$(s,t]$-Fuzzy Graphs

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Abstract

The idea of a fuzzy point and its membership to and quasi-coincidence with a fuzzy subset with respect to the standard complement have been introduced to define and study certain kinds of fuzzy topological spaces and $(\epsilon, \epsilon \vee q)$-fuzzy subgroups, respectively. This was generalized by using arbitrary complements with an equilibrium. This helped motivate the notion of an $(s,t]$-fuzzy subgroup of a group. In this paper, we introduce these ideas to fuzzy graph theory.

Keywords: fuzzy subset, fuzzy graphs, fuzzy cut node, fuzzy end node

1 Introduction

The notion of a fuzzy subset of a set was introduced by Zadeh in [9]. Rosenfeld used this notion to introduce the concept of a fuzzy graph in [7]. In [5] and [1], the idea of a fuzzy point and its membership to and quasi-coincidence with a fuzzy subset with respect to the standard complement were used to define and study certain kinds of fuzzy topological spaces and $(\epsilon, \epsilon \vee q)$-fuzzy subgroups, respectively. In [2], an arbitrary complement was used. This helps motivate the notion of an $(s,t]$-fuzzy subgroup of a group introduced in [8]. In this paper, we introduce these ideas to fuzzy graph theory.

2 $(s,t]$-Graphs

Let X be a set and let λ be a fuzzy subset of X, i.e., a function of X into the interval $[0,1]$. Let $t \in [0,1]$. Then λ^t is defined to be the set $\{x \in X \mid \lambda(x) \geq t\}$. λ^t is called a level set or t-cut of λ. We let λ^* denote the support of λ, i. e., $\lambda^* = \{x \in X \mid \lambda(x) > 0\}$. Let λ and ρ be fuzzy subsets of X. Then we write $\lambda \subseteq \rho$ if and only if $\lambda(x) \leq \rho(x)$ for all $x \in X$. Let $x \in X$ and $t \in [0,1]$. Then the fuzzy singleton x_t is the fuzzy subset of X defined by $x_t(y) = t$ if $y = x$ and $x_t(y) = 0$ if $y \neq x$, where $x, y \in V$. Then $x_t \subseteq \lambda$ if and only if $\lambda(x) \geq t$. Let $c : [0,1] \to [0,1]$ be such that (1) $c(0) = 1$ and $c(1) = 0$; (2)$\forall a, b \in [0,1], a \leq b$ implies $c(a) \geq c(b)$. The fuzzy subset $c\lambda$ of X defined by $(c\lambda)(x) = c(\lambda(x)) \forall x \in X$ is called the **complement** of λ with respect to c. We assume throughout that c has an equilibrium, i. e., a point $e_c \in [0,1]$ such that $c(e_c) = e_c$. If e_c exists, it is unique. We can think of a fuzzy singleton x_t as quasily being in λ (with respect to c) if $x_t \nsubseteq c\lambda$, i.e., $t > (c\lambda)(x)$. Throughout, we use

$\vee$ to denote maximum and $\wedge$ to denote minimum.

A graph is a pair (V, E), where V is a finite set and E is a subset of the set of all subsets of V such that for all $X \in E, |X| = 2$, i. e., X has exactly 2 elements. We use the notation (x, y) to denote an element of E. We think of V as the set of vertices and E as the set of edges. A fuzzy graph $G = (\sigma, \mu)$ is a pair such that σ is a fuzzy subset of a finite set V and μ is a fuzzy subset of E such that for all $(x, y) \in E, \mu(x, y) \leq \sigma(x) \wedge \sigma(y)$. We let G^* denote (σ^*, μ^*). Clearly G^* is a graph.

Definition 1 *$G = (\sigma, \mu)$ is called a **quasi-fuzzy graph** with respect to c if $\forall(x, y) \in E$ and $\forall t \in [0, 1], (x, y)_t \subseteq \mu$ implies either $t \leq \sigma(x) \wedge \sigma(y)$ or $t > c(\sigma(x) \wedge \sigma(y))$.*

Theorem 2 *$G = (\sigma, \mu)$ is a quasi-fuzzy graph with respect to c if and only if $\forall x, y \in V, \mu(x, y) \wedge e_c \leq \sigma(x) \wedge \sigma(y)$.*

