Generalized Toeplitz Forms and Interpolation Colligations

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Abstract. In a recent paper, V. Katsnelson related the Cotlar-Sadosky theory of Generalized Toeplitz Forms in the space $\mathcal{P}$ of trigonometric polynomials to the Katsnelson-Kheifets-Yuditskii interpolation theory. In this paper it is pointed out that the Generalized Toeplitz Forms in $\mathcal{P}$ give rise to almost commuting pairs of operators which in turn induce Potapov colligations in the same space $\mathcal{P}$ and that this leads to a new setting for the Generalized Toeplitz Forms.

1. Introduction

The classical theory of moment and interpolation colligation problems is closely related to that of sesquilinear forms $B$ satisfying the Toeplitz condition $B(\tau f, \tau g) - B(f, g) = 0$, where $\tau$ is the shift operator. Further generalizations and variants of these problems lead to more sophisticated forms of the Toeplitz condition. For instance, the abstract interpolation theory of V. Katsnelson, A. Kheifets and P. Yuditskii [KKY] deals with a remarkable approach due to V. Potapov ([EP], [KP]), the so-called Potapov colligations, where the Toeplitz condition is replaced by a Liapunov equation

$$B(\tau f, \tau f) - B(f, f) = \langle c(f), d(f) \rangle + \langle d(f), c(f) \rangle$$

(see Section 2 for definitions and motivations).

Another approach to moment problem, which arose in a Generalized Bochner Theorem (GBT) established by the second author and C. Sadosky in [CS1], also leads to a modification of the Toeplitz condition and to a class of Generalized Toeplitz Forms,

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GTFs, studied in [CS2]-[CS6], [Ar], [Br], [Do], [M], [FK], [A1], [A2] among other papers. In [CS2]-[CS6] the GBT was extended to algebraic Lax-Phillips systems and in two-parameter versions, providing new versions of the classical theorems of Nagy-Foias and Szegő. In a recent paper [K], V. Katsnelson showed that every GTF in the space \( P \) of trigonometric polynomials induces a Potapov colligation in \( P^2 \), and that through this colligation the GBT in \( P \) can be derived from the Katsnelson-Kheifets-Yuditskii (KKY) theory. This interesting result, together with a related paper by B. Fritzschke and B. Kirstein [FK], provided far-going developments of the theory of GTFs in \( P \).

It is natural to ask whether there are other interesting Potapov colligations induced by the GTFs, in the same space \( P \).

The main purpose of this paper is to point out that in \( P \) and in more general situations the GTFs give rise to almost commuting pairs of operators, which in turn induce Potapov colligations in the same space. This fact suggests a new setting for the theory of GTFs, where the Lax-Phillips condition is replaced by an “almost commuting pair” requirement (see Theorem 3.17 below). These associated Potapov colligations (in the same space) provide another version of the GBT resembling the moment problem of A. Gelfond [I].

Let us remark that the above consideration pose two interesting questions:

1) What can be said in our case about the principal function and the functional calculus of the associated almost commuting pair (as discussed in the works of Helton and Howe [HH] and Peller [P], see the end of Section 3).

2) Can the basic result in [KKY] be derived from the GBT? (This would provide a two-dimensional version of the KKY theory.)

This paper is organized as follows. In Section 2, in order to give a context for what follows, the definition of Potapov colligations is recalled and some known results are presented. Also some connections between GTFs and Potapov colligations are discussed. In Section 3, in the framework of the papers [CS1]-[CS6], a link between the two generalizations of the Toeplitz conditions is obtained, pointing out that the GTFs give rise to almost commuting pairs of operators, which in turn provide associated Potapov colligations. Also a positive answer to the corresponding question for the lifting theorem in a large class of abstract GTFs is given. Finally, Potapov colligations are associated in a canonical way to an important Riesz class of GTFs. In Section 4, an extension of the notion of interpolation colligations in the two-parameter case, which is not considered by the Katsnelson-Kheifets-Yuditskii theory, is introduced. In a particular case, a connection between this notion and the corresponding notion from [CS2] of GTFs in two-parameter algebraic scattering systems provides a lifting theorem for this kind of forms in the torus \( \mathbb{T}^2 \).

2. Invariant forms and interpolation colligations

In this section we recall and motivate the definition of Potapov colligations which will be related in next section to that of generalized Toeplitz forms. For this purpose we formulate in general terms the Toeplitz condition and corresponding commutation relations and some generalizations.
Let $V$ be a linear space and $\tau : V \to V$ be an injective linear map. A sesquilinear form $B : V \times V \to \mathbb{C}$ is called non-negative, $B \geq 0$, if $B(f,f) \geq 0$, $\forall f \in V$, and $\tau$-invariant or $\tau$-Toeplitz if the form $B_{\tau}(f,g) := B(\tau f, \tau g) - B(f, g)$ is zero, $\forall f, g \in V$.

If $B \geq 0$, then $B$ gives rise to a reproducing kernel Hilbert space $H$ as the completion of the space spanned by the set $\{K_f : V \to \mathbb{C} : K_f(g) = B(f,g), \forall f \in V\}$, and to a linear map $\gamma : V \to H$ defined by $(\gamma f)(g) = B(f, g)$; thus, $\gamma V$ is a dense subspace of $H$ and $B(f, g) = \langle \gamma f, \gamma g \rangle_H, \forall f, g \in V$. In this case $B$ is $\tau$-invariant if and only if there exists a (unique) isometric operator $\tau^\prime : H \to H$ such that $\tau^\prime \gamma f = \gamma \tau f, \forall f \in V$.

If $V_1$ and $V_2$ are two linear spaces, $\tau_1 : V_1 \to V_1$ and $\tau_2 : V_2 \to V_2$ injective maps, then a sesquilinear form $B : V_1 \times V_2 \to \mathbb{C}$ is said $(\tau_1, \tau_2)$-Toeplitz if $B(\tau_1 f_1, \tau_2 f_2) = B(f_1, f_2), \forall (f_1, f_2) \in V_1 \times V_2$. In this case we cannot speak of non-negative forms $B$ (unless $V_1 = V_2$), but if $V_1$ and $V_2$ are pre-Hilbert spaces, then of course are important the bounded forms $B$ satisfying

$$
\|B\| := \sup\{\|B(f_1, f_2)\|/\|f_1\| \cdot \|f_2\| : (f_1, f_2) \in V_1 \times V_2\} < \infty.
$$

By the Sylvester law every bounded form $B : V_1 \times V_2 \to \mathbb{C}$ gives rise to a non-negative form $\tilde{B} : \tilde{V} \times \tilde{V} \to \mathbb{C}$ (where $\tilde{V} = V_1 \times V_2$), defined by

$$
\tilde{B}([f_1, f_2], [g_1, g_2]) = \|B\| \cdot \langle f_1, g_1 \rangle_{V_1} + \|B\| \cdot \langle f_2, g_2 \rangle_{V_2} + B(f_1, g_2) + B(g_1, f_2),
$$

and $B$ is a $(\tau_1, \tau_2)$-Toeplitz form if and only if $\tilde{B}$ is a $\tilde{\tau}$-Toeplitz form, by defining $\tilde{\tau} [f_1, f_2] = [\tau_1 f_1, \tau_2 f_2]$.

Recall that, if $V_1$ and $V_2$ are Hilbert spaces, there is a one-to-one correspondence between bounded operators $A : V_1 \to V_2$ and bounded forms $B : V_1 \times V_2 \to \mathbb{C}$, defined by $B(f_1, f_2) = \langle Af_1, f_2 \rangle$; in this case, if $\tau_1, \tau_2$ are unitary operators, then $B$ is $(\tau_1, \tau_2)$-Toeplitz if and only if $A$ satisfies the commutation relation $A\tau_1 = \tau_2 A$. The expression $B_{\tau}(f, f)$ (respectively $A\tau_1 - \tau_2 A$) will be called the Toeplitz (resp. intertwining) deviation of $B$ (resp. of $A$).

Commutation relations and Toeplitz conditions play a role in moment and interpolation problems. For instance, if $V$, $B$, $\tau$, $\tau^\prime$, $\gamma$ are as at the beginning of the section, so that $\tau^\prime \gamma f = \gamma \tau f$, and $\tau$ is onto, then $\tau^\prime$ is unitary, and if in addition there exists a “cyclic element” $e \in V$ such that $V$ is spanned by the set $\{\tau^n e : n \in \mathbb{Z}\}$, then $\gamma e$ is a cyclic element of $H$ with respect to $\tau^\prime$ and the spectral theorem provides an integral representation of $B$ through a scalar measure $\mu \geq 0$. In particular, one thus obtains the following classical result.

**Theorem 2.1.** (Herglotz-Bochner theorem.) If $V$ is the space of the trigonometric polynomials in $T \sim [0, 2\pi)$ and $\tau f(t) = e^{it} f(t)$ is the shift operator, then $1 = e^{it} \in V$ is a cyclic element of $V$ and every non-negative and bounded $\tau$-Toeplitz form $B : V \times V \to \mathbb{C}$ has the integral representation $B(f, g) = \int \int f g d\mu$, where $\mu$ is a non-negative finite measure in $T$.

This theorem extends to non-negative $\tilde{\tau}$-invariant forms $\tilde{B} : V^2 \times V^2 \to \mathbb{C}$, where $V^2 = V \times V$ and $\tilde{\tau} [f_1, f_2] = [\tau f_1, \tau f_2]$, for $[f_1, f_2] \in V^2$.

The following well-known property, which can be derived from the last statement, provides another example of the application of commutation relations in moment problems.
Proposition 2.2. If $B : L^2 \times L^2 \to \Psi$ ($L^2 = L^2(\mathbb{T})$) is a bounded $\tau$-Toeplitz form, with $\tau f(t) = e^{it}f(t)$, then $B(f,g) = \int f\overline{g}dt$, with $\Phi \in L^\infty$, $\|\Phi\|_\infty = \|B\|$. Equivalently, if $A : L^2 \to L^2$ is a bounded linear operator that commutes with $\tau$, then the associated form $B(f,g) = \int (Af)\overline{g}dt = (Af,g)$ is $\tau$-invariant so that $Af = \Phi f$, $\|\Phi\|_\infty = \|A\|$, i.e. $A$ is a function of $\tau$.

Remark. If $A$ and $B$ are as in Proposition 2.2, $H^2 \subset L^2$ and $H^2 = L^2 \ominus H^2$ are Hardy subspaces of $L^2$ and $B_0$ is the restriction of $B$ to $H^2 \times H^2$, then $B_0$ satisfies $B_0(\tau f,g) = B_0(f,\tau^{-1}g)$, for $(f,g) \in H^2 \times H^2$, or equivalently $B_0(\tau f,\tau g) = B_0(f,g)$ whenever $(f,g)$ and $(\tau f,\tau g)$ are in $H^2 \times H^2$. Sesquilinear forms $B_0 : H^2 \times H^2 \to \Psi$ satisfying this weaker Toeplitz condition are called Hankel forms. If $A_0 : H^2 \to H^2$ is the associated operator to the bounded Hankel form $B_0$, then $A_0$ intertwines the compressions of $\tau$ to $H^2$ and $H^2$. The Nehari theorem claims the converse: every bounded Hankel form $B_0 : H^2 \times H^2 \to \Psi$ extends to a $\tau$-invariant form $B : L^2 \times L^2 \to \Psi$ with $\|B\| = \|B_0\|$ so that $B_0$ is also given by a function $\Phi$ (such that $\|\Phi\|_\infty = \|B_0\|$, by Proposition 2.2.

By returning to the beginning of the section, where $V$ is a linear space, $\tau : V \to V$ an injective linear map and $B : V \times V \to \Psi$ a $\tau$-invariant non-negative sesquilinear form, if $\tau$ is not onto, it has a “unitary extension”; more precisely, there exists a Hilbert space $H'$, a unitary operator $\tau' \in L(H')$ and a linear map $\Pi : V \to H'$ such that, setting $B'(f',g') = (f',g')_{H'}$,

\begin{equation}
\Pi(\tau f) = \tau'\Pi(f), \quad \forall f \in V,
\end{equation}

(2.1)

\begin{equation}
B'(\tau' f',\tau' g') = B'(f',g'), \quad \forall f',g' \in H',
\end{equation}

(2.2)

and

\begin{equation}
B'(\Pi f,\Pi f) = B(f,f), \quad \forall f \in V.
\end{equation}

(2.3)

Thus $B'$ is $\tau'$-Toeplitz and can be considered as an extension of $B$.

Even if $\gamma \tau$ is only a contraction, i.e. $B(\gamma f,\gamma f) \leq B(f,f)$, then $\tau$ still has a “unitary dilation”, i.e. there exists a unitary operator $\tau' \in L(H')$ and a linear map $\Pi : V \to H'$ such that (2.2), (2.3) and

\begin{equation}
\Pi(\tau f) = P\tau'\Pi(f)
\end{equation}

(2.4)

hold, where $P$ is the orthogonal projection of $H'$ onto $\Pi V$.

These concepts can be extended as follows.

Definition 2.3. We say that $\tau'$ is a $D$-perturbed unitary extension of $\tau$ if (2.2), (2.3) hold but (2.1) is replaced by the weaker condition

\begin{equation}
\Pi(\tau f) = \tau'\Pi(f) + Df,
\end{equation}

(2.5)

where $D : V \to H'$ is a given linear map. We say that $\tau'$ is a $u$-perturbed unitary dilation of $\tau$ if, in addition to (2.2), (2.3), we have

\begin{equation}
\Pi(\tau f) = P\tau'(\Pi(f) + u(f)) = P\tau'\Pi(f) + D_1 f,
\end{equation}

(2.6)
where $u : V \to H'$ is a given linear map and $P$ is the projection on $IV$.

Several generalizations of commutative relations induce corresponding generalizations of the Toeplitz condition. For instance, if $A \in L(H)$ is a dissipative operator, i.e. $\Im \langle Ax, x \rangle \geq 0$, then it satisfies the relation $i(IA^* - AI) \geq 0$, which generalizes the commutation relation $AI - IA^* = 0$ satisfied by self-adjoint operators. This dissipative condition can be written as $A' = CC^*$, where $A' = i(A^* - A)$ and $C = (A'^{1/2})^{-1}$. Livshitz showed that important features of the dissipative operator $A$ are reflected in the analytic function $W_A(\lambda) = I + i\lambda C^*(I - \lambda A)^{-1}C$, called the characteristic function of $A$, and established basic product and factorization theorems for this function. If $A$ is not dissipative, then $A' = (\text{sign } A') \cdot CC^*$, and one sets $W_A(\lambda) = I + i\lambda C^*(I - \lambda A)^{-1}C$. Since this definition imposes limitations in the basic theorems, Brodskii [B] replaced the above factorization of $A'$ by $A' = CJC^*$, where $C \in L(G, H)$, $J \in L(G)$, for an auxiliary Hilbert space $G$, such that $J = J^*$ and $J^2 = I$, and defined $W_A(\lambda) = I + i\lambda JC^*(I - \lambda A)^{-1}C$. This definition leads to the notion of Brodskii nodes that are special linear systems. Sakhnovich [S] replaced $AI - IA^*$ by $AS - SA^*$, for an arbitrary self-adjoint operator $S \in L(H)$, and introduced the "fundamental identity" $AS - SA^* = iCJC^*$ guided by the theory of seminormal operators studied by Putman and by the theorem of Liapunov (see [DK]). Sakhnovich extended the notion of Brodskii nodes (see [BGK]) to such commutation relations, and showed that this fundamental identity allows to develop a unified method for factorization and Wiener-Hopf equations. Since the dissipative operators are Cayley transforms of contractions, there is a corresponding notion of characteristic functions for invertible contractions $T : H \to H$, where the factorization $A - A^* = CJC^*$ is replaced by $I - TT^* = RJR^*$, and the Sakhnovich identity is replaced by $(T^{-1}S - ST^*)T = RJR^*$, $R \in L(G, H)$. When $G = \mathcal{E} \oplus \mathcal{E}$, $\mathcal{E}$ a Hilbert space, the last identity can be rewritten as a Liapunov equation

$$B(Tf, Tf) - B(f, f) = \langle c(f), d(f) \rangle_\mathcal{E} + \langle d(f), c(f) \rangle_\mathcal{E},$$

where $B$ is the sesquilinear form associated with $S$, and $c, d : H \to \mathcal{E}$.

In this way the Sakhnovich identity is transformed (through Cayley) into the following Potapov identity.

**Definition 2.4.** We say that $[V, B, \tau; \mathcal{E}, u, v]$ is an interpolation colligation if $V$ is a linear space, $B : V \times V \to \mathcal{E}$ a non-negative form, $\tau : V \to V$ a linear map, $\mathcal{E}$ a Hilbert space, called scale space, and $u, v : V \to \mathcal{E}$ linear maps such that

$$B(\tau f, \tau f) - B(f, f) = \langle u(f), u(f) \rangle_\mathcal{E} - \langle v(f), v(f) \rangle_\mathcal{E},$$

(2.7)

Here $u$ and $v$ control the Toeplitz deviation $B_\tau(f, f) = B(\tau f, \tau f) - B(f, f)$ of $B$, and (2.7) is called the fundamental identity. We will be concerned with the special case where $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_1$ and $u(f) = (c(f) + d(f)) \oplus 0$, $v(f) = c(f) \oplus d(f)$, where $c, d : V \to \mathcal{E}_1$, so that (2.7) becomes a Liapunov equation

$$B_\tau (f, f) = \langle c(f), d(f) \rangle_{\mathcal{E}_1} + \langle d(f), c(f) \rangle_{\mathcal{E}_1},$$

(2.8)

If $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $I = I_{\mathcal{E}_1}$, then (2.8) can be rewritten as the above analogue of the
Sakhnovich factorization identity

\begin{equation}
B_\tau(f, f) = |c(f), d(f)| \cdot J \cdot |c(f), d(f)|^*.
\end{equation}

**Remark.** Given a triplet \([V, B, \tau]\) as above, \(B \geq 0\), there always exists a Hilbert space \(E\) and linear maps \(u, v : V \rightarrow E\) such that \([V, B, \tau; E; u, v]\) is an interpolation colligation. In fact (2.7) is trivially satisfied if we take \(E = H = \gamma V\) and \(u = \gamma \tau, v = \gamma\). Generally there are other choices of \(E, u, v\) that lead to interesting consequences (for example, those in which \(E\) has finite dimension).

To each interpolation colligation one can associate an isometric operator and a perturbed unitary dilation of \(\tau\). In fact, condition (2.7) allows to define an isometry \(U\) in the space \(H \oplus E\) by

\begin{equation}
U(\gamma f \oplus u(f)) = \gamma \tau f \oplus v(f),
\end{equation}

with domain \(D_U = \text{closure of } \{\gamma f \oplus u(f) : f \in V\}\) and range \(R_U = \text{closure of } \{\gamma \tau f \oplus v(f) : f \in V\}\). Since \(U\) has unitary extensions of the type \(\tilde{U} \in L(H \oplus E)\) where \(H \supset H\), setting \(V' = \tilde{H} \oplus E, \tau' = \tilde{U}, B' = \text{the inner product in } V'\) and \(\Pi f = \gamma f \oplus 0, f \in V\), it is clear that (2.2), (2.3) and (2.6) hold. Thus we have the following result.

**Proposition 2.5.** Each unitary extension \(\tilde{U}\) of \(U\), with respect to the scale space \(E\), gives rise to a \(u\)-perturbed dilation of \(\tau\) where the operator \(u\) in (2.6) coincides with the operator \(u\) in the fundamental identity (2.7).

A theory of Arov and Grossman [AG] describes essentially the class \(U\) of all such unitary extensions \(\tilde{U}\) of \(U\), and therefore also the corresponding perturbed dilations of \(\tau\), through the so-called *scattering functions* with respect to the scale space \(E\), which are operator-valued analytic functions of the Schur class. Observe that in all those perturbed dilations of \(\tau\) the range of \(\Pi\) is the same subspace of \(\gamma V \oplus E\) and \(u : V \rightarrow 0 \oplus E \subset V'\) is also the same, and all \(V'\) contain \(V \oplus E\), being \(E\) the scale space.

The theory of Katsnelson, Kheifets and Yuditskii [KKY] associates to each interpolation colligation \([V, B, \tau, E, u, v]\) an abstract interpolation problem as follows.

Let \(\mathbb{B}(E)\) be the set of all operator-valued analytic functions \(w : \mathbb{D} \rightarrow L(E)\) (\(\mathbb{D}\) stands for the unit disk), such that \(\|w(\xi)\| \leq 1, \forall \xi \in \mathbb{D}\), and for each \(w \in \mathbb{B}(E)\) let \(L^2_w\) be the space of the \(E\)-valued functions \(f(t) = \begin{bmatrix} I_E & w(t) \\ w(t)^* & I_E \end{bmatrix}^{1/2} \tilde{f}(t)\), where \(\tilde{f} \in L^2(E \oplus E)\) and \(\tilde{f}(t) \perp \text{Ker } \begin{bmatrix} I_E & w(t) \\ w(t)^* & I_E \end{bmatrix}\) for a.e. \(t \in \mathbb{T}\), so that \(f \mapsto \tilde{f}\) is one-to-one and one can define \(\|f\|_{L^2_w} = \|\tilde{f}\|\). The de Branges-Rovnyak space \(H_w\) is the subspace of \(L^2_w\) consisting of the functions \(\begin{bmatrix} f_+ \\ f_- \end{bmatrix}\) with \(f_+ \in H^2_+(E), f_- \in H^2_-(E)\). The abstract interpolation problem requires to describe the set of all pairs \((w, F)\), where \(w \in \mathbb{B}(E)\) and \(F\) is a map from \(V\) to \(H_w\), such that
i) under the map \( f \mapsto Ff \), the operator \( \tau \) of the colligation is implemented by the shift of \( H_w \), i.e.
\[
(F\tau f)(t) = t \cdot (Ff)(t) - \begin{bmatrix} I_{\xi} & w(t) \\ w(t)^* & I_{\xi} \end{bmatrix} \begin{bmatrix} -u(f) \\ v(f) \end{bmatrix}, \quad \text{for a.e. } t \in \mathbb{T},
\]

ii) \( \|Ff\|_{H_w}^2 \leq B(f, f), \ \forall f \in V. \)

Katsnelson, Kheifets and Yuditskii showed that there is an injection from the set of the unitary extensions \( \tilde{U} \) into the set of the solutions of this abstract problem, and that the injection is onto, providing a parametrization of all those solutions. This abstract interpolation theory applies to most of the known interpolation problems and includes the general method of Potapov.

3. Riesz forms and the Potapov condition

The already mentioned refinement of the Toeplitz condition, and the corresponding class of generalized Toeplitz forms, GTFs, introduced in [CS1] arose from a problem concerning the Hilbert transform. In this section the definition of those forms is recalled and an important subclass of Riesz forms which gives rise to a corresponding class of Potapov colligations is singled out.

**Definition 3.1.** Let \( P \) be the linear space of the trigonometric polynomials, \( P_1 \) and \( P_2 \) the subspaces of \( P \) spanned by the sets \( \{e^{int}, \ n \geq 0\} \) and \( \{e^{int}, \ n < 0\} \), respectively, so that \( P = P_1 + P_2 \) and there exist two linear projections \( \Pi_1 : P \to P_1 \), \( \Pi_2 : P \to P_2 \) such that \( \Pi_1 f = f \) if \( f \in P_1 \), \( \Pi_2 f = f \) if \( f \in P_2 \) and \( f = \Pi_1 f + \Pi_2 f, \ \forall f \in P \). If \( \Gamma : P \to P \) is the Hilbert transform given by \( \Gamma f = -i\Pi_1 f + i\Pi_2 f \), then a measure \( \mu \geq 0 \) in \( \mathbb{T} \) is said to belong to the Riesz class, \( \mu \in \mathcal{R} \), if \( \Gamma \) is continuous in the \( L^2(\mu) \)-norm, or equivalently if \( \Pi_1 \) and \( \Pi_2 \) are continuous in this norm.

The classical M. Riesz theorem says that \( \Gamma \) is continuous in the \( L^p(dt) \)-norm for every \( 1 < p < \infty \) so that the Lebesgue measure \( dt \) belongs to the Riesz class, \( dt \in \mathcal{R} \). The problem of characterizing the class \( \mathcal{R} \) of measures was first stated and solved by Helson and Szegö. By the Herglotz-Bochner theorem, to give a measure \( \mu \geq 0 \) is the same as to give a non-negative and bounded \( \tau \)-Toeplitz form, where \( \tau(f)(t) = e^{it}f(t) \), \( B_1 : P \times P \to \mathcal{C} \), \( B_1(f, g) = \int fg d\mu \), and \( \mu \in \mathcal{R} \) if and only if the corresponding form satisfies \( B_1(\Pi_1 f, \Pi_1 f) \leq c \cdot B_1(f, f) \) and \( B_1(\Pi_2 f, \Pi_2 f) \leq c \cdot B_1(f, f) \), \( \forall f \in P \) and some fixed \( c \). Letting \( B_0 \) denote the restriction of \( (1 + \varepsilon)B_1 = \frac{c+1}{c}B_1 \) to \( P_1 \times P_2 \), this condition can be rewritten as

\[
|B_0(f, g)| \leq B_1(f, f)^{1/2}B_1(g, g)^{1/2}, \ \forall (f, g) \in P_1 \times P_2.
\]

Since \( B_0 \) is a Hankel form, Cotlar and Sadosky proposed the next more general characterization problem:

Characterize all the triplets \( B_0, B_1, B_2 \), where \( B_0 : P_1 \times P_2 \to \mathcal{C} \) is Hankel, \( B_1, B_2 : P \times P \to \mathcal{C} \) are non-negative \( \tau \)-Toeplitz forms, satisfying

\[
(3.1) \quad |B_0(f, g)| \leq B_1(f, f)^{1/2}B_2(g, g)^{1/2}, \ \forall (f, g) \in P_1 \times P_2.
\]
Defining the form $B : P \times P \to \mathcal{C}$ by

$$(3.2) \quad B(f, g) = B_1(\Pi_1 f, \Pi_1 g) + B_2(\Pi_2 f, \Pi_2 g) + B_0(\Pi_1 f, \Pi_2 g) + \overline{B_0(\Pi_1 g, \Pi_2 f)},$$

and applying the Sylvester principle, it is plain that $(3.1)$ is equivalent to $B \geq 0$. Any non-negative sesquilinear form $B : P \times P \to \mathcal{C}$ that can be written as $(3.2)$ with $B_1, B_2$ $\tau$-Toeplitz forms in $P \times P$ and $B_0 : P_1 \times P_2 \to \mathcal{C}$ Hankel form, is called from [CS1] a non-negative generalized Toeplitz form, GTF. Thus our problem can be reformulated as follows:

**Characterize all the non-negative GTFs in $P \times P$.**

An answer is given by the following theorem from [CS1], mentioned in the introduction:

**Theorem 3.2.** (Generalized Bochner Theorem.) If $B : P \times P \to \mathcal{C}$ is a non-negative GTF, then there exist $\mu_1$, $\mu_2$, $\mu_0$ measures in $\mathbb{T}$ such that

$$B(f_1, f_2) = \int f_1 \overline{f_2} d\mu_0, \quad \mu_0 \in P \times P,$$

and $|\mu_0(\Delta)|^2 \leq \mu_1(\Delta) \cdot \mu_2(\Delta)$, for all Borel set $\Delta$ in $\mathbb{T}$.

Here $\mu_1$ and $\mu_2$ are uniquely determined but not so $\mu_0$: Arocena [Ar] (see also [A1]) obtained parametrization formulas describing all such $\mu_0$. As already mentioned, Katsnelson [K] and Fritzsche and Kirstein [FK] gave an exhausting treatment of this theorem (see also [M], [Br], [Do], [N]). Observe that the theorem gives a necessary and sufficient condition for the following Nehari-type moment problem:

**Given three sequences** $s_1(n)$, $s_2(n)$, $s_0(n)$, $n \in \mathbb{Z}$, **when do there exist measures** $\mu_1$, $\mu_2$, $\mu_0$ **satisfying** $|\mu_0(\Delta)|^2 \leq \mu_1(\Delta) \mu_2(\Delta)$ **and such that** $s_1(n) = \mu_1(n)$, $s_2(n) = \mu_2(n)$ **for** $n \in \mathbb{Z}$, **while** $s_0(n) = \mu_0(n)$ **for** $n < 0$?

Moreover, Theorem 3.2 clearly implies the following property (see [CS5]).

**Theorem 3.3.** (lifting theorem in $\mathbb{T}$.) If $B : P \times P \to \mathcal{C}$ is a non-negative GTF, then there exist a Toeplitz form $B' : P \times P \to \mathcal{C}$ such that $B' = B_0$ in $P_1 \times P_2$ and $|B'(f_1, f_2)|^2 \leq B_1(f_1, f_1) \cdot B_2(f_2, f_2)$, $\forall (f_1, f_2) \in P \times P$.

If $B_1 = B_2$ and $B_0$ is the restriction of $B_1$, then we obtain essentially the moment problem solved by Herglotz-Bochner. Since the class $\mathcal{R}$ plays a role in Analysis, we single out among the GTFs in $P \times P$ the following subclass:

**Definition 3.4.** A GTF $B : P \times P \to \mathcal{C}$, $B \geq 0$, is said to belong to the class $\mathcal{R}$ if the forms $B_1 \geq 0$ and $B_0 \geq 0$ in $(3.2)$ satisfy

$$(3.3) \quad B_1(\Pi_1 f, \Pi_1 f) \leq c \cdot B_1(f, f), \quad B_2(\Pi_2 f, \Pi_2 f) \leq c \cdot B_2(f, f), \quad \forall f \in P,$$

i.e. if $B_1(f, g) = \int f \overline{g} d\mu_1$, $B_2(f, g) = \int f \overline{g} d\mu_2$ with $\mu_1, \mu_2 \in \mathcal{R}$.

On the other hand, the notion of GTFs was extended in [CS2] as follows:
Definition 3.5. We say that $[V, \tau, W_1, W_2]$ is an algebraic scattering system, a.s.s., if $V$ is a linear space, $\tau : V \to V$ a linear isomorphism, and $W_1, W_2$ are subspaces of $V$ such that

\begin{equation}
\tau W_1 \subset W_1, \quad \tau^{-1} W_2 \subset W_2, \tag{3.4}
\end{equation}

\begin{equation}
\forall f \in V, \exists n \geq 0, f_1 \in W_1, f_2 \in W_2 \text{ such that } f = \tau^{-n} f_1 = \tau^n f_2, \tag{3.5}
\end{equation}

\begin{equation}
V = W_1 \oplus W_2. \tag{3.6}
\end{equation}

Let $\Pi_1 : V \to V, \Pi_2 : V \to V$ be the projections on $W_1$ and $W_2$, respectively. Condition (3.6) expresses that there is a unique decomposition $f = \Pi_1 f + \Pi_2 f$, for each $f \in V$.

In the classical Lax-Phillips theory, systems where $V$ is a Hilbert space, $W_1$ and $W_2$ are closed subspaces such that $W_1 \perp W_2$ and $W_1 \oplus W_2 \neq V$ are considered. Instead, here the opposite condition (3.6) is of interest.

Proposition 3.6. Let $[V, \tau, W_1, W_2]$ be an a.s.s. Then, if we call $D^+ = \Pi_1 \tau \Pi_2$, $D^- = \Pi_2 \tau^{-1} \Pi_1$, we have:

\begin{align*}
\Pi_1 \tau^{n+1} - \tau^{n+1} \Pi_1 &= D^+_n, \\
\Pi_2 \tau^{n+1} - \tau^{n+1} \Pi_2 &= D^-_n,
\end{align*}

where $D^+_n = \sum_{k=0}^{n} \tau^{n-k} D^+ \tau^k, \quad D^-_n = \sum_{k=0}^{n} \tau^{-(n-k)} D^- \tau^{-k}, \quad \text{for } n \geq 0.

