Singular Solutions of Homogeneous Kinetic Equations

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Gas of bosons.

A dilute homogeneous isotropic gas of bosons is described by the Uehling-Uhlenbeck equation (in polar coordinates):

$$\text{(U-U)}\begin{cases} \dfrac{\partial f}{\partial t}(t,k_1) = \displaystyle\int\int_{D(k_1)} W\left(k_1,k_2,k_3,k_4\right) q(f) dk_3 dk_4 \\ q\left(f\right) = f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \end{cases}$$

$$D\left(k_1\right) \equiv \left\{(k_3,k_4) : k_3 > 0, k_4 > 0, k_3 + k_4 \geq k_1 > 0\right\}$$

$$W\left(k_1,k_2,k_3,k_4\right) = \frac{\min\left(\sqrt{k_1},\sqrt{k_2},\sqrt{k_3},\sqrt{k_4}\right)}{\sqrt{k_1}}, \quad k_2 = k_3 + k_4 - k_1.$$

L. W. Nordheim (1928), E. A. Uehling & G. E. Uhlenbeck (1933),
E. Zaremba, T. Nikuni, A. Griffin (1999) .

Steady States of the U-U Equation

The U-U equation preservesthe total density: $\frac{d}{dt}\int_0^\infty f(t,k)\sqrt{k}\,dk = 0$ and has a family of steady states $\mathcal{B}_\rho$ characterized by their total density $\rho > 0$:

- If $0 < \rho < \rho_0 := \int_0^\infty \frac{\sqrt{k}\,dk}{e^k - 1} \equiv \frac{\sqrt{\pi}}{2}\zeta(3/2)$, ζ: the Riemann's zeta function,

$$\mathcal{B}_\rho(k) \equiv F_\mu(k) := \frac{1}{e^{k+\mu} - 1} \quad \text{where} \quad \rho = \int_0^\infty \frac{\sqrt{k}\,dk}{e^{\mu+k} - 1},\ \mu \geq 0.$$

- If $\rho > \rho_0$: $\quad \mathcal{B}_\rho(k) \equiv \frac{1}{e^k - 1} + (\rho - \rho_0)\,\frac{\delta_0}{\sqrt{k}}, \quad \int_0^\infty \mathcal{B}_\rho(k)\sqrt{k}\,dk = \rho$

The solutions

$$F_\mu(k) = \frac{1}{e^{k+\mu} - 1}$$

are the classical Bose-Einstein equilibrium distributions if $\mu > 0$ and the Planck distribution if $\mu = 0$. The solutions

$$\mathcal{B}_\rho(k) = \frac{1}{e^k - 1} + (\rho - \rho_0)\frac{\delta_0}{\sqrt{k}}$$

are the classical distributions that describe the thermal equilibrium of a family of bosons with Bose-Einstein condensate of particles having zero momentum.

Our main contribution: To construct solutions of the U-U equation which behave like $k^{-7/6}$ as $k \sim 0$:

Singular solutions of the U-U equation.

For all initial data f_0 satisfying: $f_0(k) \le C_0 e^{-Bk}, \quad k \ge 1$

$$|f_0(k) - A\,k^{-7/6}| \le \frac{C}{k^{7/6-\delta}}, \quad 0 \le k \le 1,$$

$$|f_0'(k) + \frac{7}{6}A\,k^{-13/6}| \le \frac{C}{k^{13/6-\delta}}, \quad 0 \le k \le 1,$$

there exists a unique solution $f \in \mathbf{C}^{1,0}((0,T) \times (0,+\infty))$ and $a(t)$, satisfying:

$$0 \le f(t,k) \le L\frac{e^{-Dk}}{k^{7/6}}, \text{ if } k > 0; \quad |a(t)| \le L, \qquad \text{for } t \in (0,T)$$

$$|f(t,k) - a(t)\,k^{-7/6}| \le L\,k^{-7/6+\delta/2}, \quad k \le 1, \ t \in (0,T)$$

for some positive constant L and for some $T = T(A,B,\delta) > 0$.

Mathematical interest

There is a large literature on bounded solutions for Boltzmann type equations.
On the other hand, and more closely related:
X. Lu, J. Stat. Phys. **116** (2004)
proves global existence of weak solutions for the Uehling Uhlenbeck equation and describes the weak convergence towards the stationary solutions as $t \to \infty$.

Our result : seems to be the first example of singular solutions of a nonlinear kinetic equation with precise singular behaviour, $f \sim a(t)\, k^{-7/6}$ as $k \to 0$, for a general initial data, rigorously proveed.

A consequence of our analysis: the presence of some regularizing effect.
More precisely, write: $f(t,x) = \lambda(t) f_0(k) + g(k,t)$.
The function g satisfies an equation like:

$$\frac{\partial g}{\partial t}(t,k) = \mathcal{N}[t,k,g,\lambda] - \lambda'(t)\, f_0.$$

If the initial data g_0 satisfies:

$$||g_0||_{\alpha,\beta} = \sup_{0\le k\le 1} \{k^\alpha |g_0(k)|\} + \sup_{k\ge 1} \left\{k^\beta |g_0(k)|\right\}; \;\; \alpha = 3/2-\delta, \;\beta = 11/6-\delta,$$

for δ as small as we want, then, for some $T>0$, the solution g satisfies:

$$||g(t)||_{7/6,\beta} \le C(t,T)||g_0||_{\alpha,\beta}, \quad \forall t \in (0,T)$$

Notice that $3/2 > 7/6$.

Surprising: the structure of this equation suggests a "hyperbolic" non regularizing behaviour for its solutions. These regularizing effects are, however, restricted to the values of f at the particular point $k=0$.

Sketch of the proof: linearisation + fixed point

The main contribution in the U-U equation comes from the modified equation:

$$\text{(MU-U)} \qquad \frac{\partial f}{\partial t}(t,k) = \widetilde{Q}(f) \equiv \int_{D(k_1)} W\left(k_1,k_2,k_3,k_4\right)\tilde{q}(f)dk_3dk_4$$

$$\tilde{q}\left(f\right) = f_3f_4(f_1+f_2) - f_1f_2(f_3+f_4)$$
$$D\left(k_1\right) \equiv \left\{(k_3,k_4)) : k_3+k_4 \geq k_1\right\}$$
$$W\left(k_1,k_2,k_3,k_4\right) = \frac{\min\left(\sqrt{k_1},\sqrt{k_2},\sqrt{k_3},\sqrt{k_4}\right)}{\sqrt{k_1}}$$
$$k_2 = k_3+k_4-k_1$$

Particular stationary solutions: $\tilde{q}(1) = \tilde{q}(k^{-1}) = 0.$

General theory by V. E. Zakharov and co authors for general weakly nonlinear waves (for example: Translations of the AMS vol. 182).

Another particular solution: $\widetilde{Q}(k^{-7/6}) = 0$ but $\tilde{q}(k^{-7/6}) \neq 0.$

Consider the non radial equation for the function $n(p,t) = f(|p|^2, t)$:

$$\frac{\partial n}{\partial t}(t,p) = \mathcal{Q}(n) \equiv \int_{D(p_1)} W(p_1, p_2, p_3, p_4)\, \tilde{q}(n) dp_3 dp_4$$

The function $n(p) = |p|^{-7/3}$ satisfies the equation for all $p \neq 0$. Moreover: the flux of this solution out of the sphere $|p| < R$ is

$$\int_{|p|<R} \mathcal{Q}(|p|^{-7/3}) d\,p = C$$

where C is a positive constant independent of R. So we have actually: $\mathcal{Q}(n) = C\delta_{p=0}.$

This solution has been extensively considered in the literature on the Bose Einstein condensation. In particular by:

B.V. Svistunov: J. Moscow Phys. Soc. **1** (1991).

D.V. Semikov & I.I. Tkachev, Phys. Rev. Lett. **74** (1995)

R. Lacaze, P. Lallemand, Y. Pomeau & S. Rica: Physica D **152-153** (2001)

Proof in two steps:

Step 1. Fundamental solutions for the linearised MU-U equation.
Step 2. Fixed point argument on the nonlinear U-U equation.

The fixed point argument: similar to the standard semigroup techniques for parabolic equations.

Short description of the linearised problem:

Linearisation of MU-U

We linearise around $f(k) = k^{-7/6}$:

$$f(k,t) = k^{-7/6} + F(k,t)$$

and obtain the following equation for F:

$$\frac{\partial F}{\partial t} = \mathcal{L}(F) \equiv -\frac{a}{k^{1/3}} F(k) + \frac{1}{k^{4/3}} \int_0^\infty K\left(\frac{r}{k}\right) F(r)\, dr$$

where a is an explicit positive constant and the kernel $K(r)$ is explicit.