Proof. Suppose G is a quasi-fuzzy graph. Let $x, y \in V$. Suppose $(x, y)_{t_0} \subseteq \mu$. Then $\mu(x, y) \geq t_0$. Suppose $t_0 > \sigma(x) \wedge \sigma(y)$. Then by hypothesis, $t_0 > c(\sigma(x) \wedge \sigma(y))$. Thus $t_0 > e_c$. Suppose $\mu(x, y) \wedge e_c > \sigma(x) \wedge \sigma(y)$. (We show this leads to a contradiction.) Then $e_c > \sigma(x) \wedge \sigma(y)$. Hence $\sigma(x) \wedge \sigma(y) < e_c \leq c(\sigma(x) \wedge \sigma(y)) < t_0$ and $e_c = c(e_c) \leq c(\sigma(x) \wedge \sigma(y))$. Let t be such that $\sigma(x) \wedge \sigma(y) < t < e_c$. Then $t < t_0$ and so $\mu(x, y) \geq t$. Since $t > \sigma(x) \wedge \sigma(y)$, we have by an argument as above that $t > c(\sigma(x) \wedge \sigma(y))$ and $t > e_c$, a contradiction. Thus $\mu(x, y) \wedge e_c \leq \sigma(x) \wedge \sigma(y)$.

Conversely, suppose $\forall x, y \in V, \mu(x, y) \wedge e_c \leq \sigma(x) \wedge \sigma(y)$. Suppose $(x, y)_t \subseteq \mu$, i. e., $\mu(x, y) \geq t$. If $t \leq \sigma(x) \wedge \sigma(y)$, we have the desired result. Suppose $t > \sigma(x) \wedge \sigma(y)$. Then $\mu(x, y) > \sigma(x) \wedge \sigma(y)$. Since also $\mu(x, y) \wedge e_c \leq \sigma(x) \wedge \sigma(y), \sigma(x) \wedge \sigma(y) \geq e_c$. Hence $c(\sigma(x) \wedge \sigma(y)) \leq c(e_c) = e_c$. Thus $t > \sigma(x) \wedge \sigma(y) \geq e_c \geq c(\sigma(x) \wedge \sigma(y))$.

Given any $t \in [0, 1]$, there exists an infinite number of functions $c : [0, 1] \to [0, 1]$ with equilibrium t. This and Theorem 1.2 motivate for the following definition.

Definition 3 *$G = (\sigma, \mu)$ is called a **quasi-fuzzy graph** with respect to $t \in (0, 1]$ if $\forall(x, y) \in E, \mu(x, y) \wedge t \leq \sigma(x) \wedge \sigma(y)$.*

In fact, we consider the following definition.

Definition 4 *Let $s, t \in [0, 1]$ with $s < t$. Then $G = (\sigma, \mu)$ is called an $(s, t]$**-fuzzy graph** if G^* is a graph and $\forall x, y \in V, \mu(x, y) \wedge t \leq (\sigma(x) \wedge \sigma(y)) \vee s$.*

Let $G = (\sigma, \mu)$ be an $(s, t]$-fuzzy graph. If $s = 0$ and $t = 1$, then G is just a fuzzy graph. If $s = 0$, then G is a quasi-fuzzy graph with respect to t.

Let $G^r = (\sigma^r, \mu^r)\forall r \in [0, 1]$, where $\sigma^r$and μ^r are r-level sets. Clearly, G^r is a graph.

Theorem 5 *G is an $(s, t]$-fuzzy graph if and only if $\forall r \in (s, t], G^r$ is a subgraph of (V, E).*

Proof. Suppose G is a an $(s, t]$-fuzzy graph. Let $r \in (s, t]$. Suppose $(x, y) \in \mu^r$. Then $\mu(x, y) \geq r$. Hence $(\sigma(x) \wedge \sigma(y)) \vee s \geq \mu(x, y) \wedge t \geq r \wedge t = r$. Since $r > s$, we have $\sigma(x) \wedge \sigma(y) \geq r$. Thus $\sigma(x) \geq r$ and $\sigma(y) \geq r$. Hence $x, y \in \sigma^r$. Thus G^r is a subgraph of (V, E).

