An Extended *K*th-Best Approach for Fuzzy Linear Bilevel Problems

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Abstract:
Organizational bilevel decision-making often involves uncertain factors. The parameters shown in a bilevel programming model, either in the objective functions or constraints, are thus often imprecise, which is called the fuzzy parameter bilevel programming (FPBLP) problem. Following our previous work, this study first proposes a model of FPBLP, then gives the *K*th-best approach to solving the proposed FPBLP problem.

Keywords: Linear bilevel programming, *K*th-best approach, Fuzzy set, Optimization

1. Introduction

The bi-level programming (BLP) problem, introduced by Von Stackelberg [8] in the context of unbalanced economic markets, is a hierarchical optimization problem where a subset of the variables are constrained to be a solution of a given optimization problem parameterized by the remaining variables [1, 2]. Existing bilevel programming solving approaches mainly suppose the situation in which the objective functions and constraints are characterized with precise parameters. It has been observed that, in most real-world situations, particularly in critical resource planning, such as planning of land-use, transportation and water resources, the possible values of these parameters are often only imprecisely or ambiguously known to the experts. This results in a difficulty to use parameters in the objective functions or constraints of a bilevel programming model. With this observation, it would be certainly more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy sets of the real line known as fuzzy set theory [9]. A bilevel programming problem in which the parameters either in the objective function or in the constraints are fuzzy is called a fuzzy parameter bilevel programming (FPBLP) problem in this study.

The FPBLP problem was first researched by Sakawa et al. in 2000 [4]. Sakawa et al. formulate bilevel programming problems with fuzzy parameters from the perspective of experts' imprecision and proposes a fuzzy programming method for fuzzy bilevel programming problems. However, Sakawa's work is mainly based on the definition of a solution for bilevel programming proposed by Bard [2,3]. One deficiency of Bard's linear BLP theory is that it could not well solve a linear bilevel programming problem when the upper-level constraint functions are of arbitrary linear form. Our recent research work has extended Bard's theory of bilevel programming by proposing a new definition of the solution for linear bilevel programming which can overcome the arbitrary linear form problem indicated above [5]. We then proposed an extended Kuhn-Tucher and extended *K*th-best approach, based on our definition of the solution, for solving linear bilevel problems [6,7].

This study will follow our previous research results shown in [5-7, 12] and develops a fuzzy number-based extended *K*th-best approach to solve the proposed FPBLP problems.

Following the introduction, Section 2 presents the definition of a solution and a fuzzy number based extended *K*th-best approach for solving FPBLP problems. Conