Análisis Matemático: Física Matemática

M. Agirre[4], J. Aguirre[1], J. Apraiz[1], N. Arrizabalaga[1], J.B. Bru[1,2,4], A. Cornejo[1], C. Cuesta[1], J. Duoandikoetxea[1], L. Escauriaza[1], M. Escobedo[1], A. Fernández[5], A. García[1], F. de la Hoz[3], S. Montaner[1], O. Oruetxebarria[1], T. Ourmières-Bonafos[4], I. Parissis[1,2], C. Pérez[1,2,4], F. Pizzichillo[4], J. Rivas[1], I.P. Rivera[1], F. Vadillo[3] y L. Vega[1,4]. [1]Dpto. Matemáticas; [2]IKERBASQUE, Basque Foundation for Science; [3]Dpto. Matemática Aplicada y Estadística e I.O.; [4]Basque Center for Applied Mathematics (BCAM); [5]Dpto. Economía Aplicada V.

Universidad del País Vasco / Euskal Herriko Unibertsitatea

Discrete Uncertainty Principles

- **Classical Uncertainty Principles**
  The classical Hardy and Heisenberg uncertainty principles assert that **a function and its Fourier transform cannot be simultaneously localized.**

- **Discrete Uncertainty Principles**
  We are currently working on versions of these results in a discrete setting, where we consider a **discrete spatial variable** $k \in \mathbb{Z}$ and we replace the usual Fourier transform by the so called **discrete-time Fourier transform**. Real-variable techniques similar to those used in [1] lead to:

  Theorem 1 *([2, 3]) Let u be a solution to*

  $$\partial_t u(k,t) = i\,[u(k+1,t) - 2u(k,t) + u(k-1,t)] + V(k)u(k,t)$$

  in $\mathbb{Z} \times [0,1]$ with $V \in L^\infty$. There exists $\mu_0 > 0$ such that if $\mu > \mu_0$ and

  $$\sum_{k \in \mathbb{Z}} e^{2\mu|k|\log|k|}(|u(k,0)|^2 + |u(k,1)|^2) < +\infty;$$

  then, $u \equiv 0$. Hence, **a solution to the discrete Schrödinger equation cannot have fast decay at two different times**.

  The decay assumed for the solution at times $t = 0$ and $t = 1$ in Theorem 1 is related to the modified Bessel function I_k, as it happens in the continuous case with the Gaussian [1].

[1] L. Escauriaza, C.E. Kenig, G. Ponce, L. Vega, *On uniqueness properties of solutions of Schrödinger equations. Comm. Partial Diff. Eq., 31 no 10-12 (2006), 1811–1823.*

[2] A. Fernández-Bertolin, *A discrete Hardy's uncertainty principle and discrete evolutions.* arXiv:1506.00119. To appear in J. Anal. Math.

[3] A. Fernández-Bertolin, L. Vega, *Uniqueness properties for discrete equations and Carleman estimates arXiv:1509.08545.*

Borderline weighted estimates for commutators

- **Averages and maximal operators**
  Given a **strictly increasing** and **convex** function A such that $A(0) = 0$ and $A(t) \to \infty$ as $t \to \infty$, we define the **average** of f with respect to A over a cube Q as

  $$\|f\|_{A(L),Q} = \inf\left\{\lambda > 0 \,:\, \frac{1}{|Q|}\int_Q A\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

  We can define **maximal operators** related to those averages as follows

  $$M_{A(L)}f(x) = \sup_{x \in Q}\|f\|_{A(L),Q}.$$

  If $A(t) = t\log(e+t)^\rho$ for $\rho > 0$ we denote $M_{A(L)} = M_{L(\log L)^\rho}$. Similarly if $A(t) = t\log(e+t)\log(e+\log(e+t))^\rho$ for $\rho > 0$, we write $M_{A(L)} = M_{L\log L(\log\log(L))^\rho}$.
- **Borderline weighted estimates for commutators**
  Given $b \in BMO$ and T a Calderón-Zygmund operator we define the commutator $[b,T]$ as

  $$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

  We have obtained the following **quantitative estimate** [3]:

  $$w_{[b,T]f}(\lambda) \lesssim \frac{1}{\varepsilon^2}\int_{\mathbb{R}^n}\Phi\left(\frac{\|b\|_{BMO}|f(x)|}{\lambda}\right)M_{L(\log L)^{1+\varepsilon}}w(x)dx \quad \varepsilon > 0,$$

