Noise Suppression by Noise

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We have analyzed the interplay between an externally added noise and the intrinsic noise of systems that relax fast towards a stationary state, and found that increasing the intensity of the external noise can reduce the total noise of the system. We have established a general criterion for the appearance of this phenomenon and discussed two examples in detail.

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For a long time, noise was considered to be only a source of disorder, a nuisance to be avoided. Recently, this view has been changing due to several phenomena that show constructive facets of noise. Among them, the most widely studied is the phenomenon of stochastic resonance, where the addition of noise to a system enhances its response to a periodic force [1,2]. This counterintuitive aspect of noise has been found under a wide variety of situations, including bistable [3] and monostable [4] systems, nondynamical elements with [5] and without [6] threshold, and pattern forming systems [7]. Similar constructive outcomes are also found in other remarkable phenomena, such as noise induced transitions [8] and noise induced transport [9]. To some extent, the presence of noise is an unavoidable feature, and as one moves from macroscopic to microscopic scales that presence becomes more and more prominent. To withdraw the noise, it is customary to reduce as much as possible all of the external noise sources that affect the system since it still seems paradoxical that adding noise might result in a less noisy system.

In this Letter we show that the intrinsic noise displayed by some systems can substantially be reduced through its nonlinear interplay with externally added noise and we establish sufficient conditions for this phenomenon to occur. The systems we consider are those relaxing fast to a stationary state where their properties are completely determined by some parameters: the state of the system, denoted by $I(t, \mathbf{V})$, is a function of some input parameters, denoted by the set $\mathbf{V}$. These systems are usually called nondynamical systems. For the sake of simplicity, we consider the case of a single input parameter, i.e., $\mathbf{V} \equiv V$. The temporal dependence of the state of the system takes into account the intrinsic fluctuations. This stochastic behavior can be described by the mean value

$$\langle I(t, V)\rangle = H(V) \tag{1}$$

and the correlation function

$$\langle I(t, V) I(t + \tau, V)\rangle - \langle I(t, V)\rangle^2 = \tilde{G}(V, \tau), \tag{2}$$

where $\langle . \rangle$ indicates average over the noise. For practical purposes, it is convenient to write $I(V, t)$ as

$$I(t, V) = H(V) + \xi(t, V), \tag{3}$$

where $\xi(t, V)$, with $\langle \xi(t, V)\xi(t + \tau, V)\rangle = \tilde{G}(V, \tau)$, represents the intrinsic noise. When the characteristic time scales of the fluctuations are smaller than any other entering the system, $\tilde{G}(V, \tau) = G(V)\delta(\tau)$, and the power spectrum of the fluctuations is flat for the range of frequencies of interest. The function $H(V)$ corresponds to the deterministic response and $G(V)$ corresponds to the intensity of the intrinsic fluctuations.

We now analyze how adding external noise affects the output of the system. We consider that a random quantity $\zeta(t)$ is added to a constant input V_0, i.e., $V(t) = V_0 + \zeta(t)$. The random term is assumed to be Gaussian noise with zero mean and correlation function $\langle \zeta(t)\zeta(t + \tau)\rangle = \sigma^2 \exp(-\tau/\tau_F)$, where σ^2 defines the noise level and τ_F is the correlation time. If the correlation time τ_F is not too small, the system adapts to each value of $V(t)$ and can be described by an input-output relationship where the input is V_0. The external noise, however, changes the characteristics of the system which is then described by

$$I(t, V_0) = H_0(V_0) + \xi_0(t, V_0), \tag{4}$$

where the mean value and the total fluctuations of the output are taken into account by the terms $H_0(V_0)$ and $\xi_0(t, V_0)$, respectively. For times higher than τ_F, the noise term can again be approximated by Gaussian white noise with zero mean, but now with correlation function $\langle \xi_0(t, V_0)\xi_0(t + \tau, V_0)\rangle = G_0(V_0)\delta(\tau)$.

For small σ^2, $H_0(V_0) = \langle I[t, V(t)]\rangle$ and $G_0(V_0) = \int_{-\infty}^{\infty}\{\langle I[t, V(t)] I[t + \tau, V(t + \tau)]\rangle - \langle I[t, V(t)]\rangle^2\}\, d\tau$ can readily be obtained by expanding $H(V)$ and $G(V)$ in power series of V around V_0. Now the average $\langle . \rangle$ must be carried out over the two noises. By averaging first over the internal noise, we obtain

$$H_0(V_0) = H(V_0) + \frac{1}{2} H''(V_0)\langle \zeta(t)^2\rangle, \tag{5}$$

$$G_0(V_0) = \int_{-\infty}^{\infty}\left\{\left[G(V_0) + \frac{1}{2} G''(V_0)\langle \zeta(t)^2\rangle\right]\delta(\tau) + H'(V_0)^2\langle \zeta(t)\zeta(t + \tau)\rangle\right\} d\tau, \tag{6}$$

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where ($'$) indicates the derivative of the function with respect to its argument. The second average leads to

$$H_0(V_0) = H(V_0) + \frac{1}{2} H''(V_0)\sigma^2, \quad (7)$$

$$G_0(V_0) = G(V_0) + \left[2\tau_F H'(V_0)^2 + \frac{1}{2} G''(V_0)\right]\sigma^2. \quad (8)$$

Therefore, the output noise may be decreased by the addition of external noise when $G''(V_0)$ is negative and the correlation time of the external noise is sufficiently small.

It may also be interesting to decrease the output noise intensity for a fixed mean value of the output. For this purpose, the input V_0 must be tuned to a new value V_c so that

$$H_0(V_c) = H(V_0), \quad (9)$$

$$G_0(V_c) = G(V_0) + \Delta G(V_0). \quad (10)$$

If the noise level is small, the new value of the input is

$$V_c = V_0 - \frac{H''(V_0)}{2H'(V_0)} \sigma^2, \quad (11)$$

and the variation of the output noise intensity $\Delta G(V_0)$ is given by

$$\frac{\Delta G(V_0)}{\sigma^2} = 2\tau_F H'(V_0)^2 + \frac{1}{2} G''(V_0) - G'(V_0)H''(V_0)/2H'(V_0), \quad (12)$$

which can take negative values as well.

As a first example illustrating the applicability of our results, we will analyze a model for electrical conduction which displays saturation. In this model, $I(t, V)$ corresponds to the current intensity and V corresponds to an input voltage. To be explicit, we consider

$$H(V) = \frac{V}{R(1 + V^2)^{1/2}}, \quad (13)$$

$$G(V) = \frac{Q}{(1 + V^2)^{1/2}}, \quad (14)$$

where R and Q are constants. A well-known example of systems exhibiting this nonlinear behavior are semiconductor systems that display hot electron effects [10,11]. According to our previous analysis, the output noise for an input with mean value V_0 is given by

$$G(V_0) = G_0(V_0) + \frac{4\tau_F - QR^2(1 - 2V_0^2)(1 + V_0^2)^{1/2}}{2R^2(1 + V_0^2)^3} \sigma^2. \quad (15)$$

This indicates that, for small correlation times and small mean voltages, the output noise of the device may be decreased by the addition of noise. Similar results are obtained when the input V_0 is changed to V_c in order for the mean value of the output to have the same value as in absence of noise. For the new value of the input,

$$V_c = V_0 + \frac{3V_0}{2 + 2V_0^2} \sigma^2, \quad (16)$$

the output noise changes to

$$G(V_c) = G_0(V_0) + \frac{4\tau_F - QR^2(1 + V_0^2)^{3/2}}{2R^2(1 + V_0^2)^3} \sigma^2. \quad (17)$$

As in the previous case, the output noise can be decreased for small values of the correlation time of the input noise, but now it is not required for the applied voltage to be small. These features are illustrated in Fig. 1 for different values of the correlation time τ_F.

To extend our results to higher input noise levels, we have computed $H_0(V_0)$ and $G_0(V_0)$ by numerically averaging Eqs. (13) and (14) over the fluctuating input. In Fig. 2

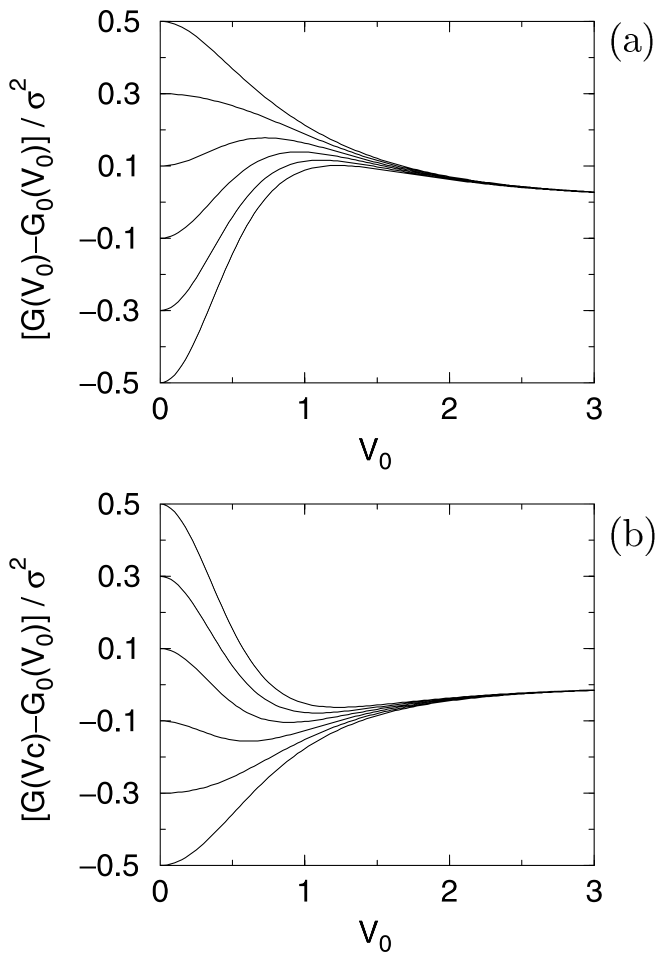

FIG. 1. Increase of the output noise [(a) constant input and (b) fixed mean value of the output] over the input noise for the nonlinear electric conduction model [Eqs. (13) and (14)] with $Q = 1$ and $R = 1$. The different lines correspond to different input noise correlation times (from bottom to top): $\tau_F = 0.0$, 0.1, 0.2, 0.3, 0.4, and 0.5. All quantities and parameters are given in arbitrary units.

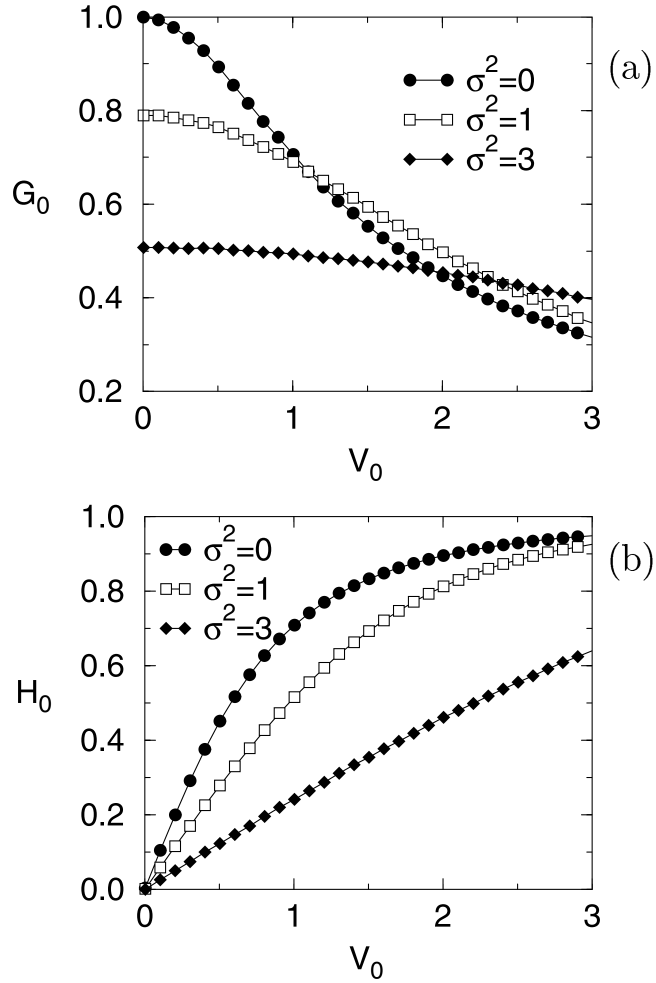

FIG. 2. Change of (a) the output mean and (b) the output fluctuations for the nonlinear electric conduction model [Eqs. (13) and (14)] with $\tau_F = 0.001$, $Q = 1$, and $R = 1$. All quantities and parameters are given in arbitrary units.

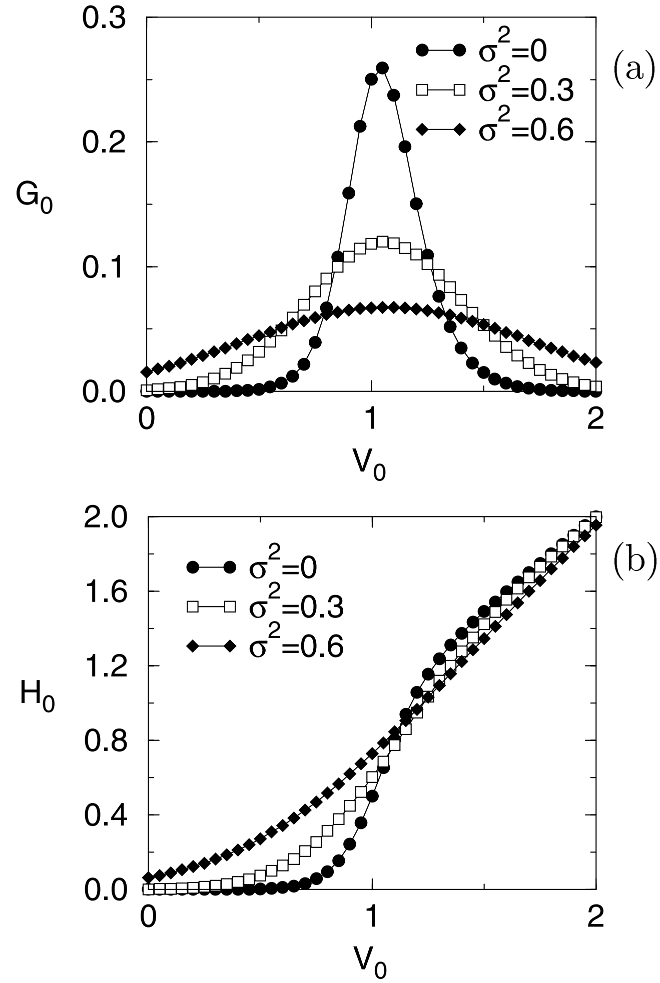

FIG. 3. Change of (a) the output mean and (b) the output fluctuations for the ionic channel model [Eqs. (18) and (19)] with $\tau_F = 0.001$, $\Delta = 10$, and $W = 1$. All quantities and parameters are given in arbitrary units.

we have shown the results for different intensities of the noise. This figure indicates that qualitatively the same phenomenon appears for higher noise intensity.

