given by:*

$$g(z,\zeta) = \frac{1}{(2\pi)^{3/2} i}\frac{\zeta}{z}\int_0^\infty \frac{M(z, \lambda - i0)}{M(z,\zeta)}\frac{d\lambda}{\lambda\,(\lambda - \zeta)} \tag{3.9}$$

where,

$$M(z,\zeta) = \exp\left[\frac{1}{2\pi i}\int_0^\infty \ln\left(\frac{\varphi(\lambda)}{z}\right)\left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right)d\lambda\right], \tag{3.10}$$

$\lambda_0 \in \mathbb{C} \setminus \mathbb{R}^+$ is arbitrary, and the logarithmic function is defined in such a way that:

$$\mathcal{I}m\left(\lim_{\lambda\to 0^+}\ln\left(\frac{\varphi(\lambda)}{z}\right)\right) = \mathcal{I}m\left(\ln\left(-\frac{a}{z}\right)\right) \in (-2\pi, 0) \tag{3.11}$$

and it is extended analytically for λ moving along the positive real line.

Remark 3.2 *The possibility to extend the function* $\ln\left(\frac{\varphi(\lambda)}{z}\right)$ *analytically as indicated in the Theorem 3.1 is not automatic but it will be obtained during the proof of this result.*

Remark 3.3 *At a first glance, the arbitrariness of the number* λ_0*, could yield several different functions* $g(z,\zeta)$*. Nevertheless, it turns out that the dependence on* λ_0 *disappears, in (3.9) as it will be seen in the proof of the Theorem 3.1.*

Proof of the Theorem 3.1. During the proof we will use several technical lemmae that we state and prove in Appendix C in order to avoid breaking the continuity of the main argument.
Since the function $\varphi(\lambda)$ does not vanish in a neighborhood of $\mathbb{R}^+$ (cf. (P-2) in Appendix B), we can define, the function $h(\lambda) \equiv \ln(\varphi(\lambda)/z)$ in such domain. Moreover, we can uniquely prescribe this function setting

$$\lim_{\lambda\to 0} \mathcal{I}\,m\,(h(\lambda)) = \mathcal{I}\,m\left(\ln(-\frac{a}{z})\right) \in (-2\pi, 0) \tag{3.12}$$

Since h is bounded in a neighborhood of $\mathbb{R}^+$, we can define the function M as in (3.10).

Notice that deforming the contour $\zeta \in [0,\infty)$ to the contour $\zeta \in [0, x-\varepsilon] \cup \left\{\zeta = \varepsilon e^{i\theta},\ \theta \in [-\pi, 0]\right\} \cup [x+\varepsilon, \infty)$, using the analiticity properties of $\log\left(\frac{\varphi(\zeta)}{z}\right)$ and taking the limit $\varepsilon \to 0^+$ we obtain:

$$\ln\left(M\left(z, x+i0^+\right)\right) = \frac{1}{2}\ln\left(\frac{\varphi(x)}{z}\right) + \frac{1}{2\pi i} PV \int_0^\infty \ln\left(\frac{\varphi(\lambda)}{z}\right)\left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right) d\lambda$$

where $M\left(z, x+i0^+\right)$ is as in (3.7). A similar argument yields:

$$\ln\left(M\left(z, x-i0^+\right)\right) = -\frac{1}{2}\ln\left(\frac{\varphi(x)}{z}\right) + \frac{1}{2\pi i} PV \int_0^\infty \ln\left(\frac{\varphi(\lambda)}{z}\right)\left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right) d\lambda$$

whence, substractint these formulae we obtain:

$$\frac{1}{z}\varphi(x) = \frac{M(z, x+i0)}{M(z, x-i0)} \tag{3.13}$$

where $M(x \pm i0)$ are defined as in (3.7).
Plugging (3.13) into (3.5) we obtain

$$M(z, x - i0)\, g(z, x - i0) = M(z, x + i0)\, g(z, x + i0) + \frac{M(z, x - i0)}{\sqrt{2\pi} z}, \quad (3.14)$$

for all $x \in \mathbb{R}^+$. We now claim that

$$\frac{M(z, x - i0)}{z} = W(z, x + i0) - W(z, x - i0), \quad \text{for any } x > 0 \quad (3.15)$$

where:

$$W(z, \zeta) = \frac{1}{2\pi i} \int_0^\infty \frac{M(z, \lambda - i0)}{z} \left(\frac{1}{\lambda - \zeta} - \frac{1}{\lambda - \lambda_0} \right) d\lambda \quad (3.16)$$

and λ_0 is an arbitrary number in D.

Formula (3.15) is a consequence of the Plemej Sojoltski formula if $M(z, \lambda - i0)$ had good boundedness properties for $\lambda \to 0$ and $\lambda \to +\infty$. Such properties, in whose proof plays a crucial role property (P-4) in Appendix B, are summarized in Proposition C.1 in Appendix C.

Using (3.15) in (3.14) we deduce:

$$M(z, x - i0) g(z, x - i0) + W(z, x - i0) = M(z, x + i0) g(z, x + i0) + W(z, x + i0), \quad (3.17)$$

for all $x \in \mathbb{R}^+$.

It then follows that the function $C(z, \cdot)$ defined by means of:

$$C(z, \cdot) \equiv M(z, \cdot) g(z, \cdot) + \frac{W(z, \cdot)}{\sqrt{2\pi}} \quad (3.18)$$

is analytic in $\mathbb{C} \setminus \{0\}$. Due to the boundedness of $g(z, \cdot)$ as well as the estimates in Proposition C.1 and Proposition C.3, the function $C(z, \zeta)$ is bounded in compact sets and growths at most as $|\zeta|^{1-\delta}$ as $|\zeta| \to +\infty$, for some $\delta > 0$. Therefore, by Liouville's theorem, $C(z, \zeta)$ does not depend on ζ i. e.

$$\forall z \in \mathbb{C} \setminus \mathbb{R}^- : \quad C(z, \zeta) = C(z), \quad (3.19)$$

whence, by (3.18):

$$g(z,\zeta) = \frac{\sqrt{2\,\pi}C(z) - W(z,\zeta)}{\sqrt{2\pi}\,M(z,\zeta)}, \tag{3.20}$$

where,

$$C(z) = \frac{1}{\sqrt{2\,\pi}} lim_{\zeta\to 0,\zeta\in D} W(z,\zeta), \tag{3.21}$$

as it can be seen taking the limit of both sides of (3.18) as $\zeta \to 0$ and using the boundedness of g as well as (C.3) in Proposition C.1. Notice that the limit at the right hand side of (3.21) exists due to (C.3). The analyticity of $g(z,\cdot)$ in D follows from the analyticity of W, M as well as the fact that M does not vanish in D as it can be checked from (3.10). Finally we compute $\sqrt{2\,\pi}\,C(z) - W(z,\zeta)$. Using (3.21) and (3.16),

$$\sqrt{2\,\pi}\,C(z) - W(z,\zeta) = \frac{1}{2\pi i}\frac{\zeta}{z}\int_0^\infty M(z,\lambda - i0)\frac{d\lambda}{\lambda(\lambda-\zeta)}.$$

Plugging this formula into (3.20) we obtain (3.9)

3.2 The solution of the Carleman equation.

Using (3.9), we can immediately solve (2.31). The change of variables (3.1)-(3.3) yields

$$G(z,\xi) = g(z,T(\xi)). \tag{3.22}$$

From (3.9) we deduce that G, the solution of (2.31), is given by:

$$G(z,\xi) = \frac{3\,i}{2\pi\,z}\int_{\mathcal{I}m\,y=\frac{5}{3}} \left(\frac{m(z,y-i0)}{m(z,\xi)}\right)\frac{dy}{(e^{6\pi(y-\xi)}-1)},$$

where

$$m(z,\xi) \;=\; M(z,T(\xi)). \tag{3.23}$$

The z dependence of the quotient $\left(\frac{m(z,y-i0)}{m(z,\xi)}\right)$ may be computed explicitly as follows. Using (3.23), (3.1) and (3.10) we can rewrite m as

$$m(z,\xi) = \mathcal{V}(\xi)\ \exp[3i \int_{\mathcal{I}m\, y=(\frac{4}{3}-\delta)} \ln\left(\frac{z}{-a}\right) \times$$

$$e^{6\pi y}\left(\frac{1}{e^{6\pi y}-e^{6\pi\xi}} - \frac{1}{e^{6\pi y}-ae^{6\pi\delta i}}\right) dy] \tag{3.24}$$

$$\mathcal{V}(\xi) = \exp[-3i \int_{\mathcal{I}m\, y=\frac{4}{3}} \ln\left(\frac{\Phi(y+i0)}{-a}\right) \times \tag{3.25}$$

$$e^{6\pi y}\left(\frac{1}{e^{6\pi y}-e^{6\pi\xi}} - \frac{1}{e^{6\pi y}-ae^{6\pi\delta i}}\right) dy].$$

The function $\mathcal{V}(\xi)$ is analytic in the region $\mathcal{I}m\xi \in (4/3, 5/3)$. Moreover, we can extend $\mathcal{V}(\cdot)$ analytically to the strip $(4/3, 5/3+\varepsilon)$ with $0 < \varepsilon < \delta$ deforming the contour of the integral in (3.25), in order to avoid singularities. For instance for $\mathcal{I}m\xi \in (4/3+\varepsilon, 5/3+\varepsilon)$, $0 < \varepsilon < \delta$ the analytic extension of $\mathcal{V}$ is given by:

$$\mathcal{V}(\xi) = \exp[-3i \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon)} \ln\left(-\frac{\Phi(y)}{a}\right) \times \tag{3.26}$$

$$e^{6\pi y}\left(\frac{1}{e^{6\pi y}-e^{6\pi\xi}} - \frac{1}{e^{6\pi y}-ae^{6\pi\delta i}}\right) dy]$$

It will be assumed in the following that the function $\mathcal{V}$ has been extended in this manner wherever it is needed.
Using (3.24) we have, for $\mathcal{I}m\, y = 5/3$, and $\mathcal{I}m\, \xi \in (4/3+\varepsilon, 5/3+\varepsilon)$:

$$\frac{m(z, y-i0)}{m(z,\xi)} = E(z,y,\xi)\,\frac{\mathcal{V}(y)}{\mathcal{V}(\xi)}$$

$$where,\ \ E(z,y,\xi) = \exp[3i \int_{\mathcal{I}m\,\eta=(\frac{4}{3}+\varepsilon)} \ln\left(-\frac{z}{a}\right) \times$$

$$e^{6\pi\eta}\left(\frac{1}{e^{6\pi\eta}-e^{6\pi y}} - \frac{1}{e^{6\pi\eta}-e^{6\pi\xi}}\right) d\eta].$$

The integral E may be computed using residues. Therefore,

$$\frac{m(z, y-i0)}{m(z,\xi)} = \exp\left[6\pi\alpha(z)\,(y-\xi)\right]\,\frac{\mathcal{V}(y)}{\mathcal{V}(\xi)} \tag{3.27}$$

where $\alpha(z)$ is defined by

$$\alpha(z) = \frac{1}{2\pi i} \ln\left(-\frac{z}{a}\right), \tag{3.28}$$

and the branch of the logarithm is determined assuming that $\mathcal{A}rg(-z) \in (0, 2\pi)$ and henceforth,

$$0 < \mathcal{R}e\,(\alpha(z)) < 1. \tag{3.29}$$

It then follows that

$$G(z,\xi) = \frac{3i}{2\pi\, z} \int_{\mathcal{I}m\, y=\frac{5}{3}} e^{6\pi\alpha(z)\,(y-\xi)} \frac{\mathcal{V}(y)}{\mathcal{V}(\xi)} \frac{dy}{(e^{6\pi(y-\xi)}-1)} \tag{3.30}$$

where $\mathcal{V}(\cdot)$ has been defined in (3.25).

4 Analysis of the fundamental solution to the Cauchy problem.

4.1 Inverting the Fourier and Laplace transforms.

The function $G(z,\xi)$, in (3.30) provides the Laplace Fourier transform of the fundamental solution of the problem (2.13). In this section we invert the Laplace and Fourier transform in order to find the solution $\mathcal{G}(t,x)$ of (2.13), as well as derive its main properties.
We recall that the inverse Laplace and inverse Fourier transform for regular functions are given respectively by

$$\mathcal{L}^{-1}(G)(t) = \mathcal{G}(t) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} e^{zt} G(z) dz \tag{4.1}$$

and

$$\mathcal{F}^{-1}(\widehat{\mathcal{G}})(x) = \mathcal{G}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+bi}^{\infty+bi} e^{ix\xi} \widehat{\mathcal{G}}(\xi) d\xi \tag{4.2}$$

where in (4.1) c is large enough to have all the singularities of G at the left of the line $\mathcal{R}ez = c$, and in (4.2) we assume that $\mathcal{G} \in \mathcal{V}(M, x_0)$ which is defined in (2.20) and we then choose b in order to have the contour of integration contained on the strip S_M defined in (2.23). Since the function $G(z,\xi)$ to which we apply $\mathcal{L}^{-1}$ and $\mathcal{F}^{-1}$ is just bounded, those operators are defined in the sense of tempered distributions (cf. [17]). Therefore, the fundamental solution of the problem (2.13) is given by:

$$\mathcal{G}(t,x) = \mathcal{F}^{-1}\left(\mathcal{L}^{-1}G\right) \tag{4.3}$$

4.2 Description of $\mathcal{G}(t,x)$ near $x=0$.

We will check below that the function $G(z,\xi)$ can be split (roughly) as

$$G(z,\xi) = G_\infty(z) + [G(z,\xi) - G_\infty(z)] \tag{4.4}$$

where

$$G_\infty(z) = \lim_{|\xi|\to+\infty} G(z,\xi).$$

In particular, this implies that $\mathcal{G}(t,x)$ can be decomposed as:

$$\mathcal{G}(t,x) = g_\infty(t)\delta(x) + \mathcal{G}_{\text{reg}}(t,x)$$

where $\mathcal{G}_{\text{reg}}$ turns out to be an integrable function. In the rest of this section we make the meaning of this decomposition precise and study the properties of g_∞, $\mathcal{G}_{\text{reg}}$. In particular we study their asymptotics as $t\to 0$, $t\to+\infty$ and $x\to\pm\infty$.