  where $w_{[b,T]f}(\lambda) = w(\{x \in \mathbb{R}^n \,:\, |[b,T]f(x)| > \lambda\})$ and $\Phi(t) = t\log(e+t)$. This estimate is a quantitative version of the main result in [2] and it is also a natural counterpart to the quantitative estimate obtained in [1] for Calderón-Zygumund operators.
- **Open questions**
  It should be possible to replace the maximal operator on the right hand side of the inequality by a smaller operator. For instance we should be able to obtain the following estimate

  $$w_{[b,T]f}(\lambda) \lesssim \frac{1}{\varepsilon^2}\int_{\mathbb{R}^n}\Phi\left(\frac{\|b\|_{BMO}|f(x)|}{\lambda}\right)M_{L\log L(\log\log L)^{1+\varepsilon}}w(x)dx \quad \varepsilon > 0.$$

  We also wonder whether the quadratic decay on ε is the best possible or if we could obtain a linear decay instead. That linear decay would lead to a better dependence on the A_1 constant in the case $w \in A_1$.

[1] T. Hytönen, C. Pérez, *The $L(\log L)^\varepsilon$ endpoint estimate for maximal singular integral operators. J. Math. Anal. Appl. 428 (2015), no. 1, 605–626.*

[2] C. Pérez, G. Pradolini, *Sharp weighted endpoint estimates for commutators of singular integral operators.* Michigan mathematical journal, (2001), V. 49, 23-37.

[3] C. Pérez, I. P. Rivera-Ríos, *Borderline weighted estimates for commutators of singular integrals.* To appear in Israel J. Math. Preprint available at http://arxiv.org/abs/1507.08568

Research Projects

IT641-13 (Gobierno Vasco, Grupos de Investigación), researcher in charge: Luis Vega. MTM2014-53145-P (Ministerio de Economía y Competitividad) researchers in charge: Luis Vega (1), Carlota Cuesta (2). MTM2014-53850-P (Ministerio de Economía y Competitividad) researchers in charge: Carlos Pérez (1), J.B. Bru (2). HADE-Harmonic Analysis and Differential Equations: new challenges (ERC-EA European Research Council Executive Agency), researcher in charge: Luis Vega.

Relativistic Quantum Mechanics

- **Shell interactions for Dirac operators**
  The **free Dirac operator** is a first-order differential operator defined by:

  $$H = -i\alpha\cdot\nabla + m\beta = \begin{pmatrix} m & 0 & -i\partial_3 & -\partial_2 - i\partial_1 \\ 0 & m & \partial_2 - i\partial_1 & i\partial_3 \\ -i\partial_3 & -\partial_2 - i\partial_1 & -m & 0 \\ \partial_2 - i\partial_1 & i\partial_3 & 0 & -m \end{pmatrix}.$$

  The **free Dirac operator** is a first order differential operator satisfying $H^2 = (-\Delta + m^2)\mathbb{I}_4$ and governs the **quantum relativistic dynamics** of an electron of mass m with no external forces acting on it.

  We study the **shell interaction of measure valued potentials**: let $\Omega \subset \mathbb{R}^3$ be a bounded regular domain, then we consider potentials V living at the boundary $\partial\Omega$ of Ω, i.e., V are $L^2(\sigma)^4$-valued potentials, where σ is the surface measure of $\partial\Omega$.

  For such potentials, in [1] we construct a domain of self-adjointness $D(H+V) \subset L^2(\mathbb{R}^3)^4$. In particular, we study **Electrostatic plus Lorentz scalar potentials**

  $$V_{es}(\varphi) = \frac{1}{2}(\lambda_e + \lambda_s\beta)(\varphi_+ + \varphi_-),\ \lambda_e, \lambda_s \in \mathbb{R} \text{ such that } \lambda_e^2 - \lambda_s^2 \neq 0, 4,$$

  where $\varphi_\pm$ are non-tangential boundary values of φ on $\partial\Omega$. We prove that $H + V_{es}$ defined on $D(H + V_{es})$ is self-adjoint and

  – If $|\lambda_e| \notin [1/C, 4C]$ for some C that depends only on the surface ($\lambda_s = 0$), then $H + V_{es}$ has no eigenvalues in $(-m, m)$.
- **Confinement**
  A potential V **generates confinement** with respect to H and $\partial\Omega$ if the particles under consideration which are initially confined in Ω at time $t = 0$ remain in Ω for all $t \in \mathbb{R}$ under the evolution $\partial_t = -i(H+V)$, i.e., $H + V$ makes $\partial\Omega$ impenetrable for particles.