Proof. Since $\tau f = \tau \Pi_1 f + \tau \Pi_2 f = (\tau \Pi_1 f + \Pi_1 \tau \Pi_2 f) + (\tau \Pi_2 f - \Pi_1 \tau \Pi_2 f) = (\tau \Pi_1 f + \Pi_1 \tau \Pi_2 f) + \Pi_2 \tau \Pi_2 f$, where $\tau \Pi_1 f + \Pi_1 \tau \Pi_2 f \in W_1, \Pi_2 \tau \Pi_2 f \in W_2$, it follows that

\begin{equation}
\Pi_1 \tau f = \tau \Pi_1 f + \Pi_1 \tau \Pi_2 f; \quad \Pi_2 \tau f = \tau \Pi_2 f - \Pi_1 \tau \Pi_2 f. \tag{3.7}
\end{equation}

Analogously,

\begin{equation}
\Pi_1 \tau^{-1} f = \tau^{-1} \Pi_1 f - \Pi_2 \tau^{-1} \Pi_2 f; \quad \Pi_2 \tau^{-1} f = \tau^{-1} \Pi_2 f + \Pi_2 \tau^{-1} \Pi_1 f. \tag{3.8}
\end{equation}

We can now obtain the result by induction. \hfill \square

From (3.4), (3.6), (3.7) and (3.8), it is easy to see the following equivalences:

Corollary 3.7. By using the same notation above, we have:

\begin{align*}
W_1 = \tau W_1 &= \tau^{-1} W_1 \iff W_2 = \tau^{-1} W_2 = \tau W_2 \iff \tau \Pi_1 = \Pi_1 \tau \\
\iff \tau \Pi_2 = \Pi_2 \tau \iff \tau^{-1} \Pi_1 = \Pi_1 \tau^{-1} \\
\iff \tau^{-1} \Pi_2 = \tau^{-1} \Pi_2 \iff D^+ = 0 \iff D^- = 0 \tag{3.9}
\end{align*}
An a.s.s. satisfying one of these conditions will be called trivial. So, we can say that \( D^+ \) and \( D^- \) estimate the deviation of the triviality.

The next basic examples show that the deviation operator \( D^+ \) can be of a special form:

**Example 3.8.** (Trigonometric scalar-valued case). Let \( V \) be the space of the trigonometric polynomials in \( \mathbb{T} \), \( W_1 \) and \( W_2 \) the subspaces of analytic and conjugate-analytic polynomials, respectively, and \((\tau f)(t) = e^{it}f(t)\) the right shift. In this case, \( D^+ f = \hat{f}(-1) \cdot e_0 \).

**Example 3.9.** (Trigonometric vector-valued case). Now, let \( V \) be the space of the \( N \)-valued trigonometric polynomials in \( \mathbb{T} \), where \( N \) is a separable Hilbert space. To be precise:

\[
f \in V \iff f(t) = \sum_{\text{finite}} e^{int} \hat{f}(n), \quad t \in \mathbb{T}, \quad \text{where } \hat{f}(n) \in N.
\]

Let \( W_1 \) and \( W_2 \) denote the subspaces of analytic and conjugate-analytic polynomials, respectively, and again \( \tau \) is the right shift operator. Under these hypotheses, \( D^+ \) is nuclear.

More generally, if \( N_1, N_1 \) are two separable Hilbert spaces, we can define the set

\[
V = \left\{ \sum_{\text{finite}} e_n(t) \hat{f}(n) : \hat{f}(n) \in N_1 \text{ if } n \geq 0, \hat{f}(n) \in N_2 \text{ if } n < 0 \right\},
\]

and consistently define \( W_1, W_2 \) and \( \tau \). So \( D^+ \) is also nuclear.

For each a.s.s. \([V, \tau, W_1, W_2]\) the associated a.s.s. \([V^{(2)}, \tau', V_1, V_2]\) defined by \( V^{(2)} = V \times V \), \( \tau'[f_1, f_2] = [\tau f_1, \tau f_2], \quad \forall [f_1, f_2] \in V^{(2)} \), and \( V_1 = V \times \{0\}, \quad V_2 = \{0\} \times V \), is trivial because \( \tau'V_1 = \{[f, 0] : f \in V\} = V_1 \) and \( \tau'^{-1}V_1 = \{[f, 0] : f \in V\} = V_1 \).

The canonical map \( \Pi : V \to V^{(2)} \) defined by \( \Pi f = [\Pi_1 f, \Pi_2 f] \), \( \forall f \in V \) is injective but in general does not intertwins \( \tau \) with \( \tau' \). Instead, by using Proposition 3.6, we have \( \Pi \tau f - \tau' \Pi f = D f = [D^+ f, -D^- f] \) and we can say that the operator \( D : V \to V^{(2)} \) gives the intertwining deviation of \( \Pi \).

On the other hand, to each sesquilinear form \( \mathcal{B} : V^{(2)} \times V^{(2)} \to \mathbb{C} \) it can be associated a unique \( 2 \times 2 \)-matrix \((B_{\alpha \beta})_{\alpha, \beta = 1, 2}\) of forms \( B_{\alpha \beta} : V \times V \to \mathbb{C} \) such that

\[
\mathcal{B}([f_1, f_2], [g_1, g_2]) = \sum_{\alpha, \beta = 1, 2} B_{\alpha \beta}(f_{\alpha}, g_{\beta}),
\]

and conversely. Moreover \( \mathcal{B} \geq 0 \) if and only if

\[
\sum_{\alpha, \beta = 1, 2} B_{\alpha \beta}(f_{\alpha}, f_{\beta}) \geq 0, \quad \forall [f_1, f_2] \in V \times V.
\]

In this case it is said that the matrix \((B_{\alpha \beta})\) is non-negative. Also \( \mathcal{B} \) is \( \tau' \)-Toeplitz if and only if \( B_{\alpha \beta} \) is \( \tau \)-Toeplitz, \( \forall \alpha, \beta = 1, 2 \), and it is said that \((B_{\alpha \beta})\) is a Toeplitz matrix.
In the same way, each sesquilinear form \( \mathcal{B} : V^\otimes 2 \times V^\otimes 2 \to \mathfrak{C} \) induces a form 
\[ B := \text{ind } \mathcal{B} : V \times V \to \mathfrak{C} \]
defined by \( B(f, g) = \text{E}(\Pi f, \Pi g) \); therefore
\begin{equation}
B(f, g) = \sum_{\alpha, \beta = 1, 2} B_{\alpha \beta}(\Pi_\alpha f, \Pi_\beta g),
\end{equation}
where \( (B_{\alpha \beta}) \) is the matrix associated with \( \mathcal{B} \) by \((3.10)\).

If \( \mathcal{B} \succeq 0 \), then \( B = \text{ind } \mathcal{B} \succeq 0 \), but the converse is not true. If \( B = \text{ind } \mathcal{B} \succeq 0 \), then
\begin{equation}
\sum_{\alpha, \beta = 1, 2} B_{\alpha \beta}(f_\alpha, f_\beta) \succeq 0, \quad \forall [f_1, f_2] \in W_1 \times W_2,
\end{equation}
and, if \((3.13)\) holds, we say that \( \mathcal{B} \) and the matrix \( (B_{\alpha \beta}) \) are weakly non-negative.

If \( \mathcal{B} \) is \( \tau' \)-Toeplitz, in general it is not true that \( B = \text{ind } \mathcal{B} \) is \( \tau \)-Toeplitz. This fact leads to a second definition of GTFs (first given in \([CS3]\)).

**Definition 3.10.** Given an a.s.s. \([V, \tau, W_1, W_2]\), a sesquilinear form \( B : V \times V \to \mathfrak{C} \)
is called a generalized Toeplitz form, GTF, if there exists a Toeplitz form \( \mathcal{B} : V^\otimes 2 \times V^\otimes 2 \to \mathfrak{C} \) such that \( B = \text{ind } \mathcal{B} \) or, equivalently, if there exists an hermitian matrix 
\[ (B_{\alpha \beta})_{\alpha, \beta = 1, 2}, \]
where, for \( \alpha, \beta = 1, 2 \), \( B_{\alpha \beta} : V \times V \to \mathfrak{C} \) is a \( \tau \)-Toeplitz form and \((3.12)\) holds.

Formula \((3.12)\) is equivalent to
\begin{equation}
B(f_\alpha, f_\beta) = B_{\alpha \beta}(f_\alpha, f_\beta), \quad \forall (f_\alpha, f_\beta) \in W_\alpha \times W_\beta \quad (\alpha, \beta = 1, 2).
\end{equation}

**Lemma 3.11.** Let \( B \) be a GTF in an a.s.s. \([V, \tau, W_1, W_2]\). If \( (B_{\alpha \beta})_{\alpha, \beta = 1, 2} \) and 
\( (B'_{\alpha \beta})_{\alpha, \beta = 1, 2} \) are two matrices associated with \( B \) by \((3.12)\), then \( B_{\alpha \beta} = B'_{\alpha \beta} \), for 
\( \alpha = 1, 2 \), but \( B_{12} = B'_{12} \) only in \( W_1 \times W_2 \).

**Proof.** It is plain that \( B_{11} = B'_{11} \) in \( W_1 \times W_1 \) because both forms coincide with 
\( B \) in \( W_1 \times W_1 \) by \((3.14)\). Moreover, by \((3.5)\), if \( f, g \in V \), there exists \( N \) such that 
\( f = \tau^{-N} f_1, g = \tau^{-N} g_1 \), where \( f_1, g_1 \in W_1 \); hence \( B_{11}(f, g) = B_{11}(\tau^{-N} f_1, \tau^{-N} g_1) = B_{11}(f_1, g_1) = B'_{11}(f_1, g_1) \).

On the other hand, \( B_{12}(f, g) = B(f, g) \) for every \( (f, g) \in W_1 \times W_2 \); therefore, only 
the values \( B_{12}(\tau^n f_1, \tau^n f_2) \), for \( (f_1, f_2) \in W_1 \times W_2 \) and \( n \in \mathbf{Z} \), are fixed. If we set 
\( B_{12}(e, e) = B_{12}(\tau^n f_1, \tau^n f_2) \) with \( (f_1, f_2) \in W_1 \times W_2 \), then \( f_1 = f_2 = e = 0 \), because 
\( f_1 = f_2 \) and \( f_1, f_2 \in W_1 \cap W_2 = \{0\} \). From this fact, we can easily deduce that if 
\( e \neq 0 \), it can be defined a Toeplitz form \( B'_{12} \) such that
\begin{equation}
B'_{12} = B_{12} \quad \text{in } W_1 \times W_2 \quad \text{and} \quad B'_{12}(e, e) \quad \text{takes any fixed value}.
\end{equation}

From the previous definitions, one can say as in \([CS2]\) that giving a non-negative GTF \( B : V \times V \to \mathfrak{C} \), \( B \sim (B_{\alpha \beta}) \), is equivalent to giving two non-negative Toeplitz 
forms \( B_1, B_2 : V \times V \to \mathfrak{C} \) and a Hankel form \( B_0 : W_1 \times W_2 \to \mathfrak{C} \), i.e., \( B_0(\tau f_1, f_2) = 
B_0(f_1, \tau^{-1} f_2) \), such that
\[ |B_0(f_1, f_2)|^2 \leq B_1(f_1, f_1) \cdot B_2(f_2, f_2), \quad \forall (f_1, f_2) \in W_1 \times W_2.\]
We single out the class $\mathcal{R}$ in the a.s.s. $[\varnothing, V, W_1, W_2]$ composed of those GTFs $B \geq 0$ where $B_1$ and $B_2$ satisfy (3.3), and in this case we write $B \sim (B_1, B_2, B_0)$.

**Lemma 3.12.** Let $[\varnothing, V, W_1, W_2]$ be an a.s.s. and $B \sim (B_1, B_2, B_0) \in \mathcal{R}$, $B \geq 0$. Then there exists a non-negative form $\tilde{B} : V \times V \to \mathfrak{F}$ such that:

i) $\tilde{B}(\tau f, \tau g) = \tilde{B}(f, g)$, $\forall f, g \in V$;

ii) $B(f, f) \leq k \cdot \tilde{B}(f, f)$, $\forall f \in V$.

**Proof.** If we set $B_{\alpha\beta} = B_\alpha$ for $\alpha = 1, 2$, $B_{12} = B_0$ and define $\tilde{B}(f, g) = B_{11}(f, g) + B_{22}(f, g)$, then i) follows since $B_{11}$ and $B_{22}$ are $\tau$-Toeplitz and ii) follows from

\[
|B(f, f)| \leq \sum_{\alpha, \beta = 1, 2} |B(\Pi_\alpha f, \Pi_\beta f)| = \sum_{\alpha, \beta = 1, 2} |B_{\alpha\beta}(\Pi_\alpha f, \Pi_\beta f)|
\]

\[
\leq \sum_{\alpha, \beta = 1, 2} B_{\alpha\alpha}(\Pi_\alpha f, \Pi_\alpha f)^{1/2} B_{\beta\beta}(\Pi_\beta f, \Pi_\beta f)^{1/2}
\]

\[
\leq c \cdot \sum_{\alpha, \beta = 1, 2} B_{\alpha\alpha}(f, f)^{1/2} B_{\beta\beta}(f, f)^{1/2} \leq 2c(B_{11} + B_{22})(f, f) = 2c\tilde{B}(f, f).
\]

This result suggests the following extension.

**Definition 3.13.** A GTF $B : V \times V \to \mathfrak{F}$ in the given a.s.s. is said to belong to the class $\mathcal{R}'$ if there exists a non-negative form $\tilde{B} : V \times V \to \mathfrak{F}$ satisfying conditions i) and ii) in Lemma 3.12.

The form $\tilde{B} \geq 0$ in Lemma 3.12 gives rise to a reproducing kernel Hilbert space $\mathcal{E}$ containing $V$ as dense subspace and by ii) there exists an operator $\Gamma \in L(\mathcal{E})$ such that $B(f, g) = \langle \Gamma f, g \rangle_{\mathcal{E}}$, and the following Liapunov condition can be obtained.

**Proposition 3.14.** If $B \sim (B_1, B_2, B_0) \in \mathcal{R}'$, by setting $c(f) = (1/2)[\varnothing, \varnothing](f) = (1/2)[\varnothing, \Gamma\tau(f) - \varnothing\Gamma(f)]$ and $d(f) = \tau f$, the Liapunov-Potapov condition

\[
B(\tau f, \tau f) - B(f, f) = \langle c(f), d(f) \rangle_{\mathcal{E}} + \langle d(f), c(f) \rangle_{\mathcal{E}}
\]

holds. If $[\Gamma, \varnothing]$ is of finite rank, we can take $\mathcal{E} \cong \mathfrak{F}^n$ in last relation.

**Proof.** Since $B(f, f) = \langle \Gamma f, f \rangle$, we have

\[
B(\tau f, \tau f) - B(f, f) = (\Gamma\tau f, \tau f) - \langle \Gamma f, f \rangle = \langle (\tau^*\Gamma\tau - \Gamma)f, f \rangle = \langle \tau^*[\Gamma, \varnothing]f, f \rangle,
\]

and since $B \geq 0$ and $B(\tau f, \tau f) - B(f, f)$ is real, it follows that

\[
B(\tau f, \tau f) - B(f, f) = (1/2)(\tau^*[\Gamma, \varnothing]f, f) + (1/2)(\tau f, \tau^*[\Gamma, \varnothing]f)
\]

\[
= (1/2)([\Gamma, \tau]f, f) + (1/2)(\tau f, [\Gamma, \tau]f) = \langle c(f), d(f) \rangle + \langle d(f), c(f) \rangle.
\]

Thus, every $B \in \mathcal{R}'$ gives rise to a Potapov interpolation colligation and if $[\Gamma, \varnothing]$ is of finite rank we may take the space $\mathcal{E} \cong \mathfrak{F}^n$, and $\mathcal{R}'$ is evidently a very large class.
shall not discuss here how large is the subclass \( \mathcal{R}' \) in the class of all GTFs, but we will consider the association of Potapov colligations to GTFs from a somewhat different point of view based on the structure of the given a.s.s.

Finally, if \( B : V \times V \to \mathcal{C} \) is a non-negative GTF, there exist infinite forms \( B : V^{(2)} \times V^{(2)} \to \mathcal{C} \) such that \( B \) is \( \tau' \)-Toeplitz and \( B = \text{ind} \, B \); by (3.13) \( B \) is weakly non-negative.

To obtain non-negative Toeplitz liftings of a non-negative GTF, we have the following important fact from [CS2], [CS4].

**Theorem 3.15.** (Lifting Theorem.) If \( [V, \tau, W_1, W_2] \) is an a.s.s. and \( B \) is a non-negative GTF, then there exists a non-negative Toeplitz form \( B : V^{(2)} \times V^{(2)} \to \mathcal{C} \), or equivalently a non-negative Toeplitz matrix \( (B_{\alpha\beta}) \), such that \( B = \text{ind} \, B \). In other words, there exist a non-negative Toeplitz matrix \( (B_{\alpha\beta}) \) satisfying (3.14).

In this setting, this lifting theorem has the following consequence.

**Corollary 3.16.** Let \( B : V \times V \to \mathcal{C} \) be a non-negative GTF in an arbitrary a.s.s. \( [V, \tau, W_1, W_2] \), and \( \tilde{B} : V^{(2)} \times V^{(2)} \to \mathcal{C} \) a non-negative \( \tau' \)-Toeplitz lifting of \( B \), where \( \tau'[f_1, f_2] = [\tau f_1, \tau f_2] \). Then \( (V^{(2)}, \tilde{B}, \tau') \) is a perturbed lifting of \( (V, B, \tau) \), i.e.,

\[
\tilde{B}(\tau' f', \tau' g') = B(f', g'), \quad \forall f', g' \in V^{(2)},
\]

\[
\tilde{B}(\Pi f, \Pi f) = B(f, f), \quad \forall f \in V,
\]

where \( \Pi f = [\Pi_1 f, \Pi_2 f] \), and

\[
\Pi \tau f = \tau' \Pi f + D f, \quad \forall f \in V
\]

where \( D = [D^+, -D^+] : V \to V^{(2)} \) is given by \( D^+ = \Pi_1 \tau \Pi_2 \).

This fact is an immediate consequence of (3.10) and of that \( B \) is \( \tau' \)-Toeplitz.

Observe that the perturbed unitary extensions of this corollary are of a special type because \( V^{(2)} \), \( D \) and \( \Pi \) are fixed.

Our next goal is to consider a more general setting for the GTFs and prove a lifting theorem by using the Potapov theory. Also, we provide a wide class of liftings through the unitary extensions of the isometry associated to an interpolation colligation.

So, let \( [V, \tau, W_1, W_2] \) be a system where \( V \) is a linear space, \( \tau : V \to V \) a linear isomorphism and \( W_1, W_2 \) subspaces of \( V \) such that \( V = W_1 \oplus W_2 \) and let \( \Pi_1, \Pi_2 \) denote the corresponding projections. Suppose that the deviation operator \( D^+ = \Pi_1 \tau - \tau \Pi_1 \) has finite rank, i.e. there exist \( e_1, \ldots, e_r \in W_1 \), \( \lambda_1, \ldots, \lambda_r : V \to \mathcal{C} \) such that

\[
(3.16) \quad D^+ f = \sum_{j=1}^{r} \lambda_j(f) e_j, \quad \forall f \in V,
\]

or more generally, that

\[
D^+ f = \sum_{j=1}^{\infty} \lambda_j(f) e_j, \quad \forall f \in V.
\]

**Theorem 3.17.** Let \( [V, \tau, W_1, W_2] \) be the system above, where \( D^+ \) satisfies (3.16). If \( B : V \times V \to \mathcal{C} \) is a non-negative GTF, then \( (V, B, \tau, \mathcal{E}, u, v) \) is an interpolation colligation, where \( \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_1, \quad \mathcal{E}_1 = \mathcal{C}^r \).
where $c(f) = (c_j(f))^j_{j=1}$, $d(f) = (d_j(f))^j_{j=1}$, and

\begin{align}
 (3.17) \quad u(f) &= (c(f) + d(f)) \oplus 0, \quad v(f) = c(f) \oplus d(f),
 \end{align}

\begin{align}
 (3.18) \quad c_j(f) &= \lambda_j(f), \\
 d_j(f) &= B_{11}(e_j, \tau \Pi_1 f) - B_{22}(e_j, \tau \Pi_2 f) + B_{12}(e_j, \tau \Pi_2 f) - B_{21}(e_j, \tau \Pi_1 f).
\end{align}

Proof. For simplicity, we will suppose that $r = 1$. Recalling that the forms $B_{\alpha \beta} : V \times V \to \mathbb{C}$ are Toeplitz, we have

\begin{align*}
 B(\tau f, \tau f) &= \sum_{\alpha, \beta = 1, 2} B_{\alpha \beta}(\Pi_\alpha \tau f, \Pi_\beta \tau f) \\
 &= \sum_{\alpha, \beta = 1, 2} B_{\alpha \beta}(\tau \Pi_\alpha f - (-1)^{\alpha} \lambda(f)e, \tau \Pi_\beta f - (-1)^{\beta} \lambda(f)e) \\
 &= I + II + III + IV,
\end{align*}

where

\begin{align*}
 I &= B_{11}(\tau \Pi_1 f + \lambda(f)e, \tau \Pi_1 f + \lambda(f)e) \\
 &= B_{11}(\Pi_1 f, \Pi_1 f) + B_{11}(\tau \Pi_1 f, \lambda(f)e) + B_{11}(\lambda(f)e, \tau \Pi_1 f) + B_{11}(\lambda(f)e, \lambda(f)e), \\
 II &= B_{22}(\tau \Pi_2 f - \lambda(f)e, \tau \Pi_2 f - \lambda(f)e) \\
 &= B_{22}(\Pi_2 f, \Pi_2 f) + B_{22}(\tau \Pi_2 f, -\lambda(f)e) + B_{22}(-\lambda(f)e, \tau \Pi_2 f) + B_{22}(\lambda(f)e, \lambda(f)e), \\
 III &= B_{12}(\tau \Pi_1 f + \lambda(f)e, \tau \Pi_2 f - \lambda(f)e) \\
 &= B_{12}(\Pi_1 f, \Pi_2 f) + B_{12}(\tau \Pi_1 f, -\lambda(f)e) + B_{12}(\lambda(f)e, \tau \Pi_2 f) - B_{12}(\lambda(f)e, \lambda(f)e), \\
 IV &= B_{21}(\tau \Pi_2 f - \lambda(f)e, \tau \Pi_1 f + \lambda(f)e) \\
 &= B_{21}(\Pi_2 f, \Pi_1 f) + B_{21}(\tau \Pi_2 f, \lambda(f)e) + B_{21}(-\lambda(f)e, \tau \Pi_1 f) - B_{21}(\lambda(f)e, \lambda(f)e).
\end{align*}

By (3.15) we can choose $B_{12}(e, e)$ arbitrarily; then we do $B_{11}(e, e) + B_{22}(e, e) - B_{12}(e, e) - B_{21}(e, e) = 0$; since $B(f, f) = \sum_{\alpha, \beta} B_{\alpha \beta}(\Pi_\alpha f, \Pi_\beta f)$, the deviation of $B$ is now

\begin{align}
 (3.19) \quad B(\tau f, \tau f) - B(f, f) &= c(f)\overline{d(f)} + d(f)\overline{c(f)},
\end{align}

where

\begin{align}
 (3.20) \quad c(f) &= \overline{\lambda(f)}, \\
 d(f) &= B_{11}(e, \tau \Pi_1 f) - B_{22}(e, \tau \Pi_2 f) + B_{12}(e, \tau \Pi_2 f) - B_{21}(e, \tau \Pi_1 f).
\end{align}

Then $B$ is an interpolation colligation with $c(f)$ and $d(f)$ given by (3.20), and also satisfies (2.7), with $E_1 = \mathbb{C}^r$, $E = E_1 \oplus E_1 = \mathbb{C}^{2r}$ and

\begin{align*}
 u(f) &= (c(f) + d(f)) \oplus 0, \quad v(f) = c(f) \oplus d(f).
\end{align*}

Analogously, for all $r \geq 1$, $E_1 = \mathbb{C}^r$, $E = \mathbb{C}^{2r}$ and

\begin{align}
 (3.21) \quad B(\tau f, \tau f) - B(f, f) &= \sum_{j=1}^r (c_j(f)\overline{d_j(f)} + c_j(f)\overline{d_j(f)})
\end{align}
holds, with \( c_j \) and \( d_j \) given by (3.18).

We thus introduce a very general class of systems \([V, \tau, W_1, W_2]\) with \( V = W_1 \oplus W_2 \) where condition \( \tau W_1 \subset W_1, \tau^{-1} W_2 \subset W_2 \) is not required but replaced by the requirement that \( \tau, \Pi_j \) is an almost commuting pair. Theorem 3.17 assures that every GTF defined in such general system still has a Potapov colligation and an associated isometry to which the KKY theory applies so that the given GTF gives rise to an interpolation problem and in particular to a matrix-valued measure. In the special case where \( V = \mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2 \) (see Example 3.8), Theorem 3.17 leads to a new variant of the generalized Bochner theorem and of the generalized Nehari moment problem, as follows.

**Lemma 3.18.** Let \( B : \mathcal{P} \times \mathcal{P} \to \mathcal{C} \) be a non-negative GTF in \( \mathcal{T} \), \( \tau \) the shift, and let \((\mathcal{P}, B, \tau, \mathcal{C}^2, u, v)\) be the corresponding Potapov colligation provided by Theorem 3.17, so that \( u(f) = (c + d)(f) \varepsilon_1, v(f) = c(f) \varepsilon_1 + d(f) \varepsilon_2 \), where \( \varepsilon_1, \varepsilon_2 \) are the canonical coordinates of \( \mathcal{C}^2 \) and \( c, d \) are given by (3.20) with \( \lambda(f) = \tilde{f}(-1), c = e_0 = 1 \). Let \( U \) be the associated isometry with domain \( D_U = \{ f \oplus u(f) : f \in H \} \), where \( H \) is the reproducing kernel Hilbert space induced by \( B \). Then:

i) If \( f \in W_1 \), then \( \| c \| = 0 \) and \( \| f \| = \| \tau^n f \|, \forall n \geq 0 \).

ii) If \( f \in W_2 \), then \( c(\tau^{-1} f) = 0 \) and \( \| f \| = \| \tau^{-n} f \|, \forall n \geq 0 \).

**Proof.** If \( f \in W_1 \), then \( \tau f \in W_1 \) and \( \Pi_1 f = f, \Pi_2 f = 0, \Pi_1(\tau f) = \tau f, \Pi_2(\tau f) = 0 \), \( D^+ f = D^+(\tau f) = 0 \) and, from (3.16) and (3.20), we have that \( c(f) = \tilde{\lambda}(f) = 0, \| u(f) \|^2 - \| v(f) \|^2 = B(\tau f, \tau f) - B(f, f) = 0 \), i.e. \( \| \tau f \| = \| f \| \). By the same way we obtain that \( \| f \| = \| \tau^n f \|, \forall n \geq 0 \).

The other item can be proved analogously.

Observe also that since \( u(f) = (c + d)(f) \varepsilon_1 \), we have \( \varepsilon_2 \perp D_U \) and therefore there are unitary extensions \( \tilde{U} \in L(\tilde{H} \oplus \mathcal{C}^2) \) of \( U \), where \( \tilde{H} \supset H \), such that \( \tilde{U}^n \varepsilon_2 \perp D_U, \forall n \in \mathbb{Z} \).

Taking into account these relations, we shall deduce the following variant of the GBT:

**Theorem 3.19.** For every non-negative GTF \( B : \mathcal{P} \times \mathcal{P} \to \mathcal{C} \) there exist measures \( \mu_1 \geq 0, \mu_2 \geq 0, \mu_0, \nu, \nu' \) such that

\[
B_{11}(e_0, e_0) = \hat{\mu}_1(0), \quad B_{11}(e_n, e_0) = \hat{\mu}_1(n) + \sum_{j=1}^{n} \lambda_{n,j}^{(1)} \hat{\nu}(j), \quad n \geq 1,
\]

\[
B_{22}(e_{-1}, e_{-1}) = \hat{\mu}_2(0), \quad B_{22}(e_{-n-1}, e_{-1}) = \hat{\mu}_2(-n) + \sum_{j=0}^{n-1} \lambda_{n,j}^{(2)} \hat{\nu}'(-j), \quad n \geq 1,
\]

\[
B_{12}(e_0, e_{-1}) = \hat{\mu}_0(0), \quad B_{12}(e_n, e_{-1}) = \hat{\mu}_0(n) + \sum_{j=1}^{n} \lambda_{n,j}^{(1)} \hat{\nu}'(j), \quad n \geq 1,
\]

where the constants \( \lambda_{n,k}^{(1)}, \lambda_{n,k}^{(2)} \) do not depend on \( B \), and \( |\mu_0(\Delta)|^2 \leq \mu_1(\Delta)\mu_2(\Delta) \).
**Remark.** Moment problems of this type appear in the work of A. O. Gelfond (cfr. [1]).

**Proof.** For $n \geq 0$, since $u(e_n) = d(e_n)e_1$, $v(e_n) = d(e_n)e_2$,

$$
\tilde{U}e_n = \tilde{U}(e_n \oplus u(e_n) - 0 \oplus u(e_n)) \\
= e_{n+1} \oplus v(e_n) - \tilde{U}(u(e_n)) = e_{n+1} + d(e_n)e_2 - d(e_n)\tilde{U}(e_1).
$$

From this fact, we obtain by induction that

$$
\tilde{U}^ne_0 = e_n - \sum_{j=0}^{n-1} \kappa^{(1)}_{nj}\tilde{U}^je_2 - \sum_{j=1}^{n} \lambda^{(1)}_{nj}\tilde{U}^j\varepsilon_1, \text{ for } n > 0.
$$

Therefore,

$$
B_{11}(e_n,e_0) = \langle e_n,e_0 \rangle = \langle \tilde{U}^ne_0,e_0 \rangle + \sum_{j=1}^{n} \lambda^{(1)}_{nj}\tilde{U}^j\varepsilon_1, \text{ for } n > 0.
$$

Letting $E$ the spectral measure of $\tilde{U}$ and defining $\mu_1 = \langle Ec_0,e_0 \rangle$, $\nu = \langle E\varepsilon_1,e_0 \rangle$, we deduce (3.22) from the spectral theorem.

Formulas (3.23) and (3.24) can be obtained similarly if we define the measures $\mu_2 = \langle Ec_{-1},e_{-1} \rangle$, $\nu' = \langle E\varepsilon_1,e_{-1} \rangle$ and $\mu_0 = \langle Ec_0,e_{-1} \rangle$.

As a direct consequence of that $|B_{12}(e_0,e_{-1})|^2 \leq B_{11}(e_0,e_0)B_{22}(e_{-1},e_{-1})$ the inequality $|\mu_0(\Delta)|^2 \leq \mu_1(\Delta)\mu_2(\Delta)$ holds.

**Remark.** As in the case of the Generalized Bochner Theorem, this theorem can be stated as a generalized Nehari moment problem. Moreover, this method may allow to construct a Schur type algorithm for this problem in a similar way to [A2].

Similarly there is a new variant of the Cotlar-Sadosky Lifting Theorem, for these general systems where $\tau,\Pi_i$ satisfy the almost commutativity condition, by using the proof of the Lifting Theorem, given in [CS6], through the Wold-Kolmogorov decomposition and using instead of the expansion into trigonometric functions $c_n(t) = e^{int}$, provided by the Wold-Kolmogorov theorem. Here we shall not detail the proof of the lifting theorem in this setting and only appoint a few remarks. Observe that the solutions of the moment problem associated with Theorem 3.2 are also solutions of the problem associated with Theorem 3.19. This means that Theorem 3.2 gives only a small part of the solutions of 3.19 and, in a certain sense, the problem for 3.19 should be considered as the natural moment problem for the GTF $B$. The KKY theory provides then a description of another large part of those solutions. The codimension of $D_U$ in $H \oplus \mathfrak{c}^2$ may play a certain role in these questions. Observe that the elements of $D_U$ are the pairs $[f,(c+d)(f)e_1]$ so that

$$
[g,\lambda^\prime\varepsilon_1 + \lambda''\varepsilon_2] \perp D_U \iff \langle f,g \rangle + \overline{\nu}(c+d)(f) = 0, \forall f \in H;
$$

then, for each $\lambda = (\lambda',\lambda'') \in \mathfrak{c}^2$, there exists at most one element $[g,\lambda] \perp D_U$ and, if $(c+d)(f)$ is continuous in $f$, then this $[g,\lambda]$ has the form $[g_\lambda,\lambda]$, and $\dim D_U^e \leq 2$. Since for $\lambda = (0,\lambda'')$ and $g_\lambda = 0$ we have $[g_\lambda,\lambda] \perp D_U$, then $\dim D_U^e \geq 1$ and, if $(c+d)(f)$ is
continuous, then this dimension is 2. Moreover, by calling \( D_U = \{ f \oplus u(f) : f \in W_1 \} \),
the following statement holds.

**Lemma 3.20.** If \( \varepsilon_1 \in D_U \), then every unitary extension \( \tilde{U} \) of \( U \) satisfies
\[
(3.25) \quad \tilde{U}\varepsilon_1 = \varepsilon_2, \quad \tilde{U}|_{W_1} = \tau, \quad \tilde{U}^{-1}|_{W_2} = \tau^{-1}.
\]

**Proof.** There exists a sequence \( f_n \in W_1 \) such that \( f_n \oplus d(f_n)\varepsilon_1 \to \varepsilon_1 = 0 \oplus \varepsilon_1 \in V \oplus \mathcal{E} \). Hence \( f_n \to 0 \) in \( V \), \( d(f_n) \to 1 \), \( d(f_n) \in \mathcal{E}_1 = \mathcal{E}' \). Since \( f_n \in W_1 \), then \( \| f_n \| = \| f_n \| \to 0 \). So, by (3.17) and (2.10), for each unitary extension \( \tilde{U} \) of \( U \), we have \( \tilde{U}\varepsilon_1 = \lim \tilde{U}(f_n \oplus d(f_n)\varepsilon_1) = \lim(\tau f_n \oplus d(f_n)\varepsilon_2) = \varepsilon_2 \).

Thus if \( \varepsilon_1 \in D_U \), \( \varepsilon_2 \in D_{U^{-1}} \), then Theorem 3.2 provides all the solutions of 3.19.

**Remark.** Above we assigned in a canonical way a Potapov colligation to each GTF in \( T \) (i.e. in \( \mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2 \)), as well as in abstract scattering systems, by using the fact that \( \tau \) and \( \Pi_1 \) form an almost commuting pair, i.e. \( D^+ = \Pi_1 \tau - \tau \Pi_1 \) is of trace class, and in fact of finite dimension. Though \( \tau \) is not selfadjoint, by (3.7) and (3.8) the selfadjoint pair \( \tau + \tau^* \) and \( \Pi_1 \) is still almost commuting. Moreover we used this fact to introduce a very general class of systems \( [V = W_1 \oplus W_2, \tau] \) where the condition \( \tau W_1 \subset W_1 \), \( \tau^{-1}W_2 \subset W_2 \) is not required but instead the pair \( \tau, \Pi_1 \) is required to be almost commuting. It was shown that each GTF defined in such general system still has a Potapov colligation and an associated isometry to which the KKY theory applies, and it may be worth to explore this general direction.

On the other hand, each almost commuting pair \( A, B \) has a principal function \( g \in L^1_0(\mathcal{E}') \) such that for any polynomials \( \varphi, \psi \),
\[
\text{trace } (\varphi \cdot \psi(A, B) - \psi \cdot \varphi(A, B)) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \left( \frac{\partial \varphi}{\partial x} \cdot \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \varphi}{\partial y} \right) g(x, y) \, dx \, dy
\]
and a functional calculus was developed by Helton-Howe, Peller and others. The above formulas for \( D^+_\bigwedge, D^-_\bigwedge \) apply in the above integral when \( \varphi, \psi \) are special functions and since the GTFs are closely related to the model operators which have a Livshitz calculus, it is natural to expect that both functional calculus are related.

### 4. Two-parameter interpolation colligations

The notion of Hankel or Toeplitz forms acting in the so-called two-parameter algebraic scattering systems was first formulated in [CS2]. Several two-parameter versions of Lifting Theorem 3.15 were obtained in [CS4]. We establish a one-to-one correspondence between the class of two-parameter GTFs and that of two-parameter Potapov colligations defined in 4.1.
Definition 4.1. We say that $[V,B,\sigma,\tau;\mathcal{E}_1,\mathcal{E}_2,u_1,v_1,u_2,v_2]$ is a two-parameter interpolation colligation if $V$ is a linear space, $B : V \times V \to \mathcal{Q}$ a non-negative form, $\sigma,\tau : V \to V$ linear maps, $\mathcal{E}_1$ and $\mathcal{E}_2$ Hilbert spaces, called scale spaces, and $u_i,v_i : V \to \mathcal{E}_i$ ($i = 1,2$) linear maps such that

\begin{equation}
\begin{aligned}
B(\sigma f,\sigma f) - B(f,f) &= \langle u_1(f),u_1(f)\rangle_{\mathcal{E}_1} - \langle v_1(f),v_1(f)\rangle_{\mathcal{E}_1},
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
B(\tau f,\tau f) - B(f,f) &= \langle u_2(f),u_2(f)\rangle_{\mathcal{E}_2} - \langle v_2(f),v_2(f)\rangle_{\mathcal{E}_2}.
\end{aligned}
\end{equation}

It is obvious that, if $[V,B,\sigma,\tau;\mathcal{E}_1,\mathcal{E}_2,u_1,v_1,u_2,v_2]$ is a two-parameter interpolation colligation, then $[V,B,\sigma;\mathcal{E}_1,u_1,v_1]$ and $[V,B,\tau;\mathcal{E}_2,u_2,v_2]$ are two one-parameter interpolation colligations.

On the other hand, we recall from [CS3] that $[V,B,\sigma,\tau;W_1,W_2]$ is a two-parameter algebraic scattering system, if both $[V,\sigma;W_1,W_2]$ and $[V,\tau;W_1,W_2]$ are one-parameter a.s.s. and $\sigma = \tau \sigma$. If a sesquilinear form $B : V \times V \to \mathcal{Q}$ if a GTF in both $[V,\sigma;W_1,W_2]$ and $[V,\tau;W_1,W_2]$, then $B$ is called a $(\sigma,\tau)$-GTF in $[V,\sigma;W_1,W_2]$.

In this setting we are going to consider the following scheme:

Let $H_0$ be a semi pre-Hilbert space, $S,T : H_0 \to H_0$ two commuting linear operators satisfying the following fundamental identities:

\begin{equation}
\begin{aligned}
\langle Sf,Sf \rangle_{H_0} - \langle f,f \rangle_{H_0} &= 0, \forall f \in H_0,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\langle Tf,Tf \rangle_{H_0} - \langle f,f \rangle_{H_0} &= \|u(f)\|^2 - \|v(f)\|^2, \forall f \in H_0,
\end{aligned}
\end{equation}

where $u,v : H_0 \to \ell^2$ are continuous linear operators.

Thus, the set $[H_0,\langle \cdot,\cdot \rangle_{H_0},S,T;\ell^2,0,0,u,v]$ is a two-parameter interpolation colligation.

Let $H$ be the reproducing kernel Hilbert space associated with $H_0$; then identities (4.3) and (4.4) allow to define two isometric operators $V_1$, $V_2 : H \oplus \ell^2 \to H \oplus \ell^2$ by

\begin{equation}
\begin{aligned}
V_1(f \oplus a) &= Sf \oplus a,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
V_2(f \oplus u(f)) &= Tf \oplus v(f).
\end{aligned}
\end{equation}

So $V_1$ is unitary but $V_2$ has domain $\bigvee \{f \oplus u(f) : f \in H\}$, and range $\bigvee \{Tf \oplus v(f) : f \in H\}$. Moreover, $V_1$ and $V_2$ commute if $u(Sf) = u(f)$ and $v(Sf) = v(f)$.

The sequence $w_n(0) = 0 \oplus (\delta_{kn})_k$ in $H \oplus \ell^2$ satisfies

\begin{equation}
\begin{aligned}
\{w_n(0)\}_{n \in \mathbb{Z}} \notin D_V \text{ and } D_V + \bigvee_n \{w_n(0)\} = H \oplus \ell^2,
\end{aligned}
\end{equation}

so that there exists $\varphi_n^{(0)} \in H_0$ such that $\varphi_n^{(0)} \oplus u(\varphi_n^{(0)}) - w_n(0) \perp D_V$.

We suppose that there exist two sequences $\{e_n,0\}_{n \in \mathbb{Z}}, \{e_{n,-1}\}_{n \in \mathbb{Z}} \subset H_0$, where $S e_{n,k} = e_{n+1,k}$, and such that $H_0$ is spanned by $\{e_{n,0} = T^k e_{n,0}, k \geq 0\}$ and $\{e_{n,-1-k} = T^{-k} e_{n,-1-k}, k \geq 0\}$, $n \in \mathbb{Z}$, and

\begin{equation}
\begin{aligned}
\langle T^k \varphi_n^{(0)},e_{p,q}\rangle_{H_0} = \langle \varphi_n^{(0)},e_{p,q-k}\rangle_{H_0}, \forall n,p,q \in \mathbb{Z}, k \geq 0.
\end{aligned}
\end{equation}
In this setting, we can associate to this interpolation colligation a GTF in a two-parameter a.s.s. in the following way.

**Theorem 4.2.** If \([H_0, \langle \cdot , \cdot \rangle_{H_0}, S, T; \ell^2, \ell^2, 0, 0, u, v]\) is the two-parameter interpolation colligation defined by (4.3) and (4.4), and satisfying the hypotheses above, then it induces an hermitian matrix \((B_{\alpha\beta})_{\alpha,\beta=1,2}\), where \(B_{\alpha\beta} : H_0 \times H_0 \to \mathbb{C}\) are \(T\) and \(S\)-invariant sesquilinear forms, and such that

\[
B_{11}(f, f) + B_{22}(g, g) + B_{12}(f, g) + \overline{B_{12}(f, g)} \geq 0.
\]

So, the form \(B = (B_{\alpha\beta})_{\alpha,\beta=1,2}\) is a non-negative GTF.

