Inclusion-Exclusion Principle for Fuzzy Sets

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Abstract

In this note we extend the classical set-theoretic principle of inclusion and exclusion to fuzzy subsets of a finite set X with N elements for a positive integer N. These discussions are preliminary in nature and are concerned with general formulations of the principle rather than with any particular ramifications. The counting technique is applied to level sets of fuzzy subsets to derive the general identities. A practical example is discussed.

keywords: Principle of inclusion-exclusion; Finite fuzzy set; Level set; Set enumeration; Counting.

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1 Introduction.

The *principle of inclusion and exclusion* as a tool in combinatorial problems, was well-known over the past hundred years, even as early as the time of Bernoullis according to Riordan [3]. It has been effectively used in the solution of many problems, in particular, a French problem by the name "le probléme des recontres"in permutations, more accurately in derangements, as early as 1713. Alternative names for the same principle are sieve method, principle of cross classification or symbolic enumeration. It is the technique of counting the number of elements in the union of a finite number of subsets of a finite set or complimentarily counting the number of those elements not in the union, each subset being defined by means of a dichotomous property. In this note we take the view that it may be useful to look at those properties which give rise to fuzzy subsets of a finite set and formulate the principle of inclusion and exclusion for a finite family of fuzzy subsets of a finite set. We use the level sets of fuzzy subsets for defining the principle.

2 Preliminaries.

In this section we recall the basic definitions and some pertinent known results on fuzzy subsets, principle of inclusion and exclusion, and in the process set up the notations that are used in the paper. Almost all books on combinatorics treat the principle of inclusion and exclusion to some extent. For a detailed treatment of this principle and Mobius functions we refer to [2]. For some combinatorial aspects of finite fuzzy sets, see [1].

2.1 Fuzzy Sets.

We use $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$, the real unit interval as a chain with the usual ordering for the degree of membership values. Throughout this paper we take X to be a non-empty finite set with N elements. If A is a subset of X then A^c denotes the complement of A in X.
A *fuzzy set* of X is described by a fuzzy prop-

erty p and is represented by a mapping $\mu : X \to \mathbf{I}$, $\mu(x)$ representing the degree of satisfaction of p by x. We denote the set of all fuzzy sets of X by $\mathbf{I}^X$. By an *α-cut* of μ for a real number $\alpha \in \mathbf{I}$, we mean the crisp subset $\mu^\alpha = \{x \in X : \mu(x) \geq \alpha\}$ of X. We remark that $(\mu \vee \nu)^\alpha = \mu^\alpha \cup \nu^\alpha$ and $(\mu \wedge \nu)^\alpha = \mu^\alpha \cap \nu^\alpha$ for all $\alpha \in \mathbf{I}$. Together with α-cut of fuzzy set μ we could also usefully consider the set which is the complement of μ^α in X. We denote this set by $\mu^{\bar{\alpha}}$ so that $\mu^{\bar{\alpha}} = \{x \in X : \mu(x) < \alpha\}$. For any fuzzy set μ, clearly $\mu^\alpha \cup \mu^{\bar{\alpha}} = X$.

2.2 The calculus of counting function N.

Here we introduce a convenient function N that counts elements of subsets of X with specified properties and discuss some implications. Assume p is a dichotomous property (that is, every element of X either has or does not have the property p) which describes a crisp subset A of X (that is, A consists of those elements of X which satisfy p and A^c, the complement of A in X, consists of those elements of X which does not satisfy p). Let $N(p) = |A|$ denote the number elements in A. If p' denotes the absence of property p, then by complementarity principle, the number of elements with the property p together with the number of elements without the property p (that is the number of elements with the property p') make up the totality of all elements of X. Hence we have $N(p) + N(p') = N = N(1)$. This can symbolically be expressed as $N(p') = N(1) - N(p) = N(1-p)$. It is useful to observe the complementarity of Boolean logic in this convention with $p + p' = 1$ and in terms of sets and their cardinalities, $|A| + |A^c| = |X|$. Similarly, by $N(pq)$ we mean the number of elements of X satisfying both properties p and q. Formally we can extend this convention further to $N(p+q) = N(p)+N(q)$ for the number of elements satisfying either p or q or both, and the number of elements with "not p" as $N(-p) = -N(p)$. For example, for two properties p and q, the number of elements satisfying neither p nor q is $N(p'q') = N((1-p)(1-q)) = N(1-p-q+pq) = N(1)-N(p)-N(q)+N(pq)$.