Conversely, suppose G^r is a subgraph of $(V, E)\forall r \in (s, t]$. Let $(x, y) \in E$ and let $\mu(x, y) = r_0$. Suppose $r_0 \leq s$. Then $\mu(x, y) \wedge t = \mu(x, y) = r_0 \leq s \leq (\sigma(x) \wedge \sigma(y)) \vee s$. Suppose $s < r_0 \leq t$. Then since G^{r_0} is a subgraph of (V, E), we have $x, y \in \sigma^{r_0}$. Hence $\sigma(x) \wedge \sigma(y) \geq r_0 = \mu(x, y)$. Thus $(\sigma(x) \wedge \sigma(y)) \vee s \geq \mu(x, y) \wedge t$. Suppose $r_0 > t$. Then $\mu(x, y) \wedge t = t$. Since $\mu(x, y) > t$, we have $(x, y) \in \mu^t$. Since G^t is a subgraph of (V, E), we have $x, y \in \sigma^t$. Thus $\sigma(x) \wedge \sigma(y) \geq t$. Hence $\mu(x, y) \wedge t \leq (\sigma(x) \wedge \sigma(y)) \vee s$. Thus G is an $(s, t]$-fuzzy graph.

Let $r, s, t \in [0, 1]$. Suppose $s < t$. Let X be a set and λ a fuzzy subset of X. Let $\lambda^r_{st} = \{x \in X \mid \lambda(x) \vee s \geq r \wedge t\}$. Then λ^r_{st} is called an r_{st}-level set (or cut) of λ. Let $G^r_{st} = (\sigma^r_{st}, \mu^r_{st})$.

Theorem 6 *G is an $(s, t]$-fuzzy graph if and only if $\forall r \in [0, 1], G^r_{st}$ is a subgraph of (V, E).*

Proof. Suppose G is an $(s, t]$-fuzzy graph. Let $r \in [0, 1]$. Suppose $(x, y) \in \mu^r_{st}$. Then $\mu(x, y) \vee s \geq$

$r \wedge t$. Now $(\sigma(x) \wedge \sigma(y)) \vee s \geq \mu(x,y) \wedge t$ and so $(\sigma(x) \wedge \sigma(y)) \vee s \geq (\mu(x,y) \wedge t) \vee s = (\mu(x,y) \vee s) \wedge (t \vee s) \geq (r \wedge t) \wedge t = r \wedge t$. Thus $\sigma(x) \vee s \geq r \wedge t$ and $\sigma(y) \vee s \geq r \wedge t$. Hence $x, y \in \sigma^r_{st}$. Thus $(\sigma^r_{st}, \mu^r_{st})$ is a subgraph of (V, E).

Conversely, suppose $(\sigma^r_{st}, \mu^r_{st})$ is a subgraph of (V, E) $\forall r \in [0,1]$. Let $(x, y) \in E$ and let $\mu(x,t) = r_0$. Then $\mu(x,y) \vee s \geq r_0 \wedge t$. Hence $(x,y) \in \mu^{r_0}_{st}$. Thus $x, y \in \sigma^{r_0}_{st}$. Hence $\sigma(x) \vee s \geq r_0 \wedge t$ and $\sigma(y) \vee s \geq r_0 \wedge t$. Thus $(\sigma(x) \wedge \sigma(y)) \vee s \geq r_0 \wedge t = \mu(x,y) \wedge t$. Hence G is an $(s,t]$-fuzzy graph.

3 $(s,t]$-Fuzzy Cut Vertices and $(s,t]$ -Fuzzy End Vertices

Let $G = (\sigma, \mu)$ be a fuzzy graph. Let $x, y \in V$. A path from x to y in G is a sequence $\pi : x = x_0, x_1, ..., x_n = y$ of points x_i in V such that $\sigma(x_i) > 0$ for $i = 0, 1, ..., n$ and $\mu(x_{i-1}, x_i) > 0$ for $i = 1, ..., n$. The **strength** of the path π is $\wedge\{\mu(x_{i-1}, x_i) \mid i = 1, ..., n\}$. Define $\mathrm{CONN}_G(x, y)$ to be the maximum of the strengths of all paths from x to y. If $x \in V$, we use the notation $G \backslash x$ to denote the fuzzy graph determined by setting $\sigma(x) = 0$ and $\mu(x, y) = 0$ for all $y \in V$ such that $(x, y) \in E$. Properties of fuzzy cut vertices and fuzzy end vertices can be found in [6] and [4], respectively.