**Proof.** Let \(U\) be a minimal unitary extension of \(V\) defined in a Hilbert space \(\tilde{H} \oplus \ell^2\) where \(\tilde{H} \supset H\), and we suppose that \(\dim \tilde{H} \oplus H = \infty\). Since \(Dv^\perp \supset \sqrt{\{\varphi_n^0 \oplus u(\varphi_n^0) - w_n^0\}}\), we can choose \(U\) such that

\[
U^k(\varphi_n^0 \oplus u(\varphi_n^0) - w_n^0) \perp H, \forall k \geq 0.
\]

Hence,

\[
(U w_n^0, e_{p,q}) = \langle U^k(\varphi_n^0 \oplus u(\varphi_n^0)), e_{p,q} \rangle, \forall k \geq 0, p, q \in \mathbb{Z}.
\]

In particular,

\[
(U w_n^0, e_{p,q}) = \langle U(\varphi_n^0 \oplus u(\varphi_n^0)), e_{p,q} \oplus 0 \rangle = \langle T(\varphi_n^0 \oplus v(\varphi_n^0)), e_{p,q} \oplus 0 \rangle
\]

\[
= \langle T(\varphi_n^0), e_{p,q} \rangle = \langle \varphi_n^0, e_{p,q} \rangle
\]

\[
= \langle \varphi_n^0 \oplus u(\varphi_n^0), e_{p,q-1} \oplus 0 \rangle = \langle w_n^0, e_{p,q-1} \rangle = 0.
\]

It can be proved by induction over \(k\) that \(\langle U^k w_n^0, e_{p,q} \rangle = 0, \forall p, q, n \in \mathbb{Z}, k \geq 0\):

In fact, if \(\langle U^{k-1} w_n^0, e_{p,q} \rangle = 0\), using (4.9) and (4.11), we have:

\[
\langle U^k w_n^0, e_{p,q} \rangle = \langle U^{k-1}(T \varphi_n^0 + v(\varphi_n^0)), e_{p,q} \rangle
\]

\[
= \langle U^{k-1}(T \varphi_n^0 \oplus u(T \varphi_n^0)), e_{p,q} \rangle
\]

\[
+ \sum_i [v(\varphi_n^0) - u(T \varphi_n^0)]_i \cdot \langle U^{k-1} w_i^0, e_{p,q} \rangle = \ldots
\]

\[
= \langle T^k \varphi_n^0 \oplus u(T^k \varphi_n^0), e_{p,q} \rangle = \langle \varphi_n^0, e_{p,q-k} \rangle = \langle w_n^0, e_{p,q-k} \rangle = 0.
\]

On the other hand,

\[
U e_{p,0} = U(e_{p,0} \oplus u(e_{p,0})) = \sum_i u_i(e_{p,0}) \cdot U w_i^0
\]

\[
= e_{p,1} \oplus v(e_{p,0}) - \sum_i u_i(e_{p,0}) \cdot U w_i^0
\]

\[
= e_{p,1} + \sum_i v_i(e_{p,0}) \cdot w_i^0 - \sum_i u_i(e_{p,0}) \cdot U w_i^0
\]

\[
= e_{p,1} + \sum_i c_i \cdot w_i^0 - \sum_i c_i \cdot U w_i^0.
\]
and also it is easy to see that there exist \( \{c_{i,0}, c_{i,1}, \ldots, c_{i,k}\} \) such that
\[
U^k e_{p,0} = e_{p,k} + \sum_{j=0}^{k} \sum_{i} c_{i,j} U^i w_i^{(0)},
\]
so that \( \langle U^k e_{p,0}, e_{r,q} \rangle = \langle e_{p,k}, e_{r,q} \rangle, \forall p, q, r \in \mathbb{Z}, k \geq 0 \) holds.

In the same way we can prove that there exist \( \{d_{i,0}, d_{i,1}, \ldots, d_{i,k}\} \) such that
\[
U^{-k} e_{p,-1} = e_{p,-k-1} + \sum_{j=0}^{k} \sum_{i} d_{i,j} U^{-i} w_i^{(0)}, \forall p \in \mathbb{Z}, k \geq 0.
\]

If we choose the extension \( U \) such that \( U^{-k} w_0^{(0)} \perp e_{r,s}, \forall n, r \in \mathbb{Z}, k, s \geq 0 \), we obtain that \( \langle U^{-k} e_{p,-1}, e_{r,s} \rangle = \langle e_{p,-k-1}, e_{r,s} \rangle \). We now define, for each \( p, r \in \mathbb{Z} \) and \( k, s \geq 0 \), the sesquilinear forms
\[
B_{11}(e_{p,k}, e_{r,s}) = \langle U^{-k} e_{p,0}, e_{r,0} \rangle,
\]
\[
B_{12}(e_{p,k}, e_{r,-s}) = \langle U^{-k+s} e_{p,0}, e_{r,-1} \rangle,
\]
\[
B_{22}(e_{p,-k-1}, e_{r,-s-1}) = \langle U^{-k+s} e_{p,-1}, e_{r,-1} \rangle.
\]

It is plain that \( B_{11} \) and \( B_{22} \) are Toeplitz, i.e. \( T \) and \( S \)-invariants, defined in \( H_0 \times H_0 \) and \( B_{12} \) is a Hankel form, defined in \( H_{0,+} \times H_{0,-} \), where \( H_{0,+} = \{e_{n,k} : n \in \mathbb{Z}, k \geq 0\} \) and \( H_{0,-} = \{e_{n,k} : n \in \mathbb{Z}, k < 0\} \).

Moreover
\[
|B_{12}(e_{p,q}, e_{r,-s})|^2 = |\langle U^k e_{p,0}, e_{r,-s} \rangle|^2 \leq \|U^k e_{p,0}\|^2 \cdot \|e_{r,-s}\|^2
\]
\[
= \langle U^k e_{p,0}, U^k e_{p,0} \rangle \cdot \langle e_{r,-s}, e_{r,-s} \rangle = B_{11}(e_{p,k}, e_{p,k}) \cdot B_{22}(e_{r,-s}, e_{r,-s}).
\]

By using the two-parameter lifting theorem from [CS4], the thesis follows. \( \square \)

In the following two-dimensional trigonometric example, we can conversely associate to each \((\sigma, \tau)\)-GTF a two-parameter interpolation colligation, thus providing a new proof of the lifting theorem in [CS4] for these GTFs. Let
\[
\mathcal{P} = \{f : \mathbb{T}^2 \to \mathbb{C} : f(s, t) = \sum_{\text{finite}} \hat{f}(m, n) e^{im s} e^{int}\}
\]
be the space of trigonometric polynomials in \( \mathbb{T}^2 \),
\[
\mathcal{P}_1 = \{f \in \mathcal{P} : \hat{f}(m, n) = 0 \text{ if } n < 0\}, \mathcal{P}_2 = \{f \in \mathcal{P} : \hat{f}(m, n) = 0 \text{ if } n \geq 0\},
\]
and
\[
\sigma f(s, t) = e^{is} f(s, t), \quad \tau f(s, t) = e^{it} f(s, t), \forall f \in \mathcal{P}.
\]

**Theorem 4.3.** Let \( B_1, B_2 : \mathcal{P} \times \mathcal{P} \to \mathcal{C} \) be two non-negative Toeplitz forms with respect to the shifts \( \sigma \) and \( \tau \), and let \( B_0 : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathcal{C} \) be a Hankel form, such that
if \( B_0 \leq (B_1, B_2) \) in \( \mathcal{P}_1 \times \mathcal{P}_2 \), i.e. \( |B_0(f,g)|^2 \leq B_1(f,f) \cdot B_2(g,g) \), \( \forall (f,g) \in \mathcal{P}_1 \times \mathcal{P}_2 \). We call \( H_0 = \left( \mathcal{P}_1 \right)_{\mathcal{P}_2} \), and define the metric given by \( B_1, B_2, B_0 \) as:

\[
\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_{H_0} = B_1(f_1, g_1) + B_2(f_2, g_2) + B_0(f_1, g_2) + \overline{B_0(g_1, f_2)}.
\]

If we define \( S, T : H_0 \to H_0 \) by \( S \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \sigma f_1 \\ P - \sigma f_2 \end{pmatrix}, T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \tau f_1 \\ P - \tau f_2 \end{pmatrix} \), where \( P_- \) is the projection from \( \mathcal{P} \) on \( \mathcal{P}_2 \), then there exist two linear maps \( u, v : H_0 \to \ell^2 \) satisfying (4.4), (4.7) and (4.8).

Proof. For every \( f \in H_0 \), it is clear that \( \langle Sf, Sf \rangle_{H_0} = \langle f, f \rangle_{H_0} \) because \( S \) is a unitary operator.

On the other hand,

\[
\left\langle T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle_{H_0} - \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle_{H_0} = B_1(\tau f_1, \tau f_1) - B_1(f_1, f_1)
+ B_2(P_- \tau f_2, P_- \tau f_2) - B_2(f_2, f_2)
+ B_0(\tau f_1, P_- \tau f_2) - B_0(f_1, f_2)
+ B_0(\tau f_1, P_- \tau f_2) - B_0(f_1, f_2),
\]

where

\[
B_1(\tau f_1, \tau f_1) = B_1(f_1, f_1);
\]

\[
B_2(P_- \tau f_2, P_- \tau f_2) = B_2(\tau f_2, P_- \tau f_2) - B_2((I - P_-)\tau f_2, P_- \tau f_2)
= B_2(f_2, \tau^{-1} P_- \tau f_2).
\]

If \( \{\eta_{k,-}\} \) is an orthonormal basis of \( \mathcal{P}_2 \oplus \tau^{-1} \mathcal{P}_2 \) with respect to \( B_2 \), then there exists \( g_2 \in \mathcal{P}_2 \) such that \( f_2 = \tau^{-1} g_2 + \sum_k a_{k,-}\eta_{k,-} \), where \( a_{k,-} = B_2(f_2, \eta_{k,-}) \). This implies that \( \tau^{-1} P_- \tau f_2 = f_2 - \sum_k a_{k,-}\eta_{k,-} \). Then

\[
B_2(f_2, \tau^{-1} P_- \tau f_2) = B_2(f_2, f_2) - \sum_k a_{k,-} B_2(f_2, \eta_{k,-}).
\]

By the same way

\[
B_0(\tau f_1, P_- \tau f_2) = B_0(f_1, \tau^{-1} P_- \tau f_2) = B_0(f_1, f_2) - \sum_k a_{k,-} B_0(f_1, \eta_{k,-}).
\]

Summing up,

\[
\left\langle T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle_{H_0} - \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle_{H_0}
= - \sum_k a_{k,-} \cdot a_{k,-} - \sum_k a_{k,-} \cdot B_0(f_1, \eta_{k,-}) - \sum_k a_{k,-} \cdot B_0(f_1, \eta_{k,-})
= \sum_k B_0(f_1, \eta_{k,-}) \cdot B_0(f_1, \eta_{k,-})
- \sum_k [a_{k,-} + B_0(f_1, \eta_{k,-})] \cdot [a_{k,-} + B_0(f_1, \eta_{k,-})].
\]
Now, we define the operators $u, v : H_0 \to \ell^2$ by $u(f) = (u_k(f))_{k \in \mathbb{Z}}$, $v(f) = (v_k(f))_{k \in \mathbb{Z}}$, where $u_k(f) = B_0(f_1, \eta_{k-1})$ and $v_k(f) = B_0(f_1, \eta_{k-1}) + B_2(f_2, \eta_{k-1})$; so, (4.4) holds.

We define the reproducing kernel Hilbert space $H$ as the completion of the space spanned by $\{K_f : H_0 \to \mathbb{C} : \langle f, g \rangle = \langle K_f, K_g \rangle_H, \forall f, g \in H_0\}$, with the inner product $\langle K_f, K_g \rangle_H = \langle f, g \rangle_{H_0}$.

The operator $V : H \oplus \ell^2 \to H \oplus \ell^2$ defined by $V(K_f \oplus u(f)) = K_T f \oplus v(f)$, is an isometric operator, due to the fundamental identity, and it has domain $D_V = \{K_f \oplus u(f), f \in \mathcal{P}\}$ and range $R_V = \{K_T f \oplus v(f), f \in \mathcal{P}\}$.

Conditions (4.7) and (4.8) hold with the elements $w_k^{(0)} = 0 \oplus u_k^{(0)}$, $k \in \mathbb{Z}$, where $u_k^{(0)} = (\Delta_{nk})_{n \in \mathbb{Z}}$. □

References


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