  In [1] we prove that V_{es} generates confinement w.r.t. H and $\partial\Omega$ $\Leftrightarrow$ $\lambda_e^2 - \lambda_s^2 = -4$, which extends previous results in [2].
- **Current research**
  The **MIT bag model** is one of the simplest relativistic models for the confinement of an electron in a box, which is a particular case of the above-mentioned confinement potentials. Continuing with the work in [3] we want to study the eigenvalue problem for this model,

  $$\begin{cases} H\psi = a\psi, & \text{in } \Omega \\ -i\beta(\alpha\cdot n)\psi = \psi, & \text{in } \partial\Omega \end{cases}$$

  where $\psi \in H^1(\Omega)^4$, $a > 0$ and n is the exterior normal vector. We are interested in self-adjointness, spectral properties and the non-relativistic limit.

[1] N. Arrizabalaga, A. Mas, L. Vega, *Shell interactions for Dirac operators: on the point spectrum and the confinement. SIAM J. Math. Anal. 47(2), (2015) 1044–1069.*

[2] J. Dittrich, P. Exner, P. Sěba, *Dirac operators with a spherically symmetric δ-shell interaction. J. Math. Phys. 30 (1989), 2875–2882.*

[3] L. LeTreust, *A variational study of some hadron bag models. Calc. Var. Partial Differential Equations 49, (2014) 753–793.*

Analytic parabolic equations and Control Theory

- **Second order analytic parabolic equations**: we consider the parabolic evolution

  $$\begin{cases} \partial_t u - \sum_{i,j=1}^n a^{ij}(x,t)\partial_{x_i x_j}u = 0, & \text{in } \Omega\times(0,T] \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T] \\ u(x,0) = u_0, & u_0 \in L^2(\Omega). \end{cases} \tag{1}$$

  We assume that the **space-time dependent coefficients** satisfy

  i) **uniform parabolicity**: for some $\lambda > 0$

  $$\sum_{j,i=1}^n a^{ij}(x,t)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \text{for } (x,t) \in \Omega\times(0,T],\ \forall\xi \in \mathbb{R}^n;$$

  ii) **real-analyticity**: there exists $\rho_0 > 0$ such that

  $$|\partial_x^\alpha\partial_t^k a(x,t)| \le \rho_0^{-|\alpha|-k}|\alpha|!k! \quad \text{for } (x,t) \in \Omega\times(0,T] \text{ and } (\alpha,k) \in \mathbb{N}^n\times\mathbb{N}.$$

- **Quantitative estimates of real-analyticity**
  Assuming that the boundary of the domain Ω can be described locally as the graph of a real-analytic function we have proved [2, 3] that solutions to (1) satisfy the following **quantitative estimate of analyticity**:

  $$|\partial_x^\alpha\partial_t^k u(x,t)| \le e^{\frac{\rho}{t}}\rho^{-|\alpha|-k}t^{-k}|\alpha|!k!\|u_0\|_{L^2(\Omega)} \tag{2}$$

  with $(x,t) \in \Omega\times(0,T]$ and $(\alpha,k) \in \mathbb{N}^n\times\mathbb{N}$. Here ρ is a constant depending on ρ_0 and the *real-analyticity* of the domain Ω.
- **Control Theory**
  The results obtained in [2, 3] are extensions of those in [1], where the estimate (2) is used to prove a **null-controllability** property from **measurable sets** for the Heat equation. In [3] we have extended the analyticity and controllability results to **higher-order parabolic equations**.

[1] J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, *Observability Inequalities and Measurable Sets. J. Eur. Math. Soc. 16 (2014) 2433-2475.*

[2] L. Escauriaza, S. Montaner, C. Zhang, *Observation from measurable sets for parabolic analytic evolutions and applications. J. Math. Pure Appl. 104 (2015) 837-867.*

[3] L. Escauriaza, S. Montaner, C. Zhang, *Analyticity of solutions to parabolic evolutions and applications. To appear. Preprint available at ArXiV.org.*