Since $G(z,\xi) - G_\infty(z)$ does not decay fast enough as $|\xi|\to+\infty$, it is convenient, instead of descomposing G as in (4.4), to split $G(z,\xi)$ in the following manner. Using the change of variables $y-\xi=\theta$ as well as Proposition C.6, we can rewrite (3.30) as:

$$G(z,\xi) = \frac{3\,i}{\sqrt{2\pi}\,z}\int_{\mathcal{I}m\,\theta=\frac{5}{3}-\mathcal{I}m\xi} e^{[6\pi\alpha(z)\,\theta]} e^{\left[3i\theta\ln\left(-\frac{\Phi(\xi)}{a}\right)+h(\xi,\theta)\right]}\,\frac{d\theta}{(e^{6\pi\theta}-1)}. \tag{4.5}$$

We have shown in formula (B.2), in Appendix B, that the function $\Phi(\xi)$ has the form:

$$\Phi(\xi) = -a + \sum_{n\in\mathbb{Z}} \frac{B_n}{\xi-\xi_n}$$

for suitable B_n, ξ_n (cf. (B.2)). We then define

$$\Psi(\xi) = -a + \sum_{|n|\geq L} \frac{B_n}{\xi - \xi_n} + \frac{1}{\xi - \xi_L} \sum_{|n|\leq L} B_n, \tag{4.6}$$

where L is chosen in order to have

$$\sum_{|n|\geq L} \frac{|B_n|}{|\xi - \xi_n|} \leq \varepsilon_1 \quad \text{in} \quad |\mathcal{I}m\, \xi| \leq 10 \tag{4.7}$$

for ε_1 small to be precised, and ξ_L satisfies

$$\left| \frac{1}{\xi - \xi_L} \sum_{|n|\leq L} B_n \right| \leq \varepsilon_2, \quad \text{in} \quad |\mathcal{I}m\, \xi| \leq 10 \tag{4.8}$$

for $\varepsilon_2 > 0$ small enough to be precised. Notice that choosing L large enough we can assume that Ψ is analytic in the strip $|\mathcal{I}m\xi| \leq 10$. Moreover, if ε_1 and ε_2 are small enough and L large enough, we have that

$$|\ln\left(\frac{\Phi(\xi)}{\Psi(\xi)}\right)| = \mathcal{O}(|\xi|^{-2}) \tag{4.9}$$

as $|\xi| \to \infty$ and $|\mathcal{I}m\xi| \leq 10$. It then follows that the function $\tilde{h}$ given by

$$\tilde{h}(\xi, \theta) = h(\xi, \theta) + 3i \ln\left(\frac{\Phi(\xi)}{\Psi(\xi)}\right) \tag{4.10}$$

also satisfies the estimates (C.27). Let us decompose G as follows

$$G(z, \xi) = \mathcal{A}_1 + \mathcal{A}_2, \tag{4.11}$$

$$\mathcal{A}_1 = \frac{3\,i}{\sqrt{2\pi}\,z} \int_{\mathcal{I}m\,\theta = \frac{5}{3} - \mathcal{I}m\xi} e^{[6\pi\alpha(z)\,\theta]} \frac{e^{\left[3i\theta \ln\left(-\frac{\Psi(\xi)}{a}\right)\right]} d\theta}{(e^{6\pi\theta} - 1)}, \tag{4.12}$$

$$\mathcal{A}_2 = \frac{3\,i}{\sqrt{2\pi}\,z} \int_{\mathcal{I}m\,\theta = \frac{5}{3} - \mathcal{I}m\xi} e^{[6\pi\alpha(z)\,\theta]} e^{\left[3i\theta \ln\left(-\frac{\Psi(\xi)}{a}\right)\right]} \frac{(e^{\tilde{h}(\xi,\theta)} - 1)\, d\theta}{(e^{6\pi\theta} - 1)}, \tag{4.13}$$

where the function, $\mathcal{A}_1$ approaches a constant value as $|\xi| \to +\infty$ and can be chosen such that $|\max \mathcal{A}_1 - \min \mathcal{A}_1|$ is as small as we need. On the other hand, $\mathcal{A}_2$ decays as $|\xi| \to +\infty$ faster than $1/|\xi|$ (cf. (B.3) and Proposition C.6), and as a consequence, its inverse Fourier transform will be a continuous bounded function.

We remark that $\mathcal{A}_1$ may be explicitly computed by using residues. To this end, it is enough to replace the integral defining $\mathcal{A}_1$ by the limit of integrations in a sequence of contours Γ_R. These contours are squares with basis $\mathcal{I}m\theta = d$, $\mathcal{R}e\theta \in (-R, R)$, with $d \in (\delta, \delta + 1/3)$ and the remaining sides are contained in the half plane $\mathcal{I}m\theta \le 0$. On these sides, the integrand in (4.12) can be estimated as:

$$\left| e^{[6\pi\alpha(z)\,\theta]} \frac{e^{\left[3i\theta \ln\left(-\frac{\Psi(\xi)}{a}\right)\right]}}{(e^{6\pi\theta}-1)} \right| \le e^{\left[-3R\ln\left(\frac{|z|}{\Psi(\xi)|}\right)\right]} \frac{e^{\left[3\mathcal{R}e\theta\left(arg(-\frac{z}{a})-arg(-\frac{\Psi(\xi)}{a})\right)\right]}}{|e^{6\pi\theta}-1|}. \tag{4.14}$$

Choosing ε_1 and ε_2 in (4.7), (4.8) small enough, it follows that $\ln(|z|/|\Psi(\xi)|) > 0$ if $|z| > 2$. On the other hand, $arg(-z/a) \in (0, 2\pi)$ and $\lim_{|\xi|\to+\infty}(arg(-\Psi(\xi)/a) = 0$, it then follows

$$\lim_{R\to+\infty} \int_{\Gamma_R \setminus \{\mathcal{I}m\theta = d\}} e^{[6\pi\alpha(z)\,\theta]} \frac{e^{\left[3i\theta \ln\left(-\frac{\Psi(\xi)}{a}\right)\right]}}{(e^{6\pi\theta}-1)} d\theta = 0. \tag{4.15}$$

Therefore $\mathcal{A}_1$ can be computed adding the residues of the integrand in (4.12):

$$\mathcal{A}_1 = \frac{1}{\sqrt{2\pi}\, z} \sum_{n=0}^{+\infty} e^{-2i\pi n\alpha(z)} e^{n\ln\left(-\frac{\Psi(\xi)}{a}\right)} = \frac{1}{\sqrt{2\pi}\, z} \left(1 + \frac{\Psi(\xi)}{z}\right)^{-1}, \tag{4.16}$$

for $|z| > 2$. The validity of (4.16) for $|z| \le 2$ follows by analytic continuation.

Proposition 4.1 *The fundamental solution $\mathcal{G}(t,x)$ defined by (3.30) and (4.3) might be decomposed in the following manner:*

$$\mathcal{G}(t,x) = \mathcal{G}_{sing}(t,x) + \mathcal{G}_{reg}(t,x) \tag{4.17}$$

where

$$\mathcal{G}_{sing}(t,x) := \mathcal{F}^{-1}\left(\mathcal{L}^{-1}\mathcal{A}_1\right), \quad \mathcal{G}_{reg}(t,x) := \mathcal{F}^{-1}\left(\mathcal{L}^{-1}\mathcal{A}_2\right). \tag{4.18}$$

The term $\mathcal{G}_{sing}$ can be written as

$$\mathcal{G}_{sing}(t,x) = e^{-at}\delta(x) + \sum_{k=1}^{5} \frac{\alpha_k(t)}{|x|^{\frac{6-k}{6}}} + \alpha_6(t)\, sign(x) + \mathcal{H}(t,x) \tag{4.19}$$

where $\mathcal{H}$ is a Hölder continuous function in x in a neighbourhood of $x = 0$. The functions α_k might be computed explicitly. The function $\mathcal{G}_{reg}$ is a continuous function in a neighbourhood of $x = 0$. The functions α_k and the Hölder constants of the function $\mathcal{H}(t, \cdot)$ are uniformly bounded in bounded intervals of t.
Moreover, the following global estimate holds:

$$|\mathcal{G}_{sing}(t,x) - e^{-at}\delta(x)| \le Ce^{-(a-\varepsilon)t}\,\varphi(x) \tag{4.20}$$

where

$$|\varphi(x)| \le \begin{cases} \dfrac{1}{|x|^{5/6}} & \textit{for } |x| \le 1 \\ e^{-10|x|} & \textit{for } |x| \ge 1. \end{cases} \tag{4.21}$$

Proof. Using (4.16) and (4.18), we obtain:

$$\mathcal{G}_{sing}(t,x) = \mathcal{F}^{-1}\left(\frac{e^{-\Psi(\cdot)t}}{\sqrt{2\pi}}\right)(x).$$

Then, for t bounded, using the Taylor expansion for the exponential function as well as (4.9) and (B.3), we obtain:

$$\mathcal{F}^{-1}\left(\frac{e^{-\Psi(\cdot)t}}{\sqrt{2\pi}}\right)(x) = e^{-at}\delta(x) + e^{-at}\sum_{k=1}^{5}\frac{\beta_k(t)}{|x|^{\frac{6-k}{6}}} + \beta_6 sign(x) + \mathcal{H}(t,x) \tag{4.22}$$

where $\mathcal{H}$ is a Hölder continuous function in x in a neighborhood of $x = 0$, and $\beta_1, \cdots \beta_6$, are polynomials in the t variable.
On the other hand, in order to derive bounds for large t we argue as follows. Let us introduce a regular cut off function χ, $\chi(s) = 1$ for $|s| \le 1$, and $\chi(s) = 0$ for $s \ge 2$. We rewrite $\mathcal{G}_{sing}(t,x)$ as

$$\begin{aligned} \mathcal{G}_{sing}(t,x) &= e^{-at}\left\{\mathcal{F}^{-1}(1) + \mathcal{F}^{-1}\left(e^{-(\Psi(\xi)-a)t}\left[1-\chi\left(\frac{|\xi|}{t^6}\right)\right]-1\right)\right. \\ &\quad \left. + \mathcal{F}^{-1}\left(e^{-(\Psi(\xi)-a)t}\chi\left(\frac{|\xi|}{t^6}\right)\right)\right\} \end{aligned} \tag{4.23}$$

The first term in the right hand side of (4.23) gives the Dirac mass. In the second term, since $\Psi(\xi)\,t$ is bounded, it is possible to linearise in the exponential. Therefore, arguing as in the derivation of (4.22) we obtain:

$$|e^{-at}\mathcal{F}^{-1}\left(e^{-(\Psi(\xi)-a)t}(1-\chi\left(\frac{|\xi|}{t^6}\right))-1\right)| \le Ce^{-(a-\varepsilon)t}\,\varphi_1(x)$$

where $\varepsilon > 0$ is arbitrarily small and φ_1 satisfies,

$$\varphi_1(x) \sim \frac{1}{|x|^{5/6}} \quad \text{as } x \to 0, \quad \varphi_1(x) \sim e^{-10|x|} \quad \text{as } |x| \to \infty.$$

Finally, for the last term in the right hand side of (4.23) we obtain the estimate

$$|e^{-at}\mathcal{F}^{-1}\left(e^{-(\Psi(\xi)-a)t}\chi\left(\frac{|\xi|}{t^6}\right)\right)| \leq Ce^{-(a-\varepsilon)t}\,e^{-10|x|}.$$

This yields (4.20) and (4.21).
We now proceed to derive the hölderianity of $\mathcal{G}_{reg}$ near the origin. To this end we compute first the inverse of the Laplace transform. Using (4.1), and classical contour deformation arguments, we obtain

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{e^{[6\pi\alpha(z)\,\theta]}}{z}\right) &= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} e^{[6\pi\alpha(z)\,\theta]}\,e^{zt}\,\frac{dz}{z} \\ &= (at)^{3i\theta}(e^{6\pi\theta}-1)\Gamma(-3i\theta), \end{aligned} \tag{4.24}$$

where Γ is the usual Gamma function. Plugging (4.24) into (4.13) (4.18) we arrive at

$$\mathcal{G}_{reg}(x,t) = \mathcal{F}^{-1}\left(\frac{3\,i}{\sqrt{2\pi}}\int_{\mathcal{I}m\,\theta=\frac{5}{3}-\mathcal{I}m\xi}\Gamma(-3i\theta)e^{3i\theta\ln\left(-\frac{\Psi(\xi)}{a}\right)}(e^{\tilde{h}(\xi,\theta)}-1)\,(at)^{3i\theta}d\theta\right). \tag{4.25}$$

Due to (4.6)-(4.8), $|e^{3i\theta\ln\left(-\frac{\Psi(\xi)}{a}\right)}| \leq e^{\varepsilon_2|\theta|}$ with $\varepsilon_2 > 0$ small. On the other hand, Stirling's formula for the Gamma function implies, $|\Gamma(-3i\theta)| \leq Ce^{-|\theta|/2}$ for $\mathcal{I}m\,\theta = \frac{5}{3} - \mathcal{I}m\xi$ and $\mathcal{I}m\xi \in (4/3, 5/3)$ and $|\theta|$ large. These estimates yield the convergence of the integral in (4.25).

The Hölder property of $\mathcal{G}_{reg}$ follows combining (4.25) and (C.27). More precisely we split the integral in (4.25), in the two regions $|\theta| \geq |\xi|$ and $|\theta| \leq |\xi|$. It follows from (C.27) that $e^{\tilde{h}(\xi,\theta)} - 1$ is bounded by $C|\xi|^{-7/6}$ when $|\theta| \leq |\xi|$. Due to the fact that the rest of the integrand decays exponentially in $|\theta|$ the resulting contribution to the integral may be bounded as $C|\xi|^{-7/6}$. In order to estimate the contribution to the integral due to the region $|\theta| \geq |\xi|$ we take into account that $|\tilde{h}(\xi,\theta)| \leq \varepsilon_3|\theta|$ as $|\theta| \to +\infty$, where ε_3 may be chosen as small as we wish provided $|\xi|$ is large enough. Using again the exponential decay of the remaining terms it follows that the contribution of this part of the integral is exponentially small as $|\xi| \to +\infty$. Therefore, $|\mathcal{A}_2(t,\xi)| = \mathcal{O}(|\xi|^{-7/6})$ as $|\xi| \to +\infty$ and the hölderianity of $\mathcal{G}_{reg}$ follows by classical Fourier analysis results.

Proposition 4.1 provides a description of the fundamental solution for x near the origin. The proof of this results actually shows that $\mathcal{G}_{reg}$ is bounded for (t,x) in compact subsets of $(0,+\infty)\times\mathbb{R}$.
We now proceed to describe the asymptotic behaviour of this fundamental solution for $x \to \pm\infty$.

4.3 Asymptotics of $\mathcal{G}$ as $x \to -\infty$.

Our starting point for this analysis is the formula (4.5). Notice that

$$\widetilde{G}(z,x) = \mathcal{F}^{-1}(G(z,\cdot) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\xi=b} e^{ix\xi}\, G(z,\xi) d\xi \tag{4.26}$$

where, for $x \neq 0$, this integral is defined in the sense of oscillatory integrals (cf. for example [20]). The main contribution of $\widetilde{G}(z,x)$ as $x \to -\infty$ is due to the closest pole of $G(z,\cdot)$ to the line $\mathcal{I}m\xi = b$ below this line.
Notice that the expression (3.30) shows that $G(z,\xi)$ is analytic in the strip $\mathcal{I}m\xi \in (4/3, 5/3)$. Moreover, deforming the contour of integration, we can extend $G(z,\cdot)$ meromorphically to the strip $\mathcal{I}m\xi \in (-2/3, 5/3)$ with poles at the points $(1+2k)\,i/6,\ k = 0,1,2,3$. Moreover, for $d \in (1/3, 5/3)$ and $d - 1/3 < \mathcal{I}m\xi < d$, $G(z,\xi)$ can be computed by means of

$$G(z,\xi) = \frac{3i}{\sqrt{2\pi}\, z} \int_{\mathcal{I}m\, y=d} e^{6\pi\alpha(z)\,(y-\xi)}\, \frac{\mathcal{V}(y)}{\mathcal{V}(\xi)} \frac{dy}{(e^{6\pi(y-\xi)}-1)} \tag{4.27}$$

The residue of $G(z,\cdot)$ at the pole $7i/6$ is given by:

$$\mathcal{R}es\left(G(z,\cdot),\ \xi = \frac{7i}{6}\right) \ = \ \frac{3ia\, e^{-7\pi\alpha(z)i}}{z\sqrt{2\pi}\,\Phi'(7i/6)\mathcal{V}(3i/2)} \int_{\mathcal{I}m\, y=d} \frac{e^{6\pi\alpha(z)\,y}\mathcal{V}(y)}{(e^{6\pi(y-7i/6)}+1)}\, dy$$

Choosing d in (4.27) close to 4/3 and moving down the contour of integration in (4.26) we obtain, using residues:

$$\widetilde{G}(z,x) \ = \ \frac{3\, a\, e^{-\frac{7x}{6}}\, e^{-7\pi\alpha(z)i}}{z\Phi'(7i/6)\mathcal{V}(3i/2)} \int_{\mathcal{I}m\, y=d} \frac{e^{6\pi\alpha(z)\,y}\mathcal{V}(y)}{(e^{6\pi y}+1)}\, dy \tag{4.28}$$

$$+\frac{1}{\sqrt{2\pi}}\int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\,G(z,\xi)d\xi.$$

where $\tilde{b}$ is an arbitrary complex number satisfying $\mathcal{I}m\tilde{b}\in(1,7/6)$.
Taking the inverse of the Laplace transform we obtain:

$$G(t,x)=\mathcal{L}^{-1}(\widetilde{G})(t,x)=\sigma(t)e^{-\frac{7}{6}x}+R_1(t,x). \tag{4.29}$$

Using (4.24) and (4.28):

$$\sigma(t)\equiv-\frac{3\,a}{\Phi'(7i/6)\mathcal{V}(3i/2)}\int_{\mathcal{I}m\,y=d}(at)^{3iy+\frac{7}{2}}\,\mathcal{V}(y)\Gamma(-3iy-\frac{7}{2})dy. \tag{4.30}$$

and

$$R_1(t,x)=\frac{1}{\sqrt{2\pi}}\mathcal{L}^{-1}\left(\int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\,G(z,\xi)d\xi\right). \tag{4.31}$$

We now proceed to derive estimates for σ and R_1.

Proposition 4.2 *The following estimates hold:*

$$\sigma(t)=At^4+\mathcal{O}(t^{4+\varepsilon})\quad as\ \ t\to0^+, \tag{4.32}$$

$$|\sigma(t)|=\mathcal{O}(t^{-(3v_0-5/2)})=\mathcal{O}(t^{-3,0206})\quad as\ \ t\to+\infty, \tag{4.33}$$

where A is a given constant, ε is an arbitrary number in $(0,1/2)$.

Proof. As a first step we prove (4.32). To this end we use again contour deformation moving down the line $\mathcal{I}my=d$.
The poles of the function $\Gamma(-3iy-7/2)$ are placed at $y=(7i/6)-(ni/3), n=0,1,\cdots$. On the other hand, by Proposition (C.5) $\mathcal{V}$ has zeros at $y=(7i/6)-(ni/3)$, for $n=0,1,2,3$. We deduce that $\mathcal{V}(y)\Gamma(-3iy-7/2)$ is analytic in the strip $\mathcal{I}my\in(-1/6,v_0+1/3)$, meromorphic in $\mathbb{C}$, and has simple poles at $-i/6$ (coming from the Gamma function) and at $\pm u_0+i(v_0+1/3)$ (cf. (P-2) in Appendix B).
Therefore, the Residues Theorem implies:

$$\sigma(t)=At^4+r(t)$$

where,

$$A=\frac{a^4\,\mathcal{V}(-1/6)\,\pi i}{12\Phi'(7i/6)\mathcal{V}(3i/2)},$$

$$r(t) = -\frac{3\,a}{\Phi'(7i/6)\mathcal{V}(3i/2)} \int_{\mathcal{I}m\,y=\tilde{d}} (at)^{3iy+\frac{7}{2}}\,\mathcal{V}(y)\Gamma(-3iy-\frac{7}{2})dy$$

where $\tilde{d}$ is an arbitrary number such that $\tilde{d} \in (-1/3, -1/6)$.
Combining (C.22), (C.29) and (P-3) in Appendix B, as well as Stirling's formula it follows that $\mathcal{V}(y)\Gamma(-3iy - \frac{7}{2})$ decays exponentially as $|y| \to +\infty$ along the contour of integration. On the other hand, in the same contour of integration, $|(at)^{3iy}| \le C_\varepsilon t^{4+\varepsilon}$, with $\varepsilon \in (0, 1/2)$, whence (4.32) follows.

In order to prove (4.33) we increase the value of d in (4.30) using contour deformation. In this process we do not meet any singularity of the integrand until $d = v_0 + 1/3$ due to the Proposition C.5 as well as the analyticity properties of the Gamma function. Arguing as in the proof of (4.32), formula (4.33) follows.

We now derive the estimates for the remainder term (4.31).

Proposition 4.3 *The following estimates hold:*

$$|R_1(t,x)| \le Ce^{-\tilde{b}x}(at)^{3(\tilde{b}-\tilde{d})} \quad \textit{for } x<0,\ \ 0<t<1, \tag{4.34}$$

$$|R_1(t,x)| \le Ce^{-\tilde{b}\,x}(at)^{-3(r-\tilde{b})}, \quad \textit{for } x<0,\ \ t\ge 1, \tag{4.35}$$

where $\tilde{b}$ is an arbitrary real number in $(1, 7/6)$, $\tilde{d}$ is an arbitrary real number in $(5/6, 1)$ and r is an arbitrary number less than $v_0 = 1.84...$.

Proof of Proposition 4.3. We use again the splitting (4.11) into (4.31):

$$R_1(t,x) = R_{1,1}(t,x) + R_{1,2}(t,x), \tag{4.36}$$

where,

$$R_{1,1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\,\mathcal{L}^{-1}\left(\mathcal{A}_1(z,\xi)\right)d\xi \tag{4.37}$$

$$R_{1,2}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\,\mathcal{L}^{-1}\left(\mathcal{A}_2(z,\xi)\right)d\xi \tag{4.38}$$

We begin estimating $R_{1,1}$. Using (4.16) it follows that

$$R_{1,1}(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\,e^{-\Psi(\xi)\,t}d\xi,$$

where from now on this integral has to be understood in the sense of oscillatory integrals. Since Ψ is analytic in the strip $|\mathcal{I}m\xi| < 10$, we can decrease the value of $\tilde{b}$ by means of contour deformation, to any $\tilde{b} > -10$. Therefore, using (4.6)- (4.10) it follows that

$$|R_{1,1}(t,x)| \leq Ce^{-(a-\varepsilon)t}e^{10\,x}, \quad \text{for} \quad x<0. \tag{4.39}$$

Moreover, computing $|R_{1,1}(t,x) - R_{1,1}(0,x)|$ we arrive at:

$$|R_{1,1}(t,x)| \leq C\,t\,e^{10\,x}, \quad \text{for } x<0,\ \ 0\leq t\leq 1. \tag{4.40}$$

We now estimate $R_{1,2}$.

$$R_{1,2}(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} e^{zt}\mathcal{A}_2(z,\xi)\,dz\,d\xi$$

where $\mathcal{A}_2$ is as in (4.13). Using (4.24) we then obtain

$$\begin{aligned} R_{1,2}(t,x) \quad &= \quad \frac{3i}{\sqrt{2\pi}}\int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\int_{\mathcal{I}m\,\eta=\frac{4}{3}}\left(e^{\tilde{h}(\xi,\eta-\xi)}-1\right)\,e^{3i(\eta-\xi)\ln\left(-\frac{\Psi(\xi)}{a}\right)}\times \\ &\quad \times (at)^{3i(\eta-\xi)}\,\Gamma(3i(\xi-\eta))\,d\eta d\xi. \end{aligned}$$

We begin estimating $R_{1,2}$ as $t\to 0$ and $x<0$. Notice that, due to (C.25) we may write $R_{1,2}$ as

$$R_{1,2} = \frac{3i}{2\pi}\int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\widehat{R}_{1,2}(t,\xi)\,d\xi \tag{4.41}$$

where,

$$\widehat{R}_{1,2}(t,\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathcal{I}m\,\eta=\frac{4}{3}}(at)^{3i(\eta-\xi)}\,\Gamma(3i(\xi-\eta))\left[\frac{\mathcal{V}(\eta)}{\mathcal{V}(\xi)} - e^{(3i(\eta-\xi)\ln\left(-\frac{\Psi(\xi)}{a}\right)}\right]\,d\eta \tag{4.42}$$

We move down the contour of integration in (4.42) as usual. Notice that $\Gamma(3i(\xi-\eta))$ has a pole for $\eta=\xi$ but this singularity cancels out with the zero of the term between brackets. Using again Proposition C.5, we can rewrite (4.42) as:

$$\widehat{R}_{1,2}(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\eta=\overline{d}} (at)^{3i(\eta-\xi)}\,\Gamma(3i(\xi-\eta)) \times \left[\frac{\mathcal{V}(\eta)}{\mathcal{V}(\xi)} - e^{\left(3i(\eta-\xi)\ln\left(-\frac{\Psi(\xi)}{a}\right)\right)}\right] d\eta \qquad (4.43)$$

where $\overline{d}$ is any real number in $(5/6,1)$. This restriction in $\overline{d}$ comes from the fact that the function $\Gamma(3i(\xi-\eta))$ has a pole at $\eta=\xi-i/3$ and that we are in the region where $\mathcal{I}m\xi \in (1,7/6)$.
Using (C.25), we estimate (4.43) as

$$e^{-\tilde{b}x}(at)^{-3(\tilde{d}-\tilde{b})} \int_{\mathcal{I}m\xi=\tilde{b}} |d\xi| \int_{\mathcal{I}m\eta=\tilde{d}} |d\eta| e^{-\pi|\xi-\eta|} |\tilde{h}(\xi,\eta-\xi)|. \qquad (4.44)$$

Using (C.27) it follows that the integral term in (4.44) is bounded whence (4.34) follows.
Finally we estimate $R_1(t,x)$ for $x<0$ and $t>0$ large. To this end we take as starting point formula (4.42). Moving up the contour of integration, we do not meet any singularity until $\mathcal{I}m\eta = v_0$ (cf. Proposition C.5). Therefore, using Proposition C.6, and (4.42)

$$|R_{1,2}(t,\xi)| \le Ce^{-\tilde{b}\,x}(at)^{-3(r-\tilde{b})} \times \int_{\mathcal{I}m\xi=\tilde{b}} |d\xi| \int_{\mathcal{I}m\eta=\tilde{d}} |d\eta| |\Gamma(3i(\xi-\eta))| \left| e^{3i(\eta-\xi)\ln(-\frac{\Psi(\xi)}{a})}\right| |\tilde{h}(\xi,\eta-\xi)|.$$

where r is an arbitrary number less than v_0. Using Stirling's formula as well as (4.6)-(4.10), (4.37) and (C.27) we arrive at (4.35).

4.4 Asymptotics of $\mathcal{G}$ as $x \to +\infty$.

Proposition 4.4 *The following estimates hold,*

$$|\mathcal{G}(t,x)| \le Ce^{-\frac{11}{6}x}t^{1-\varepsilon} \quad \textit{for } x>0,\ 0\le t\le 1, \qquad (4.45)$$

$$|\mathcal{G}(t,x)| \le Ce^{-\frac{11}{6}x}t^{-(1+3(v_0-11/6))+\varepsilon} \quad \textit{for } x>0,\ t\ge 1, \qquad (4.46)$$

where $\varepsilon>0$ is arbitrarily small. Notice that $(1+3(v_0-11/6)) = 1.0206 > 1$.

Proof of Proposition 4.4. Arguing as in the derivation of (4.36)-(4.38) and using

$$\mathcal{G}(t,x) = R_{2,1}(t,x) + R_{2,2}(t,x),$$

where,using (4.16)

$$\begin{aligned} R_{2,1}(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\, e^{-\Psi(\xi)\,t} d\xi \\ R_{2,2}(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi}\, \mathcal{L}^{-1}\left(\mathcal{A}_2(z,\xi)\right) d\xi \end{aligned}$$

Using the analyticity of Φ we can deform the contour of integration and choose $\tilde{b} = 10$. Therefore as in the proof of (4.39) and (4.40),

$$|R_{2,1}(t,x)| \le Ce^{-(a-\varepsilon)t} e^{-10\,x}, \quad \text{for} \quad x > 0. \tag{4.47}$$

$$|R_{2,1}(t,x)| \le C\,t\,e^{-10\,x}, \quad \text{for } x > 0,\ \ 0 \le t \le 1. \tag{4.48}$$

On the other hand, we may write,

$$\begin{aligned} R_{2,2}(t,x) &= \frac{3i}{2\pi} \int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi} \int_{\mathcal{I}m\,\eta=d} \left(e^{\tilde{h}(\xi,\eta-\xi)} - 1\right) e^{3i(\eta-\xi)\ln\left(-\frac{\Psi(\xi)}{a}\right)} \times \\ &\quad \times (at)^{3i(\eta-\xi)}\, \Gamma(3i(\xi-\eta))\, d\eta d\xi. \end{aligned}$$

where first, we have deformed the contour deformation in the ξ variable to make $\tilde{b} > d$. This is possible because the singularity of the Gamma function at $\eta = \xi$ cancels out with the zero of the term $\left(e^{\tilde{h}(\xi,\eta-\xi)} - 1\right)$.

$$R_{2,2}(t,x) = \frac{3i}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi} \widehat{R}_{2,2}(t,\xi)\, d\xi \tag{4.49}$$

where,

$$\widehat{R}_{2,2}(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\eta=d} (at)^{3i(\eta-\xi)}\, \Gamma(3i(\xi-\eta)) \left[\frac{\mathcal{V}(\eta)}{\mathcal{V}(\xi)} - e^{(3i(\eta-\xi)\ln\left(-\frac{\Psi(\xi)}{a}\right)}\right] d\eta \tag{4.50}$$

We now try to move up the contour deformation on ξ as much as possible, but in this deformation, we must also deform the contour on the η variable, in order to avoid the singularity of $\Gamma(3i(\xi-\eta))$ at $\xi-\eta = i/3$. The function $\mathcal{V}(\xi)$ has a zero at $\xi = 11i/6$ since Φ has a pole at $3i/2$ due to (C.22). Therefore, the integral in (4.50) has a pole at $\xi = 11\,i/6$ whose corresponding residue yields a contribution $\tilde{\sigma}(t)\, e^{-11\, x/6}$, similarly as in the derivation of (4.29), (4.30). In order to obtain estimates of the time dependence of these terms, we make contour deformation in the η variable, having in mind the following ideas. First to deduce estimates for $t \to 0$, $\mathcal{I}m\eta$ should be taken as small as possible. To obtain estimates for $t \to +\infty$, $\mathcal{I}m\eta$ as large as possible. Finally, $\mathcal{I}m(\xi - \eta)$ shoud be larger that $1/3$ in order to avoid the singularities of the Gamma function. Deforming the contours as it was made in the previous subsection we obtain (4.45), (4.46).

4.5 The proof of Theorem 2.2.

At this stage, Theorem 2.2 is just a reformulation of Proposition 4.1, Proposition 4.2, Proposition 4.3, by means of formulas (2.24)-(2.12)

A Properties of the kernel K.

A.1 Proof of Proposition 2.1.

Using (1.7) and (2.2) it follows, after elementary integrals, that

$$I_3 = -\frac{72}{k_1^{1/3}} + \int_{D_1(k_1)} \frac{\sqrt{k}_2}{\sqrt{k}_1} \frac{dk_3\, dk_4}{k_3^{7/6}\, k_4^{7/6}} - 2\int_{D_2(k_1)} \frac{\sqrt{k}_4}{\sqrt{k}_1} \frac{dk_3\, dk_4}{k_2^{7/6}\, k_3^{7/6}}, \tag{A.51}$$

where

$$D_1(k_1) = \{(k_3, k_4);\ 0 < k_3 < k_1,\, 0 < k_4 < k_1,\ k_1 < k_3 + k_4\} \tag{A.52}$$

$$D_2(k_1) = \{(k_3, k_4);\ k_3 > k_1,\, 0 < k_4 < k_1\} \tag{A.53}$$

In order to estimate the two last terms of (A.51) we use one of the coordinate transformations introduced by V. E. Zakharov in [22]:

$$\varepsilon_3 = \frac{k_1^2}{k_3}, \quad \varepsilon_4 = \frac{(k_3 + k_4 - k_1)k_1}{k_3}.$$

Therefore

$$\int_{D_2(k_1)} \frac{\sqrt{k}_4}{\sqrt{k}_1} \frac{dk_3\, dk_4}{k_2^{7/6}\, k_3^{7/6}} = \frac{1}{\sqrt{k}_1} \int_{D_1(k_1)} \frac{\sqrt{k}_2\, dk_3\, dk_4}{k_3^{7/6}\, k_4^{7/6}}$$

Whence, using (A.51), gives:

$$I_3 = -\frac{a}{k^{1/3}} \equiv -\frac{1}{k_1^{1/3}} \left(72 + \int_{D_1(1)} \frac{\sqrt{k_3 + k_4 - 1}}{k_3^{7/6}\, k_4^{7/6}}\, dk_3\, dk_4 \right). \tag{A.54}$$

The last integral can be transformed into a more symmetric form using another of the transformations introduced in [22], namely:

$$\varepsilon_3 = \frac{k_4}{k_3 + k_4 - 1}, \quad \varepsilon_4 = \frac{k_3}{k_3 + k_4 - 1},$$

whence

$$a = 72 + \int_1^\infty \int_1^\infty \frac{d\rho_1\, d\rho_2}{(\rho_2 + \rho_1 - 1)^{7/6}\, \rho_1^{7/6}\, \rho_2^{7/6}}. \tag{A.55}$$

The last integral can be computed numerically. One gets:

$$a = 72.80964399...$$

Standard calculus computations yield:

$$I_1 = \frac{1}{k_1^{4/3}} \{ \int_0^{k_1} dk_3\, F_3\, K_{11}(\frac{k_3}{k_1}) + \int_{k_1}^\infty dk_3\, F_3\, K_{21}(\frac{k_3}{k_1}) \} \tag{A.56}$$

where

$$\begin{aligned} K_{11}(\theta_3) &= 2 \int_{1-\theta_3}^1 d\theta_4 \left(\frac{\theta_2^{1/2}}{\theta_4^{7/6}} (1 + \theta_2^{-7/6}) - \theta_2^{-2/3} \right) + \\ &\quad + 2 \int_1^\infty d\theta_4 \left(\frac{\theta_3^{1/2}}{\theta_4^{7/6}} (1 + \theta_2^{-7/6}) - \frac{\theta_3^{1/2}}{\theta_2^{7/6}} \right) \end{aligned} \tag{A.57}$$

$$\begin{aligned} K_{21}(\theta_3) &= 2 \int_0^1 d\theta_4 \left(\theta_4^{-2/3} (1 + \theta_2^{-7/6}) - \frac{\theta_4^{1/2}}{\theta_2^{7/6}}) \right) + \\ &\quad + 2 \int_1^\infty d\theta_4 \left(\theta_4^{-7/6} (1 + \theta_2^{-7/6}) - \theta_2^{-7/6} \right), \end{aligned} \tag{A.58}$$

with $\theta_\ell = k_\ell / k_1$, for $\ell = 2, 3, 4$. Notice that, $\theta_2 = \theta_3 + \theta_4 - 1$. In an analogous manner,

$$I_2 = \frac{1}{k_1^{4/3}} \{ \int_0^{k_1} dk_2\, F_2\, K_{12}(\frac{k_2}{k_1}) + \int_{k_1}^\infty dk_2\, F_2\, K_{22}(\frac{k_2}{k_1}) \}, \tag{A.59}$$

$$
\begin{aligned}
K_{12}(\theta_4) &= \int_1^{1+\theta_2} d\theta_3 (\theta_2+1-\theta_3)^{1/2} \left(\theta_3^{-7/6}(1+\theta_2-\theta_3)^{-7/6} - 2\theta_3^{-7/6}\right) + \\
&+ \int_0^{\theta_2} d\theta_3 \left(\theta_3^{-2/3}(\theta_2+1-\theta_3)^{-7/6} - 2\theta_3^{-2/3}\right) + \\
&\theta_2^{1/2} \int_{\theta_2}^{1} d\theta_3 \left(\theta_3^{-7/6}(\theta_2+1-\theta_3)^{-7/6} - 2\theta_3^{-7/6}\right) + \\
K_{22}(\theta_4) &= \int_1^{\theta_2} d\theta_3 \left(\theta_3^{-7/6}(1+\theta_2-\theta_3)^{-7/6} - 2\theta_3^{-7/6}\right) + \\
&+ \int_{\theta_2}^{1+\theta_2} d\theta_3 (\theta_2+1-\theta_3)^{1/2} \left(\theta_3^{-7/6}(1+\theta_2-\theta_3)^{-7/6} - 2\theta_3^{-7/6}\right) \\
&\int_0^{1} d\theta_3 \left(\theta_3^{-2/3}(\theta_2+1-\theta_3)^{-7/6} - 2\theta_3^{-2/3}\right)
\end{aligned}
\tag{A.60}
$$

On the other hand, we can write the integrals $K_{11}, \cdots K_{22}$ in terms of the Gauss hypergeometric functions $\mathcal{F}(a, b; c; z)$ using repeatedly the formulae (cf. [1]):

$$
\mathcal{F}(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,
$$

$$
\mathcal{F}(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_1^\infty t^{1-b}(t-1)^{c-b-1}(t-z)^{-a}\,dt,
$$

that are valid for $\mathcal{R}e(c) > \mathcal{R}e(b) > 0$. After some long and tedious, but standard computations we can transform equation (2.8) in:

$$
F_t = -\frac{a}{k^{1/3}} F(k) + \frac{1}{k^{4/3}} \int_0^\infty K(\frac{r}{k})\, F(r)\, dr \tag{A.61}
$$

where a is as in (A.54) and

$$
K(r) = \begin{cases} K_1(r) & \text{if } 0 \le r < 1 \\ K_2(r) & \text{if } 1 < r, \end{cases} \tag{A.62}
$$

$$
K_1(x) : \;=\; \frac{4}{3} x^{3/2} \mathcal{F}(1, 7/6; 5/2; x) + 6x^{1/3} \mathcal{F}(1, 7/6; 4/3; x) - 18x^{1/3}
$$

$$
\begin{aligned}
&+ \quad 24x^{1/2} - \frac{8x^{1/2}\Gamma(5/6)^2\,3^{1/2}}{\Gamma(2/3)(1-x)^{4/3}} + \frac{84}{5}x^{4/3}\mathcal{F}(13/6,1;11/6;x) \\
&- \quad 6x^{-2/3} - \frac{42}{5}x^{4/3}\mathcal{F}(13/6,1;11/6;-x) - \frac{12x^{1/2}\Gamma(5/6)^2}{\Gamma(2/3)(x+1)^{4/3}} \\
&+ \quad 6x^{-2/3}\mathcal{F}(1,1/6;11/6;-x) + \frac{72}{11}x^{1/3}\mathcal{F}(2,7/6;17/6;-x) \\
&- \quad \frac{3024}{935}x^{4/3}\mathcal{F}(13/6,3;23/6;-x) + 6x^{1/3}\mathcal{F}(1,7/6;4/3;-x) \\
&- \quad \frac{4}{3}x^{3/2}\mathcal{F}(1,7/6;5/2;-x), \qquad \text{(A.63)}
\end{aligned}
$$

$$
\begin{aligned}
K_2(x): \quad = \quad & 6x^{-7/6}\mathcal{F}(1,7/6;4/3;1/x) - \frac{4}{3}x^{-7/6}\mathcal{F}(1,7/6;5/2;1/x) + 24x^{-7/6} \\
- \quad & \frac{8\Gamma(5/6)^2\,3^{1/2}}{\Gamma(2/3)}(x-1)^{-4/3} + \frac{84}{5}x^{-13/6}\mathcal{F}(1,13/6;11/6;1/x) \\
- \quad & 12\frac{\Gamma(5/6)^2}{\Gamma(2/3)}(x+1)^{-4/3} - \frac{42}{5}x^{-13/6}\mathcal{F}(1,1/6;11/6;-1/x) \\
+ \quad & \frac{504}{55}x^{-19/6}\mathcal{F}(2,7/6;17/6;-1/x) - \\
- \quad & \frac{3024}{935}x^{-25/6}\mathcal{F}(3,13/6;23/6;-1/x) - \frac{42}{5}x^{-13/6}\mathcal{F}(1,13/6;11/6;-1/x) \\
+ \quad & 6x^{-7/6}\mathcal{F}(1,7/6;4/3;-1/x) - \frac{4}{3}x^{-7/6}\mathcal{F}(1,7/6;5/2;-1/x). \qquad \text{(A.64)}
\end{aligned}
$$

Formulae (2.4)-(2.7) are a consequence of the classical asymptotic expansions for the hypergeometric functions (cf. [1]), as well as the expressions for the kernels $K_{11}\cdots K_{22}$. The numerical constants in these formulae are given by:

$$
a_1 = -\frac{3}{\pi}(3\,2^{2/3}\Gamma(2/3)^3 + 2^{5/3}\Gamma(2/3)^3\,3^{1/2} - 8\pi), \qquad a_2 = \frac{100}{3}
$$

and

$$
a_3 = a_5 = 2\int_0^\infty \frac{dx}{x^{2/3}(1+x)^{7/6}} \equiv 2B(1/3,5/6).
$$

B The Fourier transform of the Kernel.

In this Section we list some properties of the function

$$
\Phi = -a + \widehat{\mathcal{K}}
$$

used in the analysis of the solutions of (2.30). Due to (A.62), we can write $\widehat{\mathcal{K}}$ in terms of suitable Mellin transforms of the functions K_1 and K_2:

$$\hat{\mathcal{K}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^1 \rho^{i\xi} K_1(\rho)\, d\rho + \frac{1}{\sqrt{2\pi}} \int_1^\infty \rho^{i\xi} K_2(\rho)\, d\rho. \qquad \text{(B.1)}$$

The Mellin transform $\hat{\mathcal{K}}$ might be computed using formulae (A.64), (A.63) in terms of generalized hypergeometric functions, but the resulting expression is not particularly illuminating. Nevertheless, using the series expansions for those functions we arrive at the following formula:

$$\begin{aligned}\Phi(\xi) \;&=\; -a + \sum_{j=0}^{\infty} \frac{A_1(j)}{(1-6i\xi+12j)} + \sum_{j=0}^{\infty} \frac{A_2(j)}{(1-3i\xi+3j)} + \sum_{j=0}^{\infty} \frac{A_3(j)}{(3+2i\xi+2j)} + \\ &\quad + \sum_{j=0}^{\infty} \frac{A_4(j)}{(10+3i\xi+6j)} \qquad \text{(B.2)}\end{aligned}$$

where the coefficients $A_i(j)$, $i = 1 \cdots 4$, $j = 0, 1, \ldots$ are:

$$\begin{aligned}A_1(j) = \;& \frac{18(2)^{1/3}\Gamma(7/6+2j)\Gamma(2/3)}{\Gamma(5/6+2j)\pi^{5/2}\Gamma(5/2+2j)\Gamma(4/3+2j)} \cdot \\ & \big\{-3^{1/2}\Gamma(2/3)\pi^{3/2}\Gamma(4/3+2j)\Gamma(5/6+2j)+ \\ & +4\pi^2\Gamma(5/2+2j)\Gamma(5/6+2j)+ \\ & +18(2)^{1/3}\Gamma(2/3)^3\Gamma(5/2+2j)\pi^{1/2}\Gamma(4/3+2j)\big\}\,,\end{aligned}$$

$$A_2(j) := -\frac{54(3^{1/2}(-1)^j+2)\Gamma(5/6)^2\,\Gamma(4/3+j)}{\pi\Gamma(1+j)},$$

$$A_3(j) := -\frac{36(3^{1/2}(-1)^j+2)\Gamma(5/6)^2\Gamma(4/3+j)}{\pi\Gamma(1+j)} + \frac{6(1+(-1)^j)\Gamma(5/6)\Gamma(1/6+j)}{\pi^{1/2}\Gamma(3/2+j)},$$

$$A_4(j) := \frac{18\Gamma(2/3)\Gamma(19/6+2j)2^{1/3}(2\pi^2\Gamma(17/6+2j)+9(2)^{1/3}\Gamma(2/3)^3\Gamma(10/3+2j)\pi^{1/2})}{\Gamma(10/3+2j)\pi^{5/2}\Gamma(17/6+2j)}.$$

We can now list the main properties of the function Φ.