2.3 Principle of inclusion and exclusion for crisp sets.

Suppose $P = \{p_1, p_2, \cdots, p_m\}$ is a collection of properties on X each of which is a dichotomous property. That is each property p_i defines a crisp subset A_i of X, namely those elements of X which satisfy it. Then

Assertion 2.1 (Principle of inclusion and exclusion) The number of elements in the set X which is not in any of the sets A_i for $i = 1, 2, \cdots, m$ is given by,

$$|A_1^c \cap A_2^c \cap \cdots \cap A_m^c| = |X| - \sum_i |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^m |A_1 \cap \cdots \cap A_m| \tag{2.1}$$

The above assertion can be stated in terms of the properties p_i's, as

Assertion 2.2 The number of elements in the set which does not satisfy any of properties p_i for $i = 1, 2, \cdots, m$ is given by (Symbolically using the counting function $N(p)$),

$$N(1) - \sum_i N(p_i) + \sum_{i,j} N(p_i\, p_j) - \cdots + (-1)^m N(p_1\, p_2\, \cdots\, p_m)$$
$$\text{or } N(p_1'\, p_2'\, \cdots\, p_m') = N((1-p_1)(1-p_2)\cdots) \tag{2.2}$$

where i runs from 1 through to m, one at a time and i, j run through any two distinct indices from 1 through to m, two at a time etc. Either of the above expressions is called the principle of inclusion and exclusion since the counting is based on the process of including everything, excluding those not required, including those wrongly excluded, and so on. The interpretation is that the expression within the bracket on the right hand side is multiplied first and then terms are separated with appropriate signs. The advantage of this symbolic representation is that we need not confine to the whole set; instead, any subset of X suitably defined by the properties may be used. For example, $N(p'_1\, p_2\, p'_3\, p_4)$ can be expressed as

$$N((1-p_1)\, p_2(1-p_3)\, p_4) = N(p_2\, p_4)$$
$$-N(p_1\, p_2\, p_4) - N(p_2\, p_3\, p_4)$$
$$+N(p_1\, p_2\, p_3\, p_4) \qquad (2.3)$$

3 Inclusion-Exclusion

In this section we define and give some of the elementary properties of the principle of inclusion and exclusion for fuzzy sets of X. We discuss a practical example illustrating the principle.

3.1 The counting function for fuzzy sets.

Suppose now p is a fuzzy property on X, that is, it is only possible to determine upto a degree of certainity (with a number between 0 and 1) whether an element has the property p or not. Consider an α in the unit interval **I**. Since we are interested in the number of elements of subsets of X with specified property p, it is appropriate to introduce the notation $p^\alpha = \mu^\alpha$ for the elements of X that have the property p to a degree at least α. Here "*at least* "is used in the inclusive sense, that is, we include α. Complementarily we denote the complement of p^α in X by $p^{\bar{\alpha}} = \mu^{\bar{\alpha}}$. That is $p^{\bar{\alpha}}$ consists of elements of X that have the property p at most α (or alternatively, $p^{\bar{\alpha}}$ is the set of elements of X with the absence of property p to a degree at least α). When we refer to "*at most* "in this context, we use it in the exclusive sense, that is α is excluded.
By $N(p^\alpha)$ we mean the number of elements of the subset $p^\alpha$of X and similarly $N(p^{\bar{\alpha}})$ means the number of elements of $p^{\bar{\alpha}}$ where $p^{\bar{\alpha}}$ denotes the complement of p^α in X. Since every element X has the property p to a degree at least α or to a degree at most α, the equation $N(p^\alpha) + N(p^{\bar{\alpha}}) = N(1)$ is true, where $N(1)$ as before stands for the totality of all elements in the set X, so that $N(p^\alpha) = N(1) - N(p^{\bar{\alpha}}) = N(1-p^{\bar{\alpha}})$ or $N(p^{\bar{\alpha}}) = N(1) - N(p^\alpha) = N(1-p^\alpha)$. Now if we consider two properties p and q, then as before $N(p^\alpha q^\alpha)$ counts the number elements that have both the properties p and q to a degree at least α. Then the number elements that have both properties p and q at most α is given by $N(p^{\bar{\alpha}} q^{\bar{\alpha}}) = N((1-p^\alpha)(1-q^\alpha)) = N(1-p^\alpha-q^\alpha+p^\alpha q^\alpha) = N(1)-N(p^\alpha)-N(q^\alpha)+N(p^\alpha q^\alpha)$. From an α-cut prospective for fuzzy sets, we have $p^\alpha = \mu^\alpha$, $N(p^\alpha) = |\mu^\alpha|$ and $p^{\bar{\alpha}} = (\mu^\alpha)^c$, $N(p^{\bar{\alpha}}) = |(\mu^\alpha)^c|$. Since $\mu^\alpha \cup (\mu^\alpha)^c = X$, symbolically $p^\alpha + p^{\bar{\alpha}} = 1$. We wish take this view in formulating the principle of inclusion and exclusion for fuzzy sets.

3.2 The principle for fuzzy sets.

Suppose $P = \{p_1, p_2, \cdots, p_m\}$ is a collection of fuzzy properties on X with the associated fuzzy subsets $P_{FS} = \{\mu_1, \mu_2, \cdots, \mu_m\}$ of X. Let α be a real number in the unit interval. Then the α-cut of $(p_1 \vee p_2 \vee \cdots \vee p_m)^\alpha = (\mu_1 \vee \mu_2 \vee \cdots \vee \mu_m)^\alpha$ which is the union of the α-cuts $\mu_1^\alpha \vee \mu_2^\alpha \vee \cdots \vee \mu_m^\alpha = p_1^\alpha \vee p_2^\alpha \vee \cdots \vee p_m^\alpha$ of each of the p_i's for $i = 1, 2, \cdots, m$. In terms of the properties P_{FS} , the principle can be expressed as

Assertion 3.1 Symbolically using the counting function $N(p^\alpha)$, the number of elements of X satisfying each property p_i ($i = 1, 2, \cdots, m$) to a degree at most α, is given by

$$N(1) - \sum_i N({p_i}^\alpha) + \sum_{i,j} N({p_i}^\alpha\, {p_j}^\alpha) - \cdots$$
$$+(-1)^m N({p_1}^\alpha\, {p_2}^\alpha \cdots {p_m}^\alpha) \qquad (3.1)$$

which can further be written as

$$N(p_1{}^{\bar{\alpha}}\, p_2{}^{\bar{\alpha}}\, \cdots\, p_m{}^{\bar{\alpha}}) = N((1-p_1{}^{\alpha})(1-p_2{}^{\alpha})\cdots) \tag{3.2}$$

where i runs from 1 through to m one at a time and i, j run through any two distinct indices from 1 through to m two at a time etc. The advantage of this symbolic representation is that we need not confine to the whole set nor with one particular level of satisfaction, namely α for all the properties p_i as above; instead, any subset of X suitably defined by these properties may be used. Also different level of satisfaction by the elements of different properties p_i can be chosen for the counting of elements in X.

For example, suppose the level of satisfaction are specified for five different properties p, p_2, p_3, p_4, and p_5 as $\alpha, \beta, \gamma, \delta$ and λ in $\mathbf{I}$ respectively. Then the number of elements that satisfy p_1 at least to the level α, p_2 at most to β, p_3 at most to γ, p_4 at least to δ and finally p_5 to a level at least λ will be given by

$$\begin{aligned} & N(p_1{}^{\alpha}\, p_2{}^{\bar{\beta}}\, p_3{}^{\bar{\gamma}}\, p_4{}^{\delta}\, p_5{}^{\lambda}) \\ &= N(p_1{}^{\alpha}\,(1-p_2{}^{\beta})\,(1-p_3{}^{\gamma})\, p_4{}^{\delta}\, p_5{}^{\lambda}) \\ &= N(p_1{}^{\alpha}\, p_4{}^{\delta}\, p_5{}^{\lambda}) - N(p_1{}^{\alpha}\, p_2{}^{\beta}\, p_4{}^{\delta}\, p_5{}^{\lambda}) \\ & -N(p_1{}^{\alpha}\, p_3{}^{\gamma}\, p_4{}^{\delta}\, p_5{}^{\lambda}) + N(p_1{}^{\alpha}\, p_2{}^{\beta}\, p_3{}^{\gamma}\, p_4{}^{\delta}\, p_5{}^{\lambda}) \end{aligned} \tag{3.3}$$

3.3 An illustrative example.

We illustrate with a practical example below. The numbers have been chosen subjectively for illustrative purposes.

Example 3.2 Suppose we have basket of 100 tomatoes, some are big (others are medium to small - property p_1), some are red (others are green to red - property p_2) and some others have black spots and yet others are bruised. Clearly the properties "black spots" and "bruises" are dichotomous whereas "ripe", "big" or "red" are fuzzy properties. If we are given that there are 25 with black spots, 15 bruised and 5 are bruised and have black spots on them, then obviously the number of desirable tomatoes (no spots, no bruises) are 100 - 25 - 15 + 5 = 65.

Suppose we are dealing with fuzzy properties. To begin with, we need to grade the tomatoes with respect a fuzzy property. For instance, if there are 50 medium-to-big tomatoes with the grading $\frac{1}{2}$ and above, 75 small-medium-to-big with the grading of $\frac{1}{4}$ and above, and all of them make the grading of $\frac{1}{100}$ and above. On the other hand, 20 of them make the grading of $\frac{4}{5}$-red, 80 of them have the grading $\frac{1}{4}$-red and 20 have grading less than $\frac{1}{4}$, that is they are just about green. There are 65 which make up the $\frac{1}{4}$ grading for size (small-medium-to-big) and for colour (reasonably red). Any tomato that fails to make the grading of at most $\frac{1}{4}$ in size and colour simultaneously does not go on the shelf. How many do not make up to the shelf? By the principle of inclusion and exclusion we are interested in $N(p_1{}^{\bar{\frac{1}{4}}}\, p_2{}^{\bar{\frac{1}{4}}})$. This is equal to $N(1) - N(p_1{}^{\frac{1}{4}}) - N(p_2{}^{\frac{1}{4}}) + N(p_1{}^{\frac{1}{4}} p_2{}^{\frac{1}{4}}) = 100 - 75 - 80 + 65 = 10$. We may generate similar but more complicated practical examples where the principle of inclusion and exclusion are applied.

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