Definition 7 *Let $G = (\sigma, \mu)$ be a fuzzy graph and let $x \in V$. Then x is called a **fuzzy cut vertex** if there exists $y, z \in V \backslash \{x\}$ such that $CONN_G(y, z) > CONN_{G \backslash x}(y, z)$.*

Definition 8 *Let $G = (\sigma, \mu)$ be a fuzzy graph and let $x \in V$. Then x is called a **fuzzy end vertex** if there exists $y \in V \backslash \{x\}$ such that $\mu(x, y) > \vee\{\mu(x, z) \mid z \in V, z \neq y\}$.*

Definition 9 *Let $s, t \in [0, 1]$ be such that $s < t$. Let $G = (\sigma, \mu)$ be a fuzzy graph and let $x \in V$. Then x is called a $(s,t]$**-fuzzy cut vertex** if there exists $y, z \in V \backslash \{x\}$ such that $CONN_G(y, z) \wedge t > CONN_{G \backslash x}(y, z) \vee s$.*

Definition 10 *Let $s, t \in [0, 1]$ be such that $s < t$. Let $G = (\sigma, \mu)$ be a fuzzy graph and let $x \in V$. Then x is called an $(s,t]$**-fuzzy end vertex** if there exists $y \in V \backslash \{x\}$ such that $\mu(x, y) \wedge t > \vee\{\mu(x, z) \mid z \in V, z \neq y\} \vee s$.*

If $s = 0$ and $t = 1$, then $(0, 1]$-fuzzy end vertices and $(0, 1]$-fuzzy cut vertices are simply fuzzy end vertices and fuzzy cut vertices of G, respectively.

Suppose that $x \in V$ is an $(s,t]$-fuzzy end vertex. Then the $y \in V$ such that $\mu(x, y) \wedge t > \vee\{\mu(x, z) \mid z \in V, z \neq y\} \vee s$ is unique. This follows since for all $u \in V \backslash \{y\}, \mu(x, y) \geq \mu(x, y) \wedge t > \mu(x, u)$. Also, if x is an $(s,t]$-fuzzy end vertex, then x is also an $(r,w]$-fuzzy vertex for all $r \in [0, s]$ and $w \in [t, 1]$.

Example 11 *Let $V = \{x, y, z\}$ and $E = \{(x, y), (x, z), (y, z)\}$. Let $s = .05$ and $t = .95$. Define the fuzzy subset σ of V and the fuzzy subset μ of E as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(x, y) = \mu(y, z) = 1$ and $\mu(x, z) = .99$. Then x is a fuzzy end vertex, but not an $(s,t]$-fuzzy end vertex. We also have that y is a fuzzy cut vertex, but not an $(s,t]$-fuzzy cut vertex.*

Example 12 *Let $V = \{x, y, z\}$ and $E = \{(x, y), (x, z), (y, z)\}$. Let $s = .05$ and $t = .95$. Define the fuzzy subset σ of V and the fuzzy subset μ of E as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(x, y) = \mu(y, z) = .04$ and $\mu(x, z) = .01$. Then x is a fuzzy end vertex, but not an $(s,t]$-fuzzy end vertex. We also have that y is a fuzzy cut vertex, but not an $(s,t]$-fuzzy cut vertex.*

Remark 13 *Let $x \in V$. If x is an end vertex in (V, E), then x is a fuzzy end vertex and an $(s,t]$-fuzzy end vertex for all $t \in (0, 1]$ and for all $s < t \leq \mu(x, y)$, where y is the unique neighbor of x in G^*.*

Definition 14 *Let $x, y \in V$. Then y is called a neighbor of x if $\mu(x, y) > 0$. y is called a t-strong neighbor of x if y is a neighbor of x and $\mu(x, t) \wedge t \geq CONN_G(x, y) \wedge t$.*

Theorem 15 *Let $z \in V$ be an $(s,t]$-fuzzy cut vertex of G. Then z has at least two t-strong neighbors.*

Proof. Since z is an $(s,t]$-fuzzy cut vertex of G, there exists $x, y \in V \backslash \{z\}$ such that $\mathrm{CONN}_G(x, y) \wedge$