The second example we consider is an ionic channel model [12]. The characteristics of the current intensity $I(t, V)$ as a function of the input voltage are given by

$$H(V) = \frac{V}{1 + e^{\Delta(W-V)}}, \tag{18}$$

$$G(V) = \frac{V^2 e^{\Delta(W-V)}}{(1 + e^{\Delta(W-V)})^2}, \tag{19}$$

where W and Δ are constants. By using Eqs. (8) and (12), one can easily see that the input noise can reduce the output noise for a fixed average of either the input or the output. Figure 3 shows the results obtained by numerically averaging Eqs. (18) and (19). As in the previous example, there is a range of values of V_0 where the input noise can reduce the output noise.

The way noise affects the output of the two systems considered previously is depicted in Fig. 4. This figure shows the output noise intensity as a function of the mean value of the output. For the values of the parameters used in Fig. 4(a), the system can have the same average output with less fluctuations by just adding noise. Similarly, in Fig. 4(b), externally added noise greatly reduces the output noise around $H_0 = 0.5$, but, for higher values of H_0, the situation is just the opposite: noise plays its usual detrimental role.

Our analysis indicates that addition of noise to systems that have intrinsic noise can change their properties to the extent that they may display less noise. Other studies have shown that intrinsic noise can be responsible for the appearance of stochastic resonance [5,13] and aperiodic stochastic resonance [14] in nondynamical systems. In those studies, external noise enhances the response of the system to a periodic or an aperiodic signal. In our case, noise does not need to cooperate with an external signal to play a constructive role. A line of investigation for future

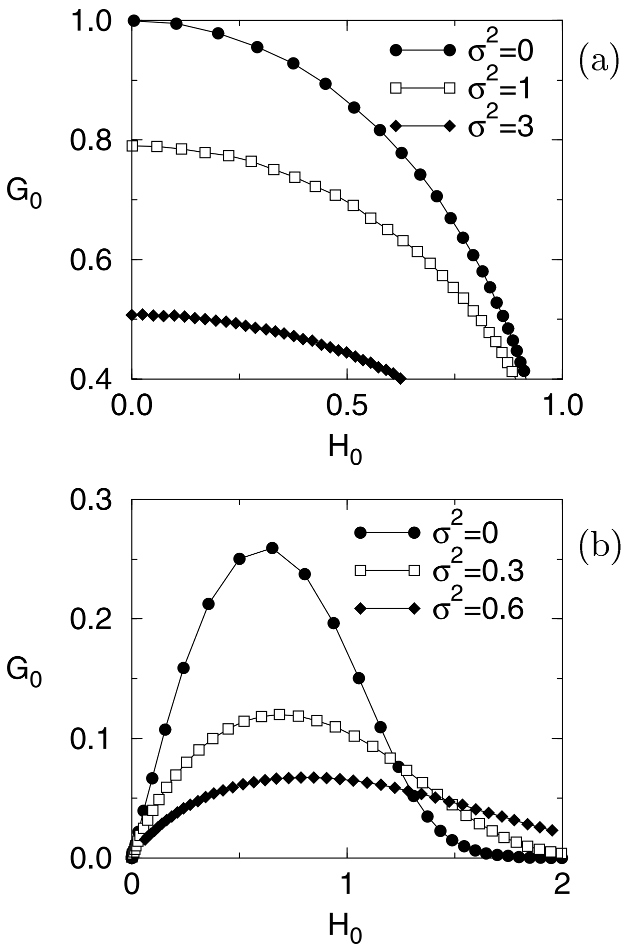

FIG. 4. Output noise intensity as a function of the mean output for (a) the nonlinear conduction model [Eqs. (13) and (14) with $\tau_F = 0.001$, $Q = 1$, and $R = 1$] and (b) the ionic channel model [Eqs. (18) and (19) with $\tau_F = 0.001$, $\Delta = 10$, and $W = 1$]. All quantities and parameters are given in arbitrary units.

work would be to extend our results to dynamical systems. This would need the development of new methodologies, in a similar way as has been done for nonlinear noise sources in stochastic differential equations [8,15]. Our work then offers new perspectives on what concerns the constructive effects of noise in general nonlinear systems.

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