Mathematical Analysis and Applications

N. Accomazzo[1], J. Apraiz[1], N. Arrizabalaga[1], J.B. Bru[1,3,4], J. Cayama[1], C.M. Cuesta[1], F. de la Hoz[2], X. Diez[1], J. Duoandikoetxea[1], L. Escauriaza[1], M. Escobedo[1], A. Fernández[1], M. Mourgoglou[1,4], O. Oruetxebarria[1], I. Parissis[1,4], C. Pérez Moreno[1,3,4], J. Rivas[1], I.P. Rivera[1,3], F. Vadillo[2], L. Vega[1,3], C. Zhang[1].

Universidad del País Vasco / Euskal Herriko Unibertsitatea

[1] Depto. De Matemáticas-UPV/EHU; [2] Depto. De Matemática Aplicada y Estadística e Investigación Operativa-EHU/UPV; [3] BCAM- Basque Center of Applied Mathematics; [4] IKERBASQUE, Basque Foundation for Science.

Fractal Conservation Laws

X. Diez, C.M. Cuesta and F. Achleitner.

Conservation laws are well-studied evolution equation which has many modelling applications in areas such as fluid dynamics, nonlinear acoustics and traffic flows. The generic scalar such equation being:

$$\begin{cases} \partial_t u(t,x) + \partial_x(f(u)) = 0, & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x) & x\in\mathbb{R}. \end{cases} \tag{1}$$

Weak solutions of (1) are not unique in general, therefore additional criteria are needed to establish uniqueness. One way of obtaining uniqueness is by regularising the equation with a viscosity term, e.g. in the linear case, adding ϵu_{xx} for $\epsilon > 0$ The solution of the new problem is unique and regular and performing the limit $\epsilon \to 0^+$ (the zero viscosity limit) gives a unique solution of (1). This coincides with what is called an **entropy solution**.

We study the following **fractal conservation law** which can be seen as a nonlocal regularisation of (1):

$$\begin{cases} \partial_t u^\epsilon(t,x) + \partial_x f(u^\epsilon) = \epsilon\, \partial_x \mathcal{D}_x^\alpha u^\epsilon, & t>0,\ x\in\mathbb{R}, \\ u^\epsilon(0,x) = u_0(x) & x\in\mathbb{R}, \end{cases} \tag{2}$$

where $u_0 \in \mathcal{L}^\infty(\mathbb{R})$, $f \in \mathcal{C}^\infty$ and the operator $\mathcal{D}_x^\alpha$ acting on the space variable x. This **fractional derivative** $\mathcal{D}^\alpha$ is the nonlocal operator

$$(\mathcal{D}^\alpha g)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{g'(y)}{(x-y)^\alpha} dy, \quad 0<\alpha<1,$$

which corresponds to a left-sided *Caputo* type fractional derivative integrated from $-\infty$ and is of order α.

This type of equation arises in fluid mechanics as a simplified model of stratified shallow-water flows, for instance. Our aim is to prove the limit $\epsilon \to 0^+$ of the family of functions parametrised by ϵ that satisfy (2).

First, we prove global existence and uniqueness of (2) for any $\epsilon > 0$. The local existence is based on a standard fixed-point argument, whereas the global existence is obtained after proving a maximum principle (see [2]).

The corresponding vanishing viscosity limit here is obtained by comparing this family of classical solutions to the entropy solution of (1). We rigorously prove that in the limit $\epsilon \to 0^+$, solutions converge to the entropy solution. To do this we use a **doubling variable technique** originally proposed in [3] for the local case.

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Commutators of singular integrals

N. Accomazzo, C. Pérez Moreno and I. Parissis.

Consider T a linear operator defined on a function space and b a "good" function (in general, this will mean locally integrable). We define a new operator called the *commutator* of T and b by $[b,T]f(x) = b(x)T(f)(x) - T(bf)(x)$. We want to study this operator on a particular setting: when the operator T is a singular integral. We can think that $T = H$ is the Hilbert transform,

$$Hf(x) = \text{p.v.} \int_{\mathbb{R}} \frac{1}{x-y} f(y)dy.$$

We will also take b in the class of *bounded mean oscillation* (BMO) defined by

$$\|b\|_{\text{BMO}} := \sup_Q \int_Q |b(x) - b_Q|\, dx < \infty,$$

where Q denotes a cube with sides parallel to the axis and $b_Q = \int_Q b(y)dy$. This turns out to be the right class for the study of commutators, since we have the following

Theorem. *If T is a Calderón Zygmund operator and b belongs to the class BMO, then $[b,T] : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded for all $1<p<\infty$. Conversely, if we have that $[b,H] : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded for some p $(1<p<\infty)$ then $b \in$ BMO.*

In the case $p=1$ we know that the commutator in general is not bounded, but we have that they satisfy an endpoint inequality when $b \in$ BMO and T is a Calderón Zygmund operator:

$$|\{x\in\mathbb{R}^n : |[b,T]f(x)| > t\}| \le \int_{\mathbb{R}^n} \phi(\|b\|_{\text{BMO}} |f(x)|/t)\, dx \tag{3}$$

where $\phi(t) = t(1+\log^+(t))$. We also have a converse for this result: if $T = H$ is the Hilbert transform and the commutator $[b,H]$ satisfies the endpoint inequality (3), we get that b has to belong to BMO.

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Some members of our group

A pseudo-spectral method for the fractional Laplacian on $\mathbb{R}$

J. Cayama, C. M. Cuesta and F. de la Hoz.

We develop a pseudo-spectral method to solve initial-value problems associated to PDEs involving the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ acting on the whole real line, where (see also [2])

$$(-\Delta)^{\alpha/2} u(x) = c_\alpha \int_{-\infty}^{+\infty} \frac{u(x)-u(y)}{|x-y|^{1+\alpha}} dy, \text{ with } c_a = -\frac{2^\alpha \Gamma(1/2+\alpha/2)}{\sqrt{\pi}\,\Gamma(-\alpha/2)}$$

We took $\alpha \in (0,2)$. We consider the representation of the fractional Laplacian given by the following lemma:

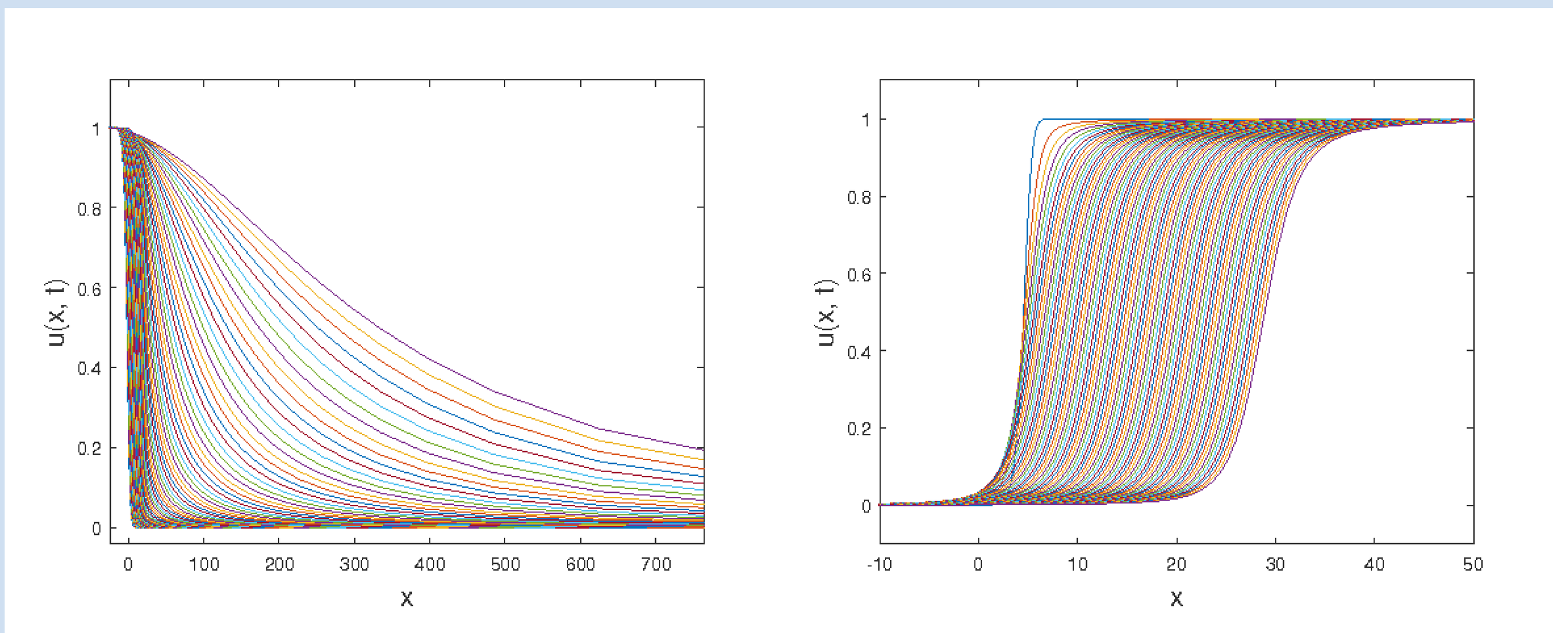
$$(-\Delta)^{\alpha/2} u(x) = \begin{cases} \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{+\infty} \frac{u'(y)}{x-y} dy, & \alpha = 1, \\ \dfrac{c_\alpha}{\alpha(1-\alpha)} \displaystyle\int_{-\infty}^{+\infty} \frac{u''(y)}{|x-y|^{\alpha-1}} dy, & \alpha \ne 1. \end{cases}$$

