Análisis Matemático y Aplicaciones: Física Matemática

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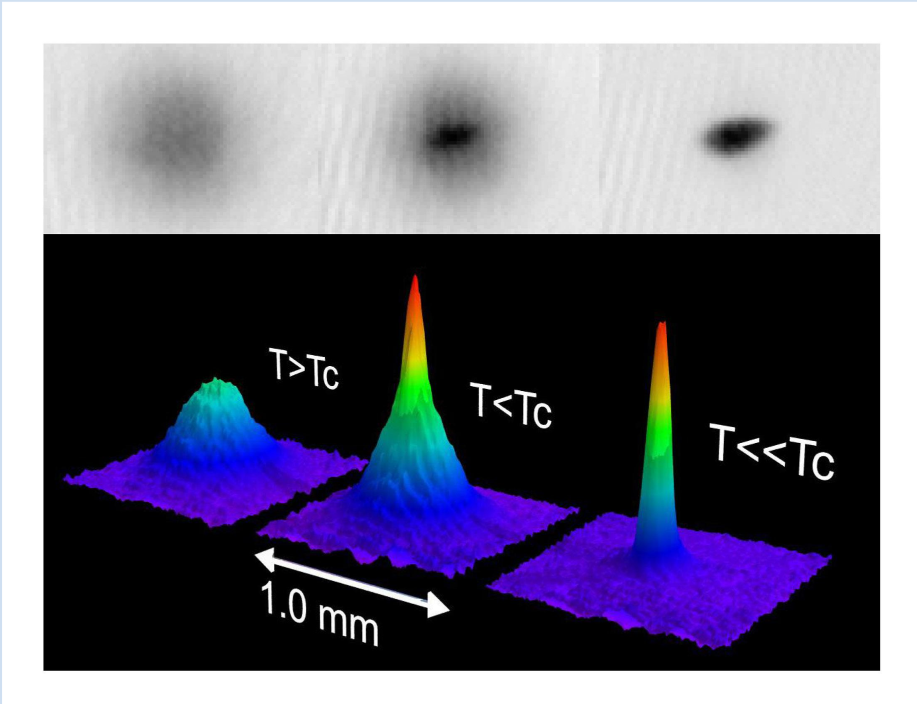
Quantum Statistical Mechanics

Mathematical Foundations of Statistical Mechanics – Quantum Many–Body Theory

- The past decade has seen a drastic increase in interest in ultra-cold atoms, driven by experiments on superconductors and Bose-Einstein condensates of ^{87}Rb, ^{7}Li, ^{23}Na, ^{85}Rb, ^{41} K, ^{133}Cs, hydrogen, metastable triplet ^{4}He, ^{174}Yb, ^{85}Rb$_2$, etc.



Bose-Einstein condensates for temperatures $\mathrm{T} < \mathrm{T_c} << 10^{-5}$K *performed for* ^{23}*Na in the MIT by the group of Wolfgang Ketterle (2001 Nobel Prize in Physics)*

- The remarkable degree of universality of quantum phase transitions allows us to focus on effective theories.
- Rigorous quantum many-body theory is however, a notoriously difficult subject.
- In fact, mathematical foundations of statistical mechanics and the quantum many-body theory involve many different fields of mathematics such as :
  (1) Functional analysis [1] – Operator theory [2]– Convex analysis [3].
  (2) Probability theory [4] – Stochastic processes [5].
  (3) Variational problems [6] – Game theory [7].
  (4) Operator algebras [8].
  (5) Differential equations [9].

Current research lines

- Mathematical Methods to diagonalize Hamiltonians [10]: Proof of global (resp. local) existence and uniqueness of solutions of the Brockett-Wegner diagonalizing flow $\dot{H}_t = [H_t, [H_t, A]]$ for bounded (resp. unbounded) operators acting on a complex Hilbert space $\mathcal{H}$.
- Diagonalization of quadratic Hamiltonians acting on a Boson Fock space by using a proof of the well-posedness of non-autonomous evolution equations, see [11].
- Mathematical description of fermion systems on lattices - as for instance electrons in solids - with long range interactions, see [7]. This gives a first answer to an old open problem in mathematical physics - first addressed by Ginibre in 1968 for bosonic systems in continuum - about the validity of the so-called Bogoliubov approximation on the level of states.
- Rigorous study of the thermodynamic impact of the Coulomb repulsion on s–wave superconductors via the strong coupling BCS–Hubbard Hamiltonian, see [12]. This analysis implies a rigorous explanation of the necessity of doping insulators to create superconductors.