The Fundamental solution

$$F_t(t,k,k_0) = -\frac{a}{k^{1/3}} F(t,k,k_0) + \frac{1}{k^{4/3}} \int_0^\infty K(\frac{r}{k})\, F(t,r,k_0)\, dr,\ \ t>0, k>0,$$

$$F(0,k,k_0) = \delta(k-k_0).$$

Theorem. *For all* $k_0 > 0$*, there exists a unique solution* $F(t,\cdot,k_0)$*such that:* $F(t,k,k_0) = \frac{1}{k_0} F(\frac{t}{k_0^{1/3}}, \frac{k}{k_0}, 1)$*. For* $k \in (0,2)$ *the function* $F(t,k,1)$ *can be written as:* $F(t,k,1) = e^{-a\,t}\delta(k-1) + \sigma(t)\, k^{-7/6} + \mathcal{R}(t,k)$ *where* $\sigma(t) = A\, t^4 + \mathcal{O}(t^{4+k})$ *as* $t \to 0$*,* $\sigma(t) = \mathcal{O}(t^{-3})$ *as* $t \to \infty$*.*
And for $k > 2$*:* $F(t,k,1) \le \beta(t)(t^3/k)^{\frac{11}{6}}$*.*

Some Remarks.

• The initial Dirac measure at $k = k_0$ persists for all time $t > 0$ and is not regularised. That is a kind of hyperbolic behaviour.

• The total mass of the Dirac measure decays exponentially fast: it is "asymptotically" regularised.

• The behaviour $k^{-7/6}$ as $k \to 0$ persists for all time.

Sketch of the proof.

Change of variables: $k = e^x$,

$$F(t, k, 1) = \mathcal{G}(t, x), \quad K(r/k) = K(e^{-(x-y)}) = e^{x-y}\mathcal{K}(x - y)$$

with $\mathcal{K}(x) = e^{-x}K(e^{-x})$ and Laplace transform in t and Fourier transform in x:

The Carleman equation.

$$zG(z,\xi) = G(z,\xi - \frac{i}{3})\Phi(\xi - \frac{i}{3}) + \frac{1}{\sqrt{2\pi}}, \tag{1}$$

where $\Phi(\xi) = -a + \widehat{\mathcal{K}}(\xi)$ and $\widehat{\mathcal{K}}$ is the Fourier transform of $\mathcal{K}$. The problem is then transformed in the following:

For any $z \in \mathbb{C}$, $\mathcal{R}ez > 0$, find a function $G(z,\cdot)$ analytic in the strip $S = \{\xi;\ \xi = u + iv,\ 4/3 < v < 11/6,\ u \in \mathbb{R}\}$ satisfying (**??**) on S.

This is solved by a kind of Wiener Hopf argument.

Physical meaning of such asymptotics.

These particle distributions have a nonzero flux of particles towards the origin. More precisely, the asymptotics $f(t,k) \sim a(t)\, k^{-7/6}$ as $k \to 0$ means that the rate gain of particles towards the zero momentum is:

$$\lim_{K\to 0} \frac{d}{dt} \left(\int_{|k_1|\leq K} \sqrt{k_1}\, f(k_1,t)\, dk_1 \right) = -\frac{(a(t))^3}{3} U'(7/6)$$

where $$U(\nu) := \int_{D(1)} h(\xi_2, \xi_3, \xi_4, \nu)\, d\xi_3\, d\xi_4$$

and $$h(\xi_2, \xi_3, \xi_4, \nu) := \left[W(\xi_1, \xi_2, \xi_3, \xi_4)\, q(\xi^{-\nu}) \right]\Big|_{\xi_1 = 1}.$$

This nonzero flux of particles towards the particles of zero momentum: makes tempting to think that the solutions constructed could provide some information about the dynamic growth of Bose-Einstein condensates.

However, this does not seem to be the case.

Why ?: because the zero momentum particles would not interact with the particles outside the condensate as they should.