• (P-1) The function $\Phi(\xi)$ is meromorphic in the complex plane $\mathbb{C}$ with poles at the points:

$$\xi \;=\; (\frac{3}{2}+j)\,i, \quad j = 0, 1, 2, \cdots$$

$$
\begin{aligned}
\xi &= (\frac{10}{3} + 2j)\, i, \quad j = 0, 1, 2, \cdots \\
\xi &= -(\frac{1}{3} + j)\, i, \quad j = 0, 1, 2, \cdots \\
\xi &= -(\frac{1}{6} + 2j)\, i, \quad j = 0, 1, 2, \cdots
\end{aligned}
$$

• (P-2) The function Φ has a simple zero at the point $\xi = 7i/6$. This is the only zero of Φ in the strip $\mathcal{I}m\xi \in (-1/6, 5/3)$. Moreover, it also has a simple zero at $\xi = 13i/6$ and two simple zeros at $\xi = \pm u_0 + iv_0$ with:

$$u_0 = 0.331..., \quad v_0 = 1.84020...$$

These are the only zeros of Φ in the strip $\mathcal{I}m\xi \in (-1/3, 5/2)$. In Figure 1 we have plotted the zeros and poles of the function Φ which play a role in the arguments used in this paper.

• (P-3) The function Φ satisfies:

$$|\Phi(\xi) - \Phi_\infty(\xi)| + |\xi||\Phi'(\xi) - \Phi'_\infty(\xi)| = \mathcal{O}(|\xi|^{-\alpha-1}) \quad \text{as } |\xi| \to +\infty \tag{B.3}$$

with $\alpha > 0$.

$$\Phi_\infty(\xi) \equiv -a + \frac{b_1}{\xi^{1/6}} + \frac{b_2}{\xi} \tag{B.4}$$

uniformly on strips of the form:

$$S_{\alpha,\beta} = \{\xi \in \mathbb{C};\ \xi = u + iv,\ \alpha < v < \beta\}.$$

• (P-4) Argument property: The function $\Phi(\lambda)$ does not make any complete turn around the origin when λ moves along any curve connecting the two extremes of the strip $S_{7/6,3/2}$ and entirely contained there. Notice that for any horizontal line, contained in this strip the number of turns is constant due to the argument principle. More precisely, since the function Φ does not vanish in the strip $S_{7/6,3/2}$, it is possible to define an analytic function $\ln \Phi(\lambda)$ on that strip (in general in a non unique way due to the multiplicity of branches of the logarithmic function). It turns out that:

$$
\begin{aligned}
arg(\Phi(-\infty + ib)) = lim_{x\to-\infty}\mathcal{I}m\,(\ln(\Phi(-\infty + ib))) = \\
lim_{x\to\infty}\mathcal{I}\,m\,(\ln(\Phi(+\infty + ib))) = arg(\Phi(+\infty + ib))
\end{aligned}
\tag{B.5}
$$

for any $b \in (7/6, 3/2)$. In Figure 4, we show a drawing of the image by Φ of the line $\mathbb{R} + 4i/3$.

Property (P-1) is just a consequence of the representation formula (B.1). The presence of a zero of Φ at $\xi = 7i/6$ follows from the fact that the function $f(k) = k^{-7/6}$ cancels the collision kernel $\mathcal{Q}(f)$ as was shown by Zakharov [22].
The property (P-3) follows from (2.6),(2.7) using standard methods for studying the asymptotics of the Fourier transform.

Concerning (P-2) and (P-4). We have numerically checked (P-2), combining the Argument Principle with a numerical computation using MAPLE V. Similar computations have been used to numerically check (P-4). Notice that due to the asymptotics (B.3) and (B.4), it is enough in order to check (P-2), (P-4), to count the numbers of rotations of Φ around the origin, when ξ moves along horizontal lines $\xi = ib + \mathbb{R}$ for suitable values of $b \in \mathbb{R}$. We show in the enclosed pictures the motion of $\Phi(\xi)$ for different values of b that in particular imply (P-2), (P-4).

Concerning the positions of the zeros of Φ some remarks are in order. The presence of a zero at $\xi = 7i/6$ is just a consequence of the fact that the nonlinear equation (1.1)-(1.4) has a family of steady states of the form $Ck^{-7/6}$ for any value of C. It turns out that the function Φ has a second zero at $\xi = 13i/6$. This is a consequence of the fact that the linearized operator in the right hand side of (2.3), cancels out the power $k^{-7/6}$ as it has been shown in [8]. The presence of this zero is a general fact that has been shown for a very general class of homogeneous kernels in [7, 8]. In general, the existence of stationary homogeneous solutions for the linearizations near equilibria of (1.1) is related to the existence of conserved quantities like energy, momentum or number of particles. In particular the existence of a homogeneous, radially symmetric solution for a linearized operator (2.3) is related to the conservation of the number of particles by the equation (1.1).
Finally the zeros at $\xi = \pm u_0 + iv_0$ does not seem to be related to any of the symmetries of the problem. Their position is determined by the whole structure of the kernel $\mathcal{K}$. We have only been able to determine them using numerical approximation.

C Auxiliary results.

In this appendix we collect several properties of the different functions used to prove the results of the paper.

C.1 Properties of the function $M(z,\zeta)$.

Proposition C.1 *Let $M(z,\zeta)$ be defined by (3.10) for $\zeta \in D \subset \mathcal{S}$, where D is defined in (3.4). Then, for $\varepsilon_0 > 0$ small enough, $M(z,\cdot)$ admits an analytic extension to the domain $D(\varepsilon_0)$ where,*

$$D(\varepsilon_0) = \left\{\zeta \in \mathcal{S};\ \zeta = |\zeta| e^{i\theta},\ \theta \in (-\varepsilon_0, 2\pi + \varepsilon_0)\right\} \tag{C.1}$$

Moreover, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any z in the region

$$\mathcal{Z}_\varepsilon \equiv \{z \in \mathbb{C};\ \mathcal{A}rg\, z \in (\pi - \varepsilon, \pi + \varepsilon)\}, \tag{C.2}$$

there holds,

$$|M(z,\zeta)| + |\zeta|\,|\frac{d}{d\zeta} M(z,\zeta)| \ \le\ C(z)|\zeta|^\delta \quad for \quad |\zeta| < 2,\ \zeta \in D(\varepsilon_0), \tag{C.3}$$

$$|M(z,\zeta)| \ \le\ C(z)|\zeta|^{1-\delta} \quad for \quad |\zeta| > 2,\ \zeta \in D(\varepsilon_0), \tag{C.4}$$

where $C(z)$ is a positive constant, which depends on z.

The proof of Proposition C.1 is based on the following technical Lemma.

Lemma C.2 *Suppose that f is analytic in the cone*

$$C(2\varepsilon_0) \equiv \{\zeta \in \mathbb{C}; \zeta = |\zeta| e^{i\theta},\ \theta \in (-2\varepsilon_0, 2\varepsilon_0)\}$$

for some $\varepsilon_0 > 0$. Let us also assume that:

$$\int_0^\infty \frac{|f(re^{i\theta})|}{1+r^2}\, dr < +\infty,\ for\ any\ \theta \in (-2\varepsilon_0, 2\varepsilon_0) \tag{C.5}$$

$$\lim_{\lambda \to 0 \lambda \in C(2\varepsilon_0)} f(\lambda) = L_1, \quad \lim_{\lambda \to \infty,\, \lambda \in C(2\varepsilon_0)} f(\lambda) = L_2, \quad L_i \in \mathbb{C},\ i = 1, 2. \tag{C.6}$$

$$|f'(\lambda)| = o(\frac{1}{\lambda}), \quad as\ \lambda \to 0,\ \lambda \to +\infty,\ \lambda \in C(2\varepsilon_0). \tag{C.7}$$

Then, for any $\lambda_0 \in \mathbb{C} \setminus C(2\varepsilon_0)$*, the function*

$$F(\zeta) = \frac{1}{2\pi i} \int_0^\infty f(\lambda) \left(\frac{1}{\lambda - \zeta} - \frac{1}{\lambda - \lambda_0} \right) d\lambda \tag{C.8}$$

is analytic in the domain $D(\varepsilon_0) \subset \mathcal{S}$ *defined in (C.1). Moreover:*

$$F(\zeta) = -\frac{L_1}{2\pi i} \ln \zeta + o\left(\ln|\zeta|\right), \quad \textit{as } \zeta \to 0,\ \zeta \in D(\varepsilon_0). \tag{C.9}$$

$$F(\zeta) = -\frac{L_2}{2\pi i} \ln \zeta + o\left(\ln|\zeta|\right), \quad \textit{as } \zeta \to +\infty,\ \zeta \in D(\varepsilon_0). \tag{C.10}$$

Proof. The analyticity of F in $D(\varepsilon_0)$ follows from standard complex variable theory combined with deformation of the contour of integration from $\mathbb{R}^+$ to the rays $e^{i\theta}\,\mathbb{R}^+$ with $|\theta| \leq \varepsilon_0$. These deformations can be performed due to the analyticity properties of f and (C.5).

We now proceed to prove (C.9). We describe in detail the proof only for those values of $\zeta \in D$. For arbitrary values of $\zeta \in D(\varepsilon_0)$ it is possible to argue in a similar manner, after deformation of the contour of integration from $\mathbb{R}^+$ to $e^{i\theta}\,\mathbb{R}^+$, $|\theta| \leq \varepsilon_0$. Let us define,

$$S(\lambda, \zeta, \lambda_0) = \frac{1}{2\pi i} \left(\frac{1}{\lambda - \zeta} - \frac{1}{\lambda - \lambda_0} \right).$$

Then,

$$\begin{aligned} F(\zeta) &= L \int_0^1 S(\lambda, \zeta, \lambda_0)\, d\lambda + \int_0^1 \left(f(\lambda) - L\right) S(\lambda, \zeta, \lambda_0)\, d\lambda \\ &\quad + \int_1^\infty f(\lambda) S(\lambda, \zeta, \lambda_0)\, d\lambda \qquad \text{(C.11)} \\ &\equiv J_1(\zeta) + J_2(\zeta) + J_3(\zeta). \qquad \text{(C.12)} \end{aligned}$$

The term $J_1(\zeta)$ can be explicitly computed and it readily follows that

$$J_1(\zeta) = -\frac{L}{2\pi i} \ln \zeta + \mathcal{O}(1), \text{ as } \zeta \to 0,\ \zeta \in D. \tag{C.13}$$

On the other hand,

$$J_3(\zeta) = \mathcal{O}(1), \text{ as } \zeta \to 0,\ \zeta \in D. \tag{C.14}$$

Finally, we estimate J_2. To this end we rewrite it as:

$$\begin{aligned} J_2(\zeta) &= \frac{1}{2\pi i}\int_0^{|\zeta|/2}(f(\lambda)-L)\,\frac{d\lambda}{\lambda-\zeta}+\frac{1}{2\pi i}\int_{|\zeta|/2}^{2|\zeta|}(f(\lambda)-L)\,\frac{d\lambda}{\lambda-\zeta} \\ &+\frac{1}{2\pi i}\int_{2|\zeta|}^{1}(f(\lambda)-L)\,\frac{d\lambda}{\lambda-\zeta}+\mathcal{O}(1),\ \text{as } \zeta\to 0,\ \zeta\in D. \end{aligned} \quad (C.15)$$

Splitting the region of integration of the third integral in two parts $(2|\zeta|,\delta)$ and $(\delta,1)$, with $\delta>0$ small, we obtain .

$$\begin{aligned} I \equiv |\int_{2|\zeta|}^{1}(f(\lambda)-L)\ \frac{d\lambda}{\lambda-\zeta}|+|\int_0^{|\zeta|/2}(f(\lambda)-L)\,\frac{d\lambda}{\lambda-\zeta}| \\ \leq \sup_{\lambda\in(0,\delta)}|f(\lambda)-L||\ln\zeta|+\frac{C}{\delta}\sup_{\lambda\in(0,1)}|f(\lambda)| \quad \text{as } \zeta\to 0,\, \zeta\in D. \end{aligned}$$

Therefore, by (C.6), we deduce:

$$I=o(|\ln\zeta|),\ \text{as}\ \zeta\to 0,\,\zeta\in D. \quad (C.16)$$

In order to estimate the second term in the right hand side of (C.15), after integrating by parts,

$$|\int_{|\zeta|/2}^{2|\zeta|}(f(\lambda)-L)\,\frac{d\lambda}{\lambda-\zeta}| \ \leq\ |\int_{|\zeta|/2}^{2|\zeta|}f'(\lambda)\ln(\lambda-\zeta)d\lambda|+C\sup_{(0,2|\zeta|)}|f(\lambda)-L||\ln\zeta|.$$

Using now (C.7) it follows that

$$|\int_{|\zeta|/2}^{2|\zeta|}f'(\lambda)\ln(\lambda-\zeta)d\lambda|=o(\ln|\zeta|) \quad \text{as } |\zeta|\to 0$$

henceforth:

$$|\int_{|\zeta|/2}^{2|\zeta|}(f(\lambda)-L)\,\frac{d\lambda}{\lambda-\zeta}|=o(\ln|\zeta|) \quad \text{as } |\zeta|\to 0 \quad (C.17)$$

Combining (C.13)-(C.17) we arrive at (C.9). The proof of (C.10) can be made along similar lines.