$t > \mathrm{CONN}_{G-z}(x, y) \vee s$. Hence there exists a path P from x to y and passing through z such that $s(P) \wedge t = \mathrm{CONN}_G(x, y) \wedge t$. Since $x \neq z \neq y, z$ has two neighbors in P, say u and v. Since (u, z) and (z, v) are on $P, \mu(u, z) \wedge t \geq s(P) \wedge t$ and $\mu(z, v) \wedge t \geq s(P) \wedge t$. Suppose $\mu(u, z) \wedge t < \mathrm{CONN}_G(u, z) \wedge t$. Then there exists a path Q from u to z such that $s(Q) \wedge t > \mu(u, z) \wedge t$. Let u' in Q be a neighbor of z. We show $u' \neq v$. Suppose $u' = v$. Let Q' be the path $x...u...u'...v...y$. Then $s(Q') \wedge t \geq s(P) \wedge t$, but Q' does not pass through z. However this is impossible since $\mathrm{CONN}_{G-z}(x, y) \wedge t < \mathrm{CONN}_G(x, y) \wedge t$. Let P' denote the path $x...u...u'zv...y$. Then $s(P') \wedge t \geq s(P) \wedge t = \mathrm{CONN}_G(x, y) \wedge t$. Suppose $\mu(u', z) \wedge t < \mathrm{CONN}_G(u, z) \wedge t$. Then we can apply the above process again. Now length(P') > length(P). Hence this process must end in a finite number of steps yielding a neighbor u^* of z such that $\mu(u^*, z) \wedge t \geq \mathrm{CONN}_G(u, z) \wedge t$. As above $u^* \neq v$. Hence a similar argument can be used to yield a neighbor v^* of z such that $u^* \neq v^*$ and $\mu(v^*, z) \wedge t \geq \mathrm{CONN}_G(u, z) \wedge t$. Thus z has two t-strong neighbors, namely u^* and v^*.

Definition 16 [3] *Let $x \in V$. Then x is called a **weak fuzzy end node** if there exists $t \in (0, h(\mu)]$ such that x is an end node in G^t, where $h(\mu)$ is the height of μ, i. e., $h(\mu) = \vee\{\mu(x, y)|(x, y) \in E\}$.*

Definition 17 [3] *Let $x \in V$. Then x is called a **partial fuzzy end node** if x is an end node in G^t for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$, where $d(\mu)$ is the depth of μ, i. e., $d(\mu) = \wedge\{\mu(x, y)|(x, y) \in E\}$.*

Proposition 18 *Let $x \in V$. Then x is an $(0, 1]$-fuzzy end vertex if and only if x is a weak fuzzy end vertex.*

Proof. . Suppose x is an $(0, 1]$-fuzzy end vertex. Then there exists $y \in V$ such that $\mu(x, y) = \mu(x, y) \wedge 1 > \vee\{\mu(x, z)|z \in V, z \neq y\}$. Thus x is an end vertex in G^a, where $a = \mu(x, y)$. Hence x is a weak fuzzy end vertex. Conversely, suppose x is a weak fuzzy end vertex. Then x is an end node in G^b for some $b \in (0, h(\mu)]$. Hence there exists a unique $y \in V$ such that $\mu(x, y) > b$ and for all other $z \in V, \mu(x, z) < b$. Thus x is an $(0, 1]$-fuzzy end vertex.

Corollary 19 *Let $x \in V$. Then x is a partial fuzzy end vertex if and only if x is a $(d(\mu), 1]$-fuzzy end vertex.*

Corollary 20 *Let $x \in V$. Then x is an end vertex of G^* if and only if x is a $(0, d(\mu)]$-fuzzy end vertex.*

References:

[1] S. K. Bhakat and P. S. Das, On the definition of a fuzzy subgroup, *Fuzzy Sets and Systems*, 51 (1992) 235-241.

[2] K. R. Bhutani, R. L. Crist, and J. N. Mordeson, Free $(s, t]$-fuzzy subgroups, submitted for publication.

[3] K. R. Bhutani, J. N. Mordeson, and A. Rosenfeld, On degress of end nodes and cut nodes in fuzzy graphs, *Iranian J. Fuzzy Systems* (2004) 57-64.

[4] K. R. Bhutani and A. Rosenfeld, Fuzzy end nodes in fuzzy graphs, *Inf. Sci.* 152 (2003) 323-326.

[5] P. P. Ming and L. Y. Ming, Fuzzy topology I: Neighbourhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.* 76 (1980) 571-599.

[6] J. N. Mordeson and P. S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica-Verlag, Heidleberg, 2000.

[7] A. Rosenfeld, Fuzzy graphs, In: L. A. Zadeh, K. S. Fu, M. Shimura Eds. Fuzzy Sets and Their Applications, 77-95, Academic Press, 1975.

[8] Xuechai Yuan, Cheng Zheng, Yonghong Ren, $(\overline{\beta}, \overline{\alpha})$-fuzzy subgroup, R-fuzzy subgroup, and $(\lambda, \mu]$-fuzzy subgroup, *Fuzzy Sets and Systems*, to appear.

[9] L. A. Zadeh, Fuzzy sets, *Inform. Control* 8 (1965) 338-353.