Model where the condensate interacts with non condensed particles:

E. Zaremba, T. Nikuni, A. Griffin: J. Low Temp. Phys. **116** (1999) 277–345

H. T. C. Stoof in *J. Low. Temp. Phys.* **114**, (1999)

R. Lacaze, P. Lallemand, Y. Pomeau, S. Rica in *Physica D* **152-153** (2001).

R. Baier, T. Stockkamp: hep-ph/0412310.

Scenario for B-E condensation: start with the U-U equation + Initial data:

$$f(0,k) = f_0(k) \quad k > 0, \quad \int_0^\infty f_0(k)\sqrt{k}dk > \rho_0$$

(dilute, homogeneous, isotropic gas of bosons at fixed temperature and pressure).

Numerical calculations show that at a positive and finite time t^* the solution f develops a singularity at $k = 0$ (Semikoz et al. , Lacaze et al.)

After t^* a macroscopic fraction of the set of particles occupies the lowest quantum state. This is the Bose-Einstein condensate. After the condensation the gas+condensate is described by a system of two coupled equations:

The coupled system

The density distribution is now $f(t,x) + n_0(t)\delta_0$ (noncondensed+condensed):

$$\partial_t f(t) = Q_1[f] + n_0(t)\,Q_2[f], \qquad \partial_t n_0(t) = n_0(t)\,Q_3[f] \tag{2}$$

$$\begin{aligned}
Q_2[f] &= \frac{1}{\sqrt{k}}\int_0^k \{f(k-k')f(k') - f(k)\,(1+f(k-k')+f(k'))\}\,dk' \\
&\quad + \frac{2}{\sqrt{k}}\int_k^\infty \{f(k')\,(1+f(k)+f(k'-k)) - f(k)f(k'-k)\}\,dk', \\
Q_3[f] &= \int_0^\infty (f(k)\,f(k') - f(k+k')(1+f(k)+f(k')))\,dk\,dk'.
\end{aligned}$$

The system (**??**) has only one relevant stationary solution: $(e^k-1)^{-1}$.

Final Remark : Similar result with the same method may be obtained for the system. The main contribution comes now from:

$$\begin{aligned}\frac{\partial f}{\partial \tau} &= \frac{1}{\sqrt{k}}\int_0^k \{f(k-k')f(k') - f(k)\left(f(k-k') + f(k')\right)\}\,dk' \\ &+ \frac{2}{\sqrt{k}}\int_k^\infty \{f(k')\left(f(k) + f(k'-k)\right) - f(k)f(k'-k)\}\,dk',\end{aligned}$$

where $\tau = \int_0^t n_0(s)\,ds.$ Stationary solution : $\frac{1}{k}$.

Linearisation : $f(k,\tau) = k^{-1} + F(k,\tau)$

$$\begin{aligned}\frac{\partial F}{\partial \tau} &= \frac{2}{k^{3/2}}\int_0^k \frac{\left(F(k') - F(k)\right)k'}{k-k'}\,dk' - \frac{2}{\sqrt{k}}F(k) + 2\sqrt{k}\int_k^\infty \frac{F(k') - F(k)}{k'(k'-k)}\,dk' \\ &+ \frac{2}{\sqrt{k}}\int_k^\infty \left\{F(k'-k)\,\frac{k-k'}{k'\,k} + F(k')\frac{k'+k}{k'\,k}\right\}dk'\end{aligned}$$

In particular one has the corresponding fundamental solution:

Theorem. *For all $k_0 > 0$, there exists a unique solution*

$$F(\tau, k, k_0) = \frac{1}{k_0} F(\frac{\tau}{\sqrt{k_0}}, \frac{k}{k_0}, 1), \quad F(0, k, k_0) = \delta(k - k_0).$$

For $k \in (0, 1)$ the function $F(\tau, k, 1)$ can be written as:

$$F(\tau, k, 1) = \frac{\sigma(\tau)}{k} + \text{lower order terms} \qquad \text{for } 0 < k < 1$$

where $\sigma \in \mathbf{C}[0, +\infty)$ satisfies, for some explicit numerical constant A and any $\delta > 0$ arbitrarily small :

$$\sigma(\tau) = \begin{cases} A\sqrt{\tau} + \mathcal{O}(\tau^{1/2+\delta}) & \text{as } \tau \to 0^+, \\ \mathcal{O}(\tau^{-3}) & \text{as } \tau \to +\infty, \end{cases}$$