Proof of Proposition C.1. Since the function h satisfies all the assumptions of the function f in Lemma C.2, it follows, that $M(z,\cdot)$ is well defined and analytic in $D(3\varepsilon_0/2)$.
On the other hand, Lemma C.2 also implies that

$$M(z,\zeta) = \exp\{-\frac{1}{2\pi i}\left(\ln\left(-\frac{a}{z}\right)\right)\ln|\zeta| + o(\ln|\zeta|)\},$$

as $|\zeta| \to 0$. Using (3.11), we obtain that:

$$|M(z,\zeta)| \le C(z)|\zeta|^{\delta} \quad \text{for} \quad |\zeta| < 2,\ \zeta \in D(\varepsilon_0)$$

for $z \in \mathcal{Z}_\varepsilon$ and some $\delta = \delta(\varepsilon) > 0$. Since the function $M(z,\cdot)$ is analytic in $D(\varepsilon_0)$, (C.3) follows using Cauchy estimates.

It only remains to prove (C.4). To this end, notice that by (C.10) ,

$$M(z,\zeta) = \exp\{-\frac{1}{2\pi i}\lim_{\zeta\to+\infty}\left(\ln\left(-\frac{\varphi(\zeta)}{z}\right)\right)\ln|\zeta| + o(\ln|\zeta|)\},$$

as $\zeta \to +\infty$. Notice that, due to (P-4) of Appendix B,

$$\lim_{\zeta\to+\infty}\left(\ln\left(-\frac{\varphi(\zeta)}{z}\right)\right) = \lim_{\zeta\to 0}\left(\ln\left(-\frac{\varphi(\zeta)}{z}\right)\right) = \ln\left(-\frac{a}{z}\right)$$

where,

$$\mathcal{I}m\left(\ln\left(-\frac{a}{z}\right)\right) \in (-2\pi, 0).$$

Therefore, for $z \in \mathcal{Z}_\varepsilon$, and some $\delta = \delta(\varepsilon) > 0$.

$$|M(z,\zeta)| \le C(z)|\zeta|^{1-\delta} \quad \text{for} \quad |\zeta| > 2,\ \zeta \in D(\varepsilon_0).$$

C.2 Properties of the function $W(z,\zeta)$.

Proposition C.3 *The function $W(z,\cdot)$ defined in (3.16) is analytic in the region D defined by (3.4). Moreover, for any $z \in \mathcal{Z}_\varepsilon$ there exists $\delta > 0$ such that:*

$$|W(z,\zeta)| \le C(z) \quad \textit{if } |\zeta| \le 1,\ \zeta \in D \tag{C.18}$$

$$|W(z,\zeta)| \le C(z)|\zeta|^{1-\delta} \quad \textit{if } |\zeta| \ge 1,\ \zeta \in D \tag{C.19}$$

where $C(z)$ depends on z.

Proof. The integral defining $W(z,\zeta)$ converges due to (C.3) and (C.4). We begin proving (C.18). The only difficulty is to estimate $W(z,\zeta)$ for $\zeta \to 0$. We give the argument for $\mathcal{A}rg(\zeta) \in (\varepsilon_0, 2\pi - \varepsilon_0)$, since for arbitrary $\zeta \in D$ a similar argument can be made after suitable contours deformations that use the analyticity of $M(z,\cdot)$ in $D(\varepsilon_0)$.
We write:

$$W(z,\zeta) = \int_0^{2|\zeta|} (\cdots)\, d\lambda + \int_{2|\zeta|}^1 (\cdots)\, d\lambda + \int_1^\infty (\cdots)\, d\lambda. \tag{C.20}$$

The last integral in (C.20) is trivially bounded as $\zeta \to 0$. In order to estimate the first integral in the right hand side of (C.20), we use the fact that $|\lambda - \zeta| \geq \varepsilon_0/2\,|\zeta|$, for $\mathcal{A}rg(\zeta) \in (\varepsilon_0, 2\pi - \varepsilon_0)$. Using (C.3) we can estimate that term as $C|\zeta|^\delta$. Finally, to estimate the second term of the right hand side of (C.20) we use for $\lambda \in (2|\zeta|, 1)$, $|\lambda - \zeta| \geq |\lambda|/2$. Henceforth, that integral is bounded as $\zeta \to 0$. From all these estimates, (C.18) follows.

To prove (C.19) we split the integral defining W as:

$$W(z,\zeta) = \int_0^1 (\cdots)\, d\lambda + \int_1^\infty (\cdots)\, d\lambda. \tag{C.21}$$

The first integral in the right hand side of (C.21) is uniformly bounded for large ζ. On the other hand the second one, might be estimated as

$$|\int_1^\infty (\cdots)\, d\lambda| \leq C \int_1^\infty |\lambda|^{1-\delta} \frac{1}{|\lambda|} \frac{|\zeta|}{|\lambda - \zeta|}\, d\lambda \leq \frac{C}{\varepsilon_0} |\zeta|^{1-\delta}$$

for $\mathcal{A}rg(\zeta) \in (\varepsilon_0, 2\pi - \varepsilon_0)$. Henceforth, (C.19) and Proposition C.3 follow.

C.3 Properties of the function $\mathcal{V}$.

In this subsection we study several properties of the function $\mathcal{V}$ that are needed in the following.

Proposition C.4 *The function $\mathcal{V}$, defined by means of (3.25) (3.26) in $\mathcal{I}m\eta \in (4/3 - \delta, 5/3)$ for some $\delta > 0$, satisfies*

$$\mathcal{V}(\eta - \frac{i}{3}) = -\mathcal{V}(\eta)\, \frac{\Phi(\eta - i/3)}{a}. \tag{C.22}$$

for all $\eta \in \mathbb{C}$ such that $\mathcal{I}m\,\eta \in (4/3-\delta, 5/3)$. The function $\mathcal{V}$ can be extended analytically to the strip $\mathcal{I}m\eta \in (-1/3, 11/6)$ and meromorphically to $\mathbb{C}$ using (C.22)

Proof. The analyticity properties of $\mathcal{V}$ would follow from (C.22) in the strip $\mathcal{I}m\,\eta \in (4/3-\delta, 5/3)$ deforming slightly the contour of integration. Indeed, Φ is a meromorphic function in $\mathbb{C}$, without poles in the strip $\mathcal{I}m\eta \in (1, 4/3)$ and with a zero in $\eta = 7i/6$. We then restrict our attention to the proof of (C.22).
Using (3.25),

$$\mathcal{V}(\eta - \frac{i}{3}) = \exp[-3i \int_{\mathcal{I}m\,y=\frac{4}{3}} \ln\left(-\frac{\Phi(y+i0)}{a}\right) e^{6\pi y}\left(\frac{1}{e^{6\pi y}-e^{6\pi\eta}} - \frac{1}{e^{6\pi y}-ae^{6\pi\delta i}}\right)dy] \tag{C.23}$$

Deforming the contour of integration to $\mathcal{I}m\,y = \frac{4}{3}-\delta+\varepsilon$ with $5/3+\delta-\mathcal{I}m\eta < \varepsilon < \delta$, we pass through the pole $y = \eta - i/3$ of the integrand. Then, using residues:

$$\begin{aligned}
&\int_{\mathcal{I}m\,y=\frac{4}{3}} \ln\left(-\frac{\Phi(y+i0)}{a}\right) e^{6\pi y}\left(\frac{1}{e^{6\pi y}-e^{6\pi\eta}} - \frac{1}{e^{6\pi y}-ae^{6\pi\delta i}}\right)dy \\
&= \int_{\mathcal{I}m\,y=(\frac{4}{3}-\delta+\varepsilon)} \ln\left(-\frac{\Phi(y)}{a}\right) e^{6\pi y}\left(\frac{1}{e^{6\pi y}-e^{6\pi\eta}} - \frac{1}{e^{6\pi y}-ae^{6\pi\delta i}}\right)dy \\
&+\frac{i}{3}\ln\left(-\frac{\Phi(\eta-i/3)}{a}\right).
\end{aligned}$$

Plugging this formula into (C.23) we obtain (C.22) and Proposition C.4 follows.

Proposition C.5 *The only zeros of the function $\mathcal{V}$ in the strip $\mathcal{I}m\eta \in (-1/3, 11/6)$ are located at $\eta = i\,\frac{(1+2k)}{6}, k = 0,1,2,3$. These zeros are simple and*

$$\lim_{\eta\to\frac{7i}{6}} \frac{\mathcal{V}(\eta)}{(\eta-7i/6)} = -\frac{\mathcal{V}(3i/2)\,\Phi'(7i/6)}{a} \neq 0 \tag{C.24}$$

Moreover the only pole of the function $\mathcal{V}$ in $\mathcal{I}m\eta \in (-1/3, 2)$ is $\eta = 11i/6$.

Proof. The results concerning the poles are just a consequence of formula (C.22) and properties (P-1), (P-2) in Appendix B. The fact the only zeros of $\mathcal{V}$ in that strip are $i\,\frac{1+k}{6}$ for $k = 0, 1, 2, 3$ is a consequence of (C.22) as well as property (P-2) of the function Φ in Appendix B. Formula (C.24) follows from (C.22), the fact that $\mathcal{V}(3i/2) \neq 0$ is a consequence of (3.24). Finally, by (P-2) in Appendix B, $\Phi'(7i/6) \neq 0$.

In order to describe the asymptotic behaviour of the function $G(z, \xi)$ as $|\xi| \to \infty$ we will use the following

Proposition C.6 *For any $\xi \in \mathbb{C}$ such that $\mathcal{I}m\xi \in (4/3, 5/3)$ and $\eta \in \mathbb{C}$ such that $\mathcal{I}m\eta = 5/3$,*

$$\frac{\mathcal{V}(\eta)}{\mathcal{V}(\xi)} = e^{\left[3i(\eta-\xi)\ln\left(-\frac{\Phi(\xi)}{a}\right)\right]} e^{[h(\xi,\eta-\xi)]} \tag{C.25}$$

where the function

$$h(\xi, \eta - \xi) := -3i \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon)} \ln\left(\frac{\Phi(y)}{\Phi(\xi)}\right) \frac{\left(e^{6\pi(\eta-y)} - e^{6\pi(\xi-y)}\right) dy}{\left(1 - e^{6\pi(\eta-y)}\right)\left(1 - e^{6\pi(\xi-y)}\right)} \tag{C.26}$$

satisfies:

$$|h(\xi, \eta - \xi)| \leq C \left\{\min\{|\xi - \eta|^2 |\xi|^{-7/6}, |\xi - \eta||\xi|^{-1/6}\} + |\xi|^{-7/6}\right\}. \tag{C.27}$$

Moreover, there exist $\alpha > 0$ and $C > 0$ such that

$$C^{-1} e^{-\alpha|\xi|^{5/6}} \leq |\mathcal{V}(\xi)| \leq C e^{\alpha|\xi|^{5/6}}, \tag{C.28}$$

Proof. Using (3.26),

$$\frac{\mathcal{V}(\eta)}{\mathcal{V}(\xi)} = \exp[-3i \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon)} \ln\left(-\frac{\Phi(y)}{a}\right) e^{6\pi y} \left(\frac{1}{e^{6\pi y} - e^{6\pi\eta}} - \frac{1}{e^{6\pi y} - e^{6\pi\xi}}\right) dy]$$

where $\varepsilon > 0$ has been chosen such that $4/3 + \varepsilon < \mathcal{I}m\xi$. We write now:

$$\frac{\mathcal{V}(\eta)}{\mathcal{V}(\xi)} \;=\; \exp[-3i \ln\left(-\frac{\Phi(\xi)}{a}\right) \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon)} \frac{\left(e^{6\pi(\eta-y)} - e^{6\pi(\xi-y)}\right) dy}{\left(1 - e^{6\pi(\eta-y)}\right)\left(1 - e^{6\pi(\xi-y)}\right)}] e^{h(\xi,\eta)}.$$

Elementary calculus yields

$$\int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon)} \frac{\left(e^{6\pi(\eta-y)} - e^{6\pi(\xi-y)}\right) dy}{\left(1 - e^{6\pi(\eta-y)}\right)\left(1 - e^{6\pi(\xi-y)}\right)} = \eta - \xi,$$

and (C.25) follows.
In order to finish the proof of Proposition C.6 it only remains to prove that h satisfies (C.27). To this end we need the following Lemma.

Lemma C.7 *There exists a constant $C > 0$ such that, for all ξ and y in $\mathbb{C}$ satisfying $\mathcal{I}m\xi \in (4/3, 5/3)$, $\mathcal{I}my \in (4/3, 5/3)$ there holds:*

$$|\ln\left(\frac{\Phi(y)}{\Phi(\xi)}\right)| \leq C\,\frac{|y-\xi|}{|\xi|^{7/6}+|y|^{7/6}}.$$

Proof. We distinguish the two following cases $|\xi - y| \geq 2|\xi|$ and $|\xi - y| \leq 2|\xi|$. In the first case, $|y| \geq |\xi|$ and

$$|\Phi(\xi)-\Phi(y)| \leq |\Phi(\xi)-1| + |\Phi(y)-1| \leq \frac{C}{|\xi|^{1/6}} \leq \frac{C\,|\xi-y|}{|\xi|^{7/6}}$$

where in the second inequality we use the property (P-3) in the Appendix B. On the other hand, if $|\xi - y| \leq 2|\xi|$, we use that

$$\Phi(y)-\Phi(\xi) = \int_{\xi}^{y} \Phi'(\lambda)\,d\lambda,$$

where the contour of integration is a segment connecting ξ and y. Using again (P-3) we obtain

$$|\Phi(\xi)-\Phi(y)| \leq C\int_{\xi}^{y}\frac{d\lambda}{|\lambda|^{7/6}} \leq \frac{C\,|\xi-y|}{|\xi|^{7/6}}.$$

Using that $|\ln(\Phi(y)/\Phi(\xi))| \leq C|\Phi(y)-\Phi(\xi)|/|\Phi(\xi)|$ as well as the fact that $|\Phi(\xi)|$ is bounded from below for $\mathcal{I}m\xi \in (4/3, 5/3)$ we deduce

$$|\ln\left(\frac{\Phi(y)}{\Phi(\xi)}\right)| \leq C\,\frac{|y-\xi|}{|\xi|^{7/6}}.$$

Exchanging the role of the variables ξ and y, Lemma (C.7) follows.

End of the proof of Proposition C.6. Using Lemma C.7 it follows that,

$$|h(\xi,\eta-\xi)| \leq C\int_{\mathcal{I}m\,y=(\frac{4}{3}+\varepsilon)}\frac{|y-\xi|}{|\xi|^{7/6}+|y|^{7/6}}\,\frac{\left|e^{6\pi(\eta-y)}-e^{6\pi(\xi-y)}\right|\,|dy|}{|1-e^{6\pi(\eta-y)}|\,|1-e^{6\pi(\xi-y)}|}.$$

Suppose that $\mathcal{R}e\xi \geq \mathcal{R}e\eta$. Then,

$$|h(\xi,\eta-\xi)| \leq C \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon)} \frac{|y-\xi|}{|\xi|^{7/6}+|y|^{7/6}} \frac{\left|e^{6\pi(\xi-y)}\right|\,|dy|}{|1-e^{6\pi(\eta-y)}|\,|1-e^{6\pi(\xi-y)}|}.$$

The terms $|1-e^{6\pi(\eta-y)}|$ and $|1-e^{6\pi(\xi-y)}|$ in the denominator do not vanish due to the choice of the contour of integration. Moreover they can be estimated from below independently of ε, by means of a suitable choice of ε. Therefore,

$$\begin{aligned} |h(\xi,\eta-\xi)| &\leq C \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon),\mathcal{R}ey\leq\mathcal{R}e\xi} \frac{|y-\xi|}{|\xi|^{7/6}+|y|^{7/6}} \frac{|dy|}{|1-e^{6\pi(\eta-y)}|} \\ &+ C \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon),\mathcal{R}ey\geq\mathcal{R}e\xi} \frac{|y-\xi|\, e^{6\pi\mathcal{R}e(\xi-y)}}{|\xi|^{7/6}+|y|^{7/6}}\,|dy| \end{aligned}$$

Then,

$$\begin{aligned} |h(\xi,\eta-\xi)| &\leq C \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon),\mathcal{R}ey\leq\mathcal{R}e\eta} \frac{|y-\xi|}{|\xi|^{7/6}} \frac{|dy|}{|1-e^{6\pi(\eta-y)}|} \\ &+ C \int_{\mathcal{I}m\, y=(\frac{4}{3}+\varepsilon),\mathcal{R}e\eta\leq\mathcal{R}ey\leq\mathcal{R}e\xi} \frac{|y-\xi|}{|\xi|^{7/6}+|y|^{7/6}}|dy| \\ &+ \frac{C}{1+|\xi|^{7/6}}. \end{aligned} \tag{C.29}$$

Using $|y-\xi| \leq |y-\eta|+|\eta-\xi|$, the first term in the right hand side of (C.29) can be estimated as $C(1+|\eta-\xi|)/|\xi|^{7/6}$.
In order to estimate the second integral, we use first that $|y-\xi| \leq C|\eta-\xi|$. The remaining integral may then be estimated as $C|\xi-\eta||\xi|^{-1/6}$. On the other hand, this second term might be also bounded as $C|\xi-\eta|^2/|\xi|^{7/6}$. Therefore, combining the two inequalities we obtain:

$$|h(\xi,\eta-\xi)| \leq C \left\{ \min\{|\xi-\eta|^2/|\xi|^{7/6}, |\xi-\eta||\xi|^{-1/6}\} + |\xi|^{-7/6} \right\}.$$

The argument for $\mathcal{R}e\eta \geq \mathcal{R}e\xi$ is similar using the symmetry of the integrand. Finally, (C.29) is an immediate consequence of (C.25), (C.27) and (P-3) in Appendix B.

Acknowledgements. We acknowledge several illuminating discussions with Professor M. A. Valle. This work has been partially supported by *HYKE*

HPRN-CT-2002-00282. M.E. is supported by CICYT Grant BFM 2002-03345, S. M. is supported by and J.L.V. is supported by CICYT Grant MTM2004-05634. Finally, we acknowledge the hospitality of the Ecole Normale Supérieure de Paris.

References

[1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, Dover Publications, Inc., New York (1972).

[2] A. Ajeizer, S. Peletminsky, Les méthodes de la physique statistique, MIR, Moscou (1980).

[3] A. M. Balk, *On the Kolmogorov-Zakharov spectra of weak turbulence*, Physica D **139**, 137–157, (2000).

[4] A. M. Balk, V. E. Zakharov, *Stability of Weak-Turbulence Kolmogorov Spectra* in Nonlinear Waves and Weak Turbulance, V. E. Zakharov Ed., A. M. S. Translations Series 2, Vol. 182, 1998, 1-81.

[5] M. Escobedo, S. Mischler, J. J. L. Velazquez, *Singular Solutions for the Uehling Uhlenbeck Equation*, preprint 2006.

[6] C. Josserand and Y. Pomeau, *Nonlinear aspects of the theory of Bose-Einstein condensates*, Nonlinearity **14**, R25-R62 (2001)

[7] A. V. Kats, V. M. Kantorovich, Symmetry properties of the collision integral and nonisotropic stationary solutions in weak turbulence theory, JETP **37** (1973) 80–85.

[8] A. V. Kats, V. M. Kantorovich, Anysotropic turbulent distributions for waves with a nondecay dispersion law, JETP **38** (1974) 102–107.

[9] R. Lacaze, P. Lallemand, Y. Pomeau and S. Rica, *Dynamical formation of a Bose-Einstein condensate.*, Physica D **152-153** (2001) 779–786.

[10] L. D. Landau, E. M. Lifshitz, Course of theoretical physics, Pergamon Press, 1977. Third Edition.

[11] E. Levich and V. Yakhot, *Time evolution of a Bose system passing through the critical point*, Phys. Rev. B **15**, (1977), 243–251 .

[12] E. Levich and V. Yakhot, *Time development of coherent and superfluid properties in a course of a $\lambda-$transition*, J. Phys. A, Math. Gen., **11**, (1978), 2237–2254.

[13] X. G. Lu, *A modified Boltzmann equation for Bose-Einstein partiacles: isotropic solutions and long time behavior*, J. Stat. Phys. **98**, (2000), 1335–1394.

[14] X. G. Lu, *On isotropic distributional solutions to the Boltzmann equation for Bose-Einstein particles*, J. Stat. Phys. **116**, (2004), 1597–1649.

[15] N. I. Muskhelishvili, Singular integral equations, Dover Publications, Inc. New York, 1992.

[16] L. W. Nordheim, *On the Kinetic Method in the New Statistics and its Applications in the Electron Theory of Conductivity*, Proc. R. Soc. London A **119**, (1928), 689–698.

[17] M. Reed, B. Simon, Methods of Mathematical Physics, Academic Press, New York, Vol.2: Fourier Analysis and Self Adjointness, 1975.

[18] D.V. Semikov, I.I. Tkachev,*Kinetics of Bose condensation*, Phys. Rev. Lett. **74** (1995) 3093-3097.

[19] D.V. Semikov, I.I. Tkachev, *Condensation of Bosons in the kinetic regime* , Phys. Rev. D **55**, 2, (1997) 489-502.

[20] M. E. Taylor, Pseudodifferential Operators, Princeton University Press, Princeton, New Jersey, 1981.

[21] E. A. Uehling, G. E. Uhlenbeck, *Transport phenomena in Einstein-Bose and Fermi-Dirac gases*, Physical Review **43** (1933) 552-561.

[22] V. E. Zakharov, *Collapse of Langmuir Waves* , Sov. JETP **35** (1972) 908–914.

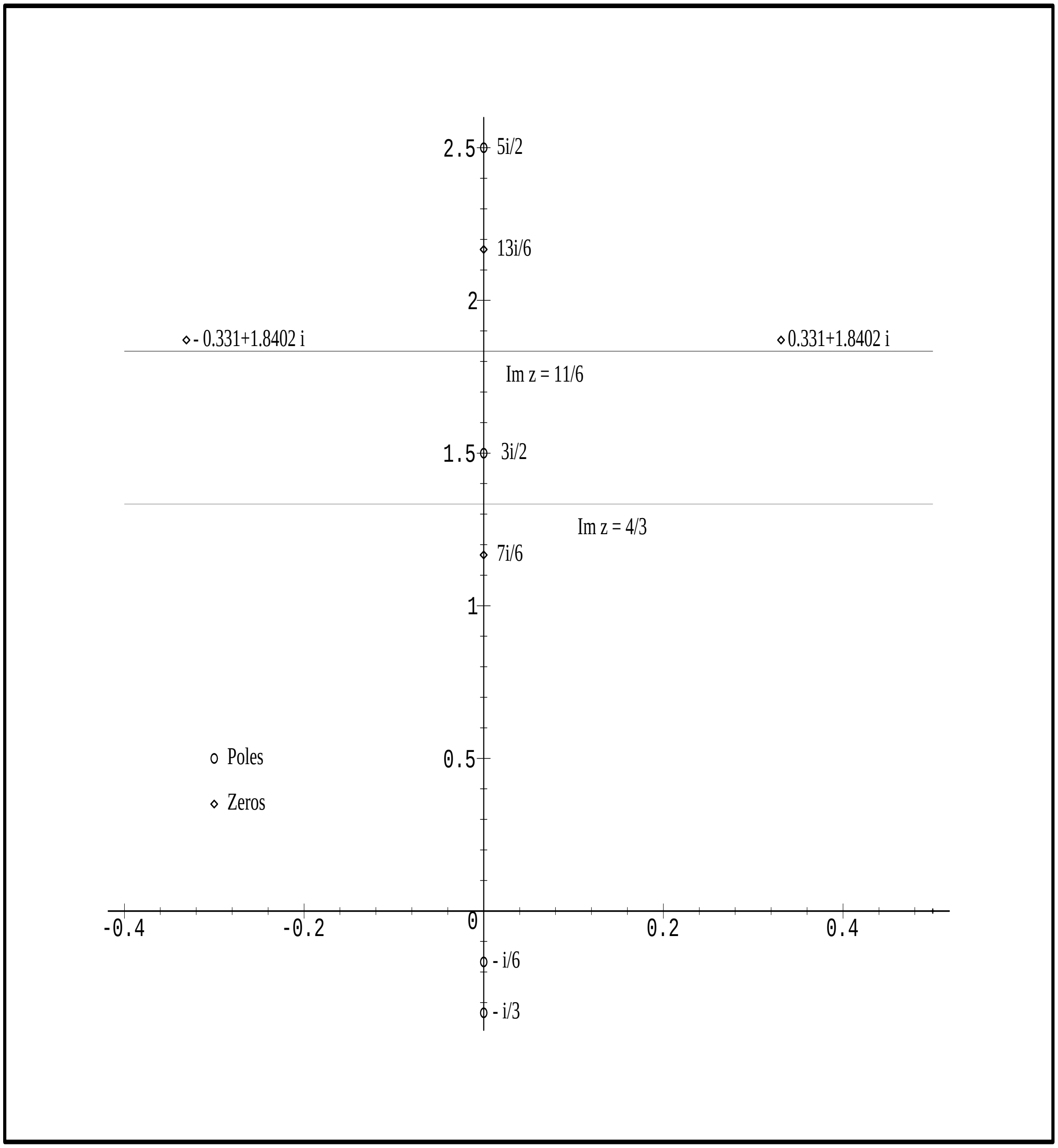

Figure 1: **Some of the zeros and poles of the function Φ.**

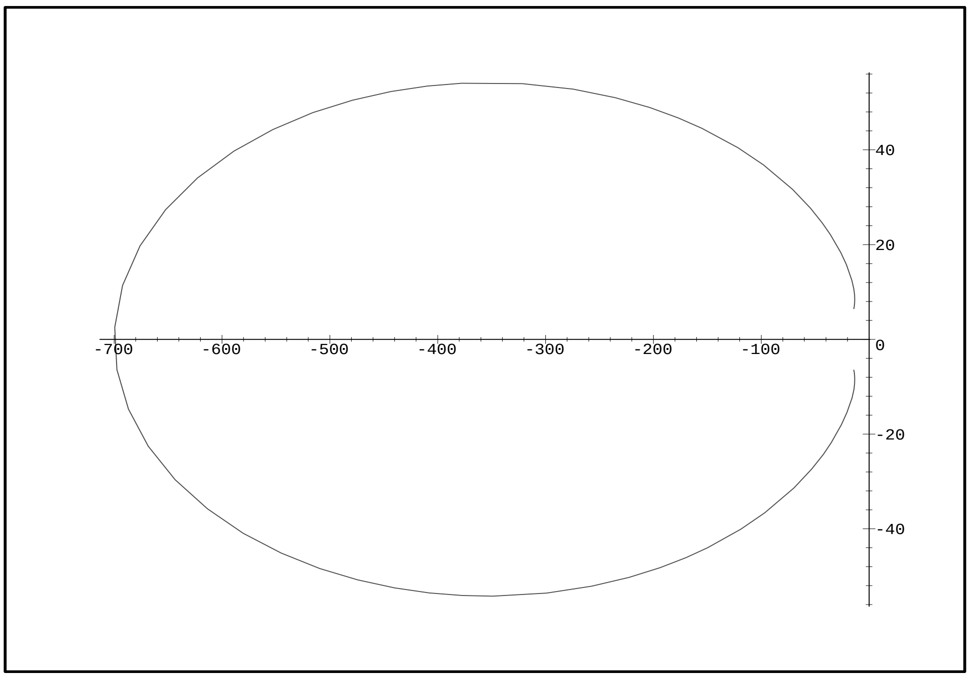

Figure 2: **b=-1/4**.

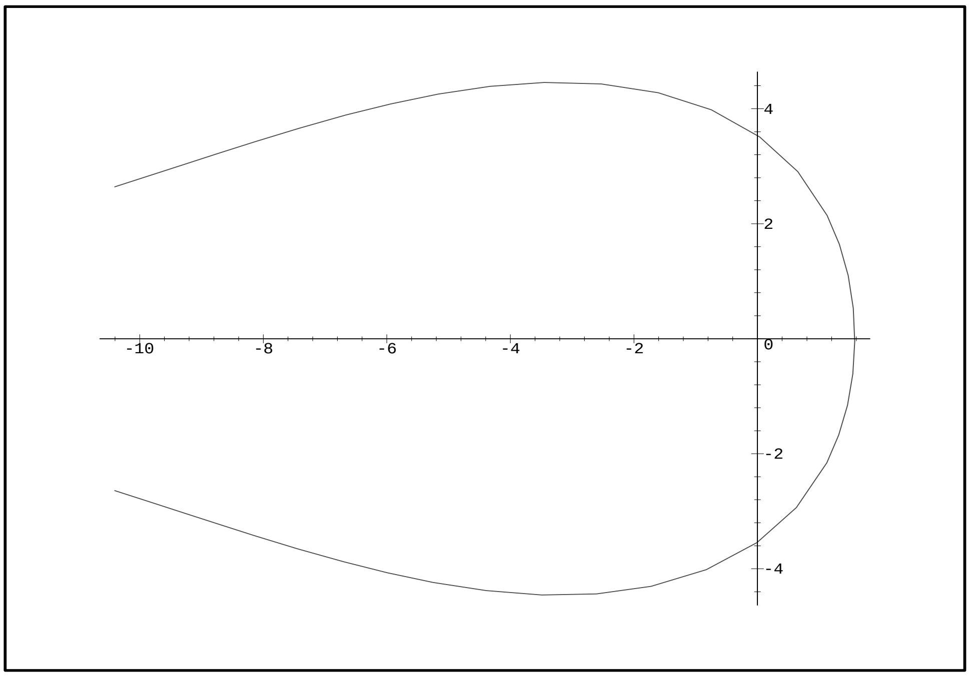

Figure 3: **b=1**.

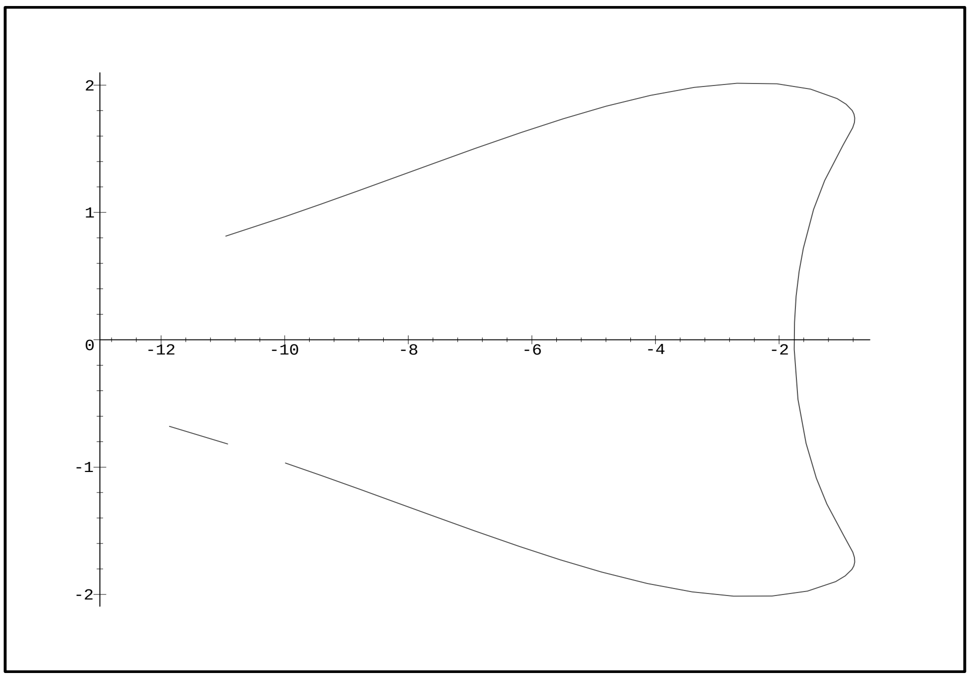

Figure 4: **b=4/3**.

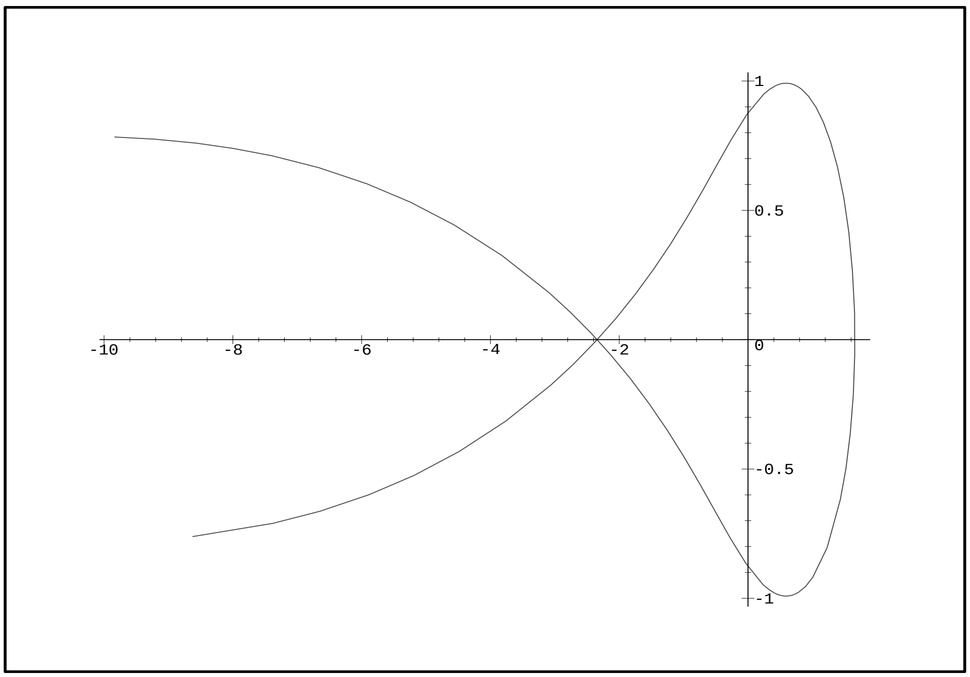

Figure 5: **b=5/3, r runs from -5 to 5**.

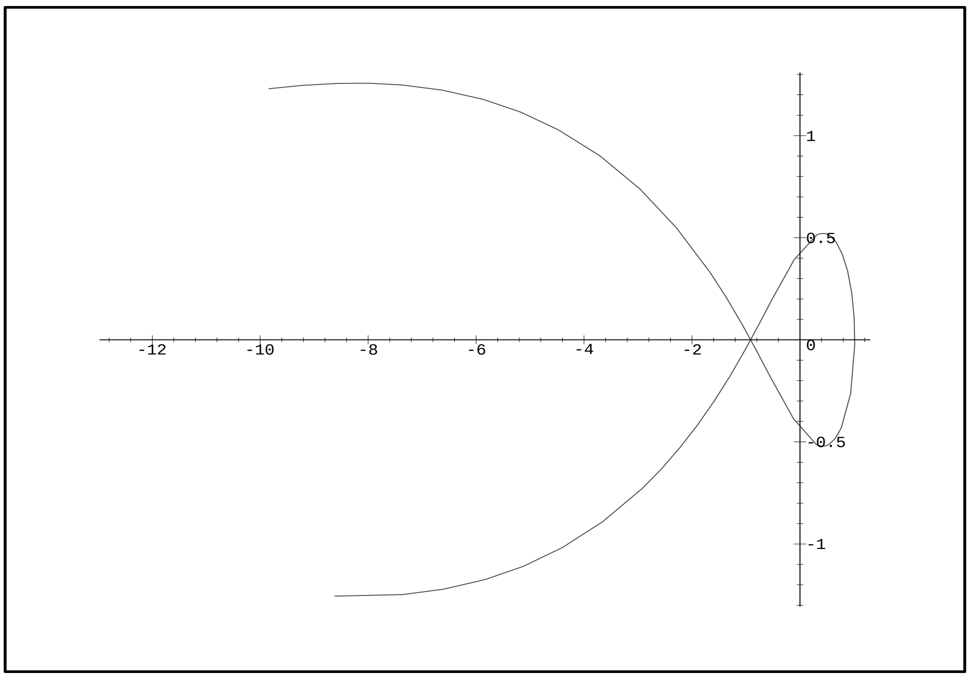

Figure 6: **b=21/12=1.75, r runs from -5 to 5**.

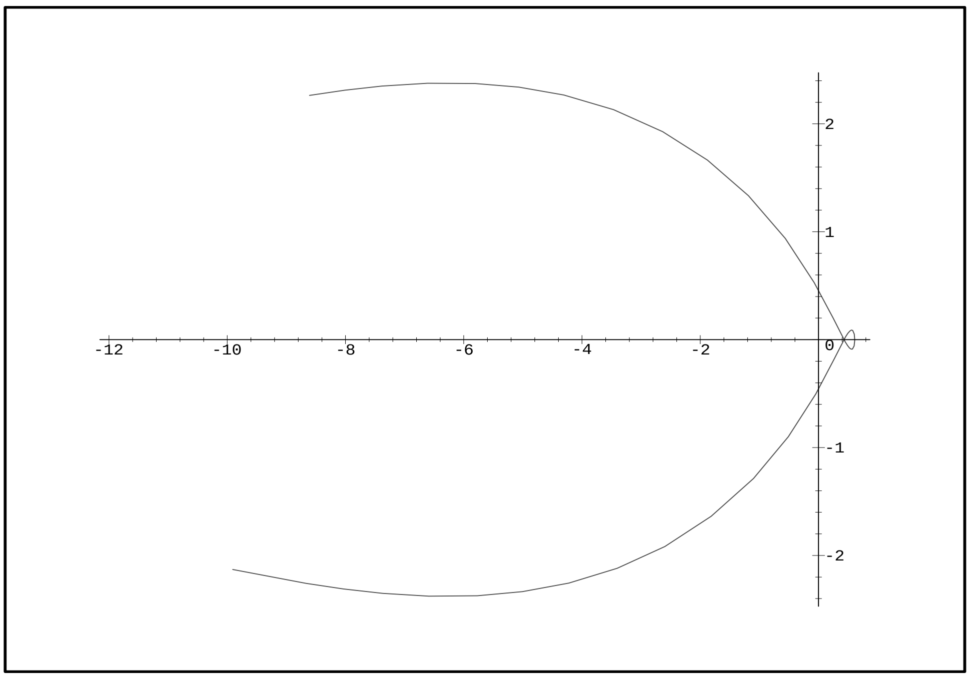

Figure 7: **b=23/12=1.9166666667, r runs from -5 to 5**.

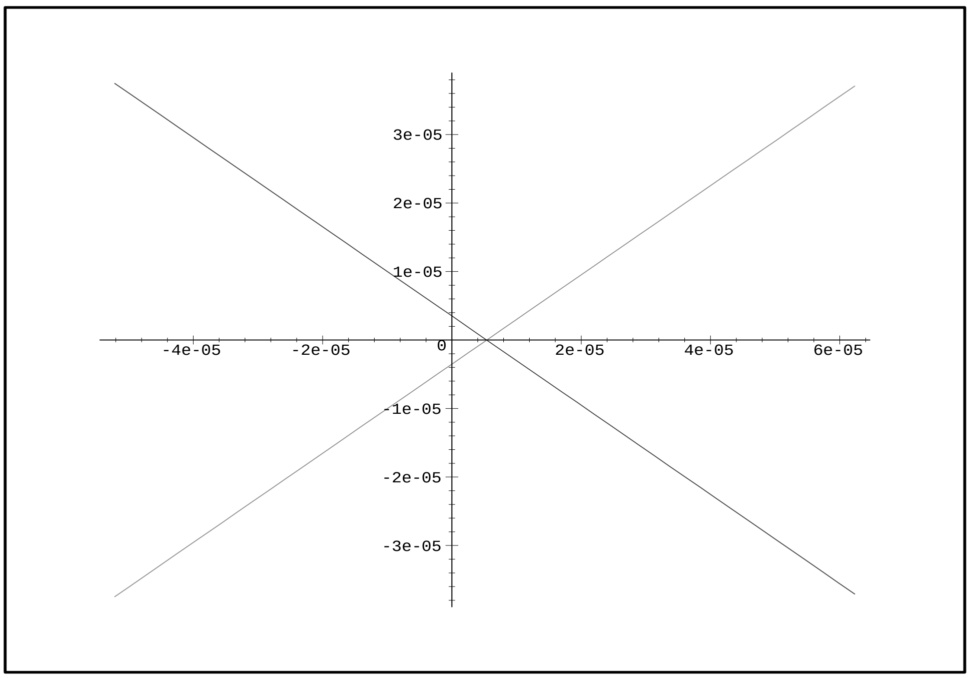

Figure 8: **b=1.8402088125**,r runs from -0.331301 to -0.331269 and from 0.331269 to 0.331301.

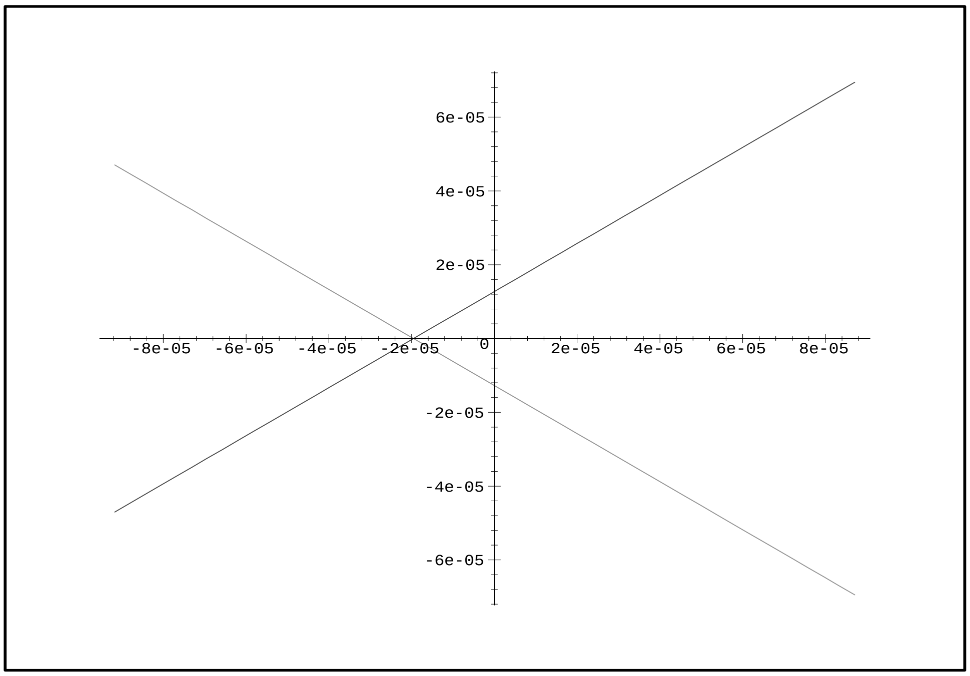

Figure 9: **b=1.840205625**, r runs from -0.33131 to -0.33126 and from 0.33126 to 0.33131..