Then, we perform the change of variable $x = L\cot(s)$, $L>0$, to transform $\mathbb{R}$ into the interval $[0,\pi]$, where a Fourier expansion of $u(x(s))$ can be applied. We approximate $(-\Delta)^{\alpha/2} e^{iks}$, $k \in \mathbb{Z}$, by means of the midpoint quadrature rule, improving the results with Richardson's extrapolation, similarly as in [1].

This method deals accurately and efficiently with problems posed on $\mathbb{R}$, and avoids truncating the domain (which requires introducing artificial boundary conditions). In order to illustrate its applicability, we have simulated the evolution of the following non-local Fisher-KPP equation:

$$\partial_t u + (-\Delta)^{\alpha/2} u = f(u), \quad x\in\mathbb{R},\ t\ge 0.$$

On the one hand, taking $f(u) = u(1-u)$, and $u(x,0) = [-\tanh(x/(2\sqrt{6}))/2 + 1/2]^2$, the solution behaves like a front whose propagation speed increases exponentially with time (see the left-hand side figure, with $\alpha = 1.89$). On the other hand, taking $f(u) = u(1-u)(u-0.75)$, and $u(x,0) = 1 - [\tanh(-x+5)/2 + 1/2]^2$, it behaves like a front that propagates with constant speed (see the right-hand side figure, with $\alpha = 1.51$).



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Research Projects

IT641-13 (Gobierno Vasco, Grupos de Investigación), researcher in charge: Luis Vega. MTM2014-53145-P (Ministerio de Economía y Competitividad) researchers in charge: Luis Vega (1), Carlota Cuesta (2). MTM2017-82160-C2 (Ministerio de Economía y Competitividad) researchers in charge: Carlos Pérez. MTM2017-82160-C2-2-P (Ministerio de Economía y Competitividad) researchers in charge: J.B. Bru (1), M. Mourgoglou (2). MTM2014-52347-C2-1-R (Ministerio de Economía y Competitividad) researchers in charge: M. Escobedo (1), J.C. Peral (2). HADE-Harmonic Analysis and Differential Equations: new challenges (ERC-EA European Research Council Executive Agency), researcher in charge: Luis Vega.