Análisis Matemático y Aplicaciones: Física Matemática

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Relativistic Quantum Mechanics: The Dirac equation

The Dirac equation has played a fundamental role in various areas of modern physics and mathematics. The **free Dirac equation** defined as

$$i\frac{\partial}{\partial t}\psi(t,x) = H_0\psi(t,x),$$

where $\psi(x,t):\mathbb{R}^{3+1}\to\mathbb{C}^4$, describes a relativistic electron or positron which moves freely as if there were no external fields or other particles. The **free Dirac operator** H_0 is defined as

$$H_0 = -i\alpha\cdot\nabla + \beta m = \begin{pmatrix} mc^2\mathbb{I}_2 & -i\sigma\cdot\nabla \\ -i\sigma\cdot\nabla & -mc^2\mathbb{I}_2 \end{pmatrix},$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ are triplets of matrices and $\alpha_k, \beta \in \mathcal{M}_{4\times4}(\mathbb{C})$, $k = 1,2,3$, are the *Dirac matrices*

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix},$$

defined in terms of the *Pauli matrices* $\sigma_k \in \mathcal{M}_{2\times2}(\mathbb{C})$, given by

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and m is a positive real constant describing the mass of the particle. We neglect the physical constants.

- **Self-adjointness of Dirac operators with electromagnetic potentials**

The property of being self-adjoint is a fundamental information to study operators in quantum mechanics. Therefore we study the self-adjointness of the Dirac operator with an Hermitian matrix potential $\mathbb{V} = \mathbb{V}(x) : \mathbb{R}^3 \to \mathcal{M}_{4\times4}(\mathbb{C})$ such that $|x||\mathbb{V}(x)| < 1$, where $|\mathbb{V}(x)| = \sup_{\|\psi\|=1}\langle\mathbb{V}\psi,\mathbb{V}\psi\rangle$. We prove that the Dirac operator $H = -i\alpha\cdot\nabla + m\beta - \mathbb{V}$ with domain

$$\mathcal{D}(H) = \{\psi \in L^2(\mathbb{R}^3,\mathbb{C}^4) : H\psi \in L^2(\mathbb{R}^3,\mathbb{C}^4)\}$$

is self-adjoint in $L^2(\mathbb{R}^3,\mathbb{C}^4)$. Moreover, we see that the self-adjoint extension is characterized by the fact that

$$\mathcal{D}(H) \subset H^{1/2}(\mathbb{R}^3,\mathbb{C}^4) \cap \left\{\psi : \int_{\mathbb{R}^3} |\psi|^2\frac{dx}{|x|} < +\infty\right\}.$$

An explicit example of an electromagnetic operator that satisfies the above assumption is

$$\mathbb{V} = \begin{pmatrix} \frac{\nu}{|x|} & \sigma\cdot A \\ \sigma\cdot A & \frac{\nu}{|x|} \end{pmatrix},$$

where $|\nu| + |x||A(x)| < 1$ for $A = A(x) : \mathbb{R}^3 \to \mathbb{R}^3$. This work has been developed in [2].

- **Dispersion of the electromagnetic Dirac equation**

The 3D Dirac equation can be listed as a **dispersive equation**. It is interesting to quantify dispersive phenomena for the perturbed flows by using Strichartz estimates. However, those estimates fail for some potentials. We construct some **counterexamples to Strichartz estimates** for the **magnetic Dirac equation**. That is, for potentials of the form

$$A(x) = |x|^{-\delta}Mx, \qquad 1 < \delta < 2$$

where

$$M := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we prove that the solution of the magnetic Dirac equation

$$\begin{cases} i\partial_t u(t,x) + Hu(t,x) = 0 \\ u(0,x) = f(x), \end{cases}$$

for $H = -i\alpha\cdot(\nabla - iA) + m\beta$ and with initial datum $f \in H^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}$, does not satisfy the Strichartz estimates

$$\|e^{itH}f\|_{L^p_t L^q_x} \le C\|f\|_{H^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}},$$

for any couple (p,q) satisfying the Schrödinger admissibility condition

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \qquad 2 \le p \le \infty, \qquad 2 \le q \le 6.$$

See [3] for the details.

Electromagnetic Helmholtz equation

Free Helmholtz equation

- The Helmholtz equation is the elliptic second order partial differential equation

$$\Delta u(x) + k^2u(x) = f(x)$$

in $\mathbb{R}^d$, where $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian, k is the wave number and u is the amplitude. It is encountered in many branches of mathematical physics as in the theory of elasticity or theory of electromagnetic waves.

- In order to solve this equation uniquely, there must be additional restrictions on the behavior of the solution at infinity. These restrictions are the so-called Sommerfeld radiation conditions, which are typically read by

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{2}}\left(\frac{\partial u(x)}{\partial|x|} \pm iku(x)\right) = 0 \quad \text{uniformly in} \quad \frac{x}{|x|} \in S^{d-1}.$$

Electromagnetic Schrödinger Hamiltonian

$$H_A = \nabla_A^2 + V = (\nabla + iA(x))^2 + V(x) \qquad \text{in} \quad \mathbb{R}^d \quad (d \ge 3).$$

- $A:\mathbb{R}^d \to \mathbb{R}^d$ magnetic (vector) potential $\to$ describes the interaction of a free particle with an external magnetic field, which is defined as the $d\times d$ anti-symmetric matrix

$$B = B(x) = (B_{jk}), \qquad B_{jk} = \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j}.$$

The trapping component of B is $B_\tau(x) = \frac{x}{|x|}B(x)$, the projection of B on the tangential space in x to the sphere of radio $|x|$. $V:\mathbb{R}^d \to \mathbb{R}$ is the electric (scalar) potential.

- We study the electromagnetic Helmholtz equation

$$H_Au(x) + \lambda u(x) = f(x), \quad x \in \mathbb{R}^d, \quad \lambda \in \mathbb{R}\backslash\{0\}.$$

Under suitable assumptions on the potential V and the trapping component B_τ, we prove some a priori estimates and Sommerfeld radiation conditions for the electromagnetic Helmholtz equation:

$$\sup_{R\ge1}\frac{1}{R}\int_{|x|\le R}|u(x)|^2dx < \infty$$

$$\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x|^2}dx \le \frac{4}{(d-2)^2}\int_{\mathbb{R}^d}\left|\nabla_A(e^{-i\lambda^{1/2}|x|}u)\right|^2dx < \infty$$

$$\sup_{R>0}R\int_{|x|\ge R}\left|\nabla_Au(x) - i\lambda^{1/2}\frac{x}{|x|}u(x)\right|^2dx < \infty,$$

where $\nabla_A = \nabla + iA(x)$. This result can be found in [4].

- We then deduce some applications to the evolution equation and the scattering theory associated to this equation.

Null–Control and measurable sets

The control for the heat equation in a smooth and bounded domain Ω in $\mathbb{R}^n$ for a time interval $(0,T)$, $T>0$ and for a distributed control f we consider

$$\begin{cases} \triangle u - \partial_t u = f(x,t)\chi_\omega(x), & \text{in } \Omega\times(0,T), \\ u = 0, & \text{on } \partial\Omega\times[0,T], \\ u(0) = u_0, & \text{in } \Omega. \end{cases} \tag{1}$$

Here, $\omega \subset \Omega$ is an interior control region. The null controllability of this equation is about the existence for any u_0 in $L^2(\Omega)$ of a control f in $L^2(\omega\times(0,T))$ with

$$\|f\|_{L^2(\omega\times(0,T))} \le N\|u_0\|_{L^2(\Omega)}, \tag{2}$$

such that $u(T)=0$. Next are our main results for the interior null–controllability case of some parabolic evolutions with controls acting over measurable sets. The results can be found in [1].

Theorem 1. *Let $n \ge 2$. Then, $\triangle - \partial_t$ is null-controllable at all positive times, with distributed controls acting over a measurable set $\omega \subset \Omega$ with positive Lebesgue measure, when*

$$\triangle = \nabla\cdot(\mathbf{A}(x)\nabla\,\cdot) + V(x),$$

is a self-adjoint elliptic operator, the coefficients matrix $\mathbf{A}$ is smooth in $\overline{\Omega}$, V is bounded in Ω and both are real-analytic in an open neighborhood of ω. The same holds when $n=1$,

$$\triangle = \frac{1}{\rho(x)}\left[\partial_x\left(a(x)\partial_x\right) + b(x)\partial_x + c(x)\right]$$

and a, b, c and ρ are measurable functions in $\Omega = (0,1)$.

Theorem 2. *Let $n \ge 2$. Then, $\triangle - \partial_t$ is null-controllable at all times $T > 0$ with boundary controls acting over a measurable set $\gamma \subset \partial\Omega$ with positive surface measure when*

$$\triangle = \nabla\cdot(\mathbf{A}(x)\nabla\,\cdot) + V(x)$$

is a self-adjoint elliptic operator, the coefficients matrix $\mathbf{A}$ is smooth in $\overline{\Omega}$, V is bounded in Ω and both are real-analytic in an open neighborhood of γ in $\overline{\Omega}$.

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Proyectos

IT-305-07 (Gobierno Vasco, Grupos de Investigación) MTM 2007-72186 (Ministerio de Educación y Ciencia).   Investigador Principal: Luis Vega