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Dispersive Equations in Quantum Mechanics

- **Electromagnetic Schrödinger flow**

Consider a self-adjoint Schrödinger Hamiltonian $H = (i\nabla + A(x)) + V(x)$ acting on $H^1(\mathbb{R}^n)$, with a fixed magnetic potential $A : \mathbb{R}^n \to \mathbb{R}^n$ and electric potential $V : \mathbb{R}^n \to \mathbb{R}$ and look to the Schrödinger flow generated by the group $S(t) = e^{itH}$. The dispersive properties of $S(t)$ were extensively investigated in the last years by the members of the group. We summarize them:

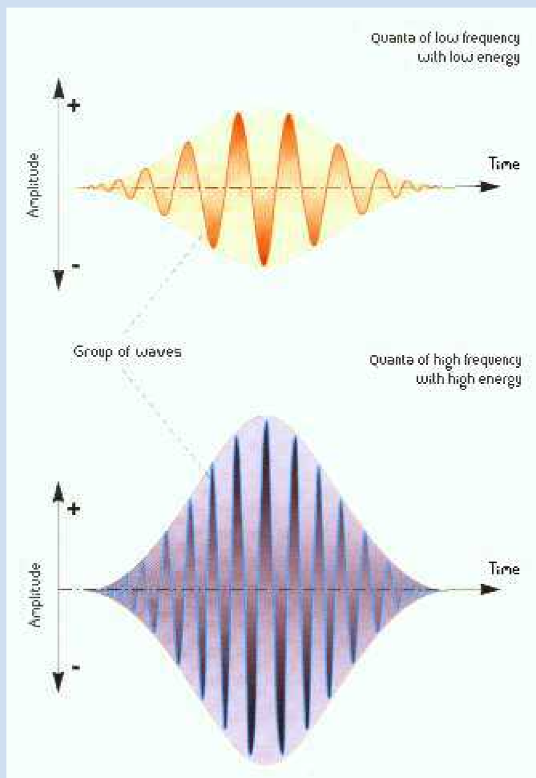
1. Define by $B = dA$ the magnetic field and $B_\tau = \frac{x}{|x|}B$ its tangential component to the sphere (*trapping component*). If $B_\tau, (\partial_r V)_+$ are *small*, then weak dispersive properties hold for $S(t)$ (Fanelli-Vega [4]).
2. If B_τ and $(\partial_r V)_+$ are *small* and A, V are short-range, then endpoint Strichartz estimates hold for $S(t)$
$$\|S(t)u_0\|_{L^p_t L^q_x} \lesssim \|u_0\|_2,$$
for any Schrödinger admissible Strichartz pair (p, q) (D'Ancona-Fanelli-Vega-Visciglia [1]).
3. Strichartz estimates in general fail for long range potentials A, V (Fanelli-Garcia [4] and Goldberg-Vega-Visciglia [6]).

The contribution given in this field by the group also permitted to understand the time-propagation of solutions of the electromagnetic wave, Klein-Gordon and Dirac equations. The relation between dispersive equations and the Helmholtz-type equation
$$Hu - (k^2 \pm i\epsilon)u = 0$$
is an object of investigation since the paper [7], in the case $A \equiv 0$, and [3] when $A \neq 0$.
The main aim is to completely understand the linear theory about Hamiltonian of the form H, which is the fundamental tool in order to study Physical models described by nonlinear perturbations of $S(t)$.

- **Uncertainty Principles**

One cannot simultaneously localize both a function and its Fourier transform. Some versions:

1. *Heisenberg*: let f be a function such that $\|f\|_2 = 1$; then
$$\left(\int_{-\infty}^{+\infty}(t-a)^2|f(t)|^2\,dt\right)\cdot\left(\int_{-\infty}^{+\infty}(\xi-a)^2|\hat{f}(\xi)|^2\,d\xi\right) \geq \frac{1}{16\pi^2}.$$
2. *Hardy*: let $|f(x)| \leq Ae^{-\pi\alpha x^2}$ and $|\hat{f}(\xi)| \leq Be^{-\pi\beta\xi^2}$ for some constants A, B, α and β. If $\alpha\beta > 1$, then $f \equiv 0$.



Uncertainty and the Schrödinger equation (Escauriaza, Kenig, Ponge and Vega [2])

1. Let u be a solution of $\partial_t u = i\Delta u$ on $\mathbb{R}^n \times [0,T]$. If $|u(x,0)| \leq Ae^{-|x|^2/\beta^2}$ and $|u(x,T)| \leq Be^{-|x|^2/\alpha^2}$, with $\alpha\beta < 4T$, then $u \equiv 0$.
2. Let u be a solution of $\partial_t u = i(\Delta u + V(x,t)u)$ in $\mathbb{R}^n \times [0,T]$, with a potential $V \in L^\infty$. Assume that, for $\alpha, \beta > 0$ with $\alpha\beta < 4T$ it holds:
   (i) $\|e^{|x|^2/\beta^2}u(0)\|_{L^2(\mathbb{R}^n)} < +\infty, \quad \|e^{|x|^2/\alpha^2}u(T)\|_{L^2(\mathbb{R}^n)} < +\infty$
   (ii) $\sup_{[0,T]}\|e^{|x|^2/(\alpha t+\beta(1-t))^2}V(t)\|_{L^\infty(\mathbb{R}^n)} < +\infty$

   Then $u \equiv 0$.

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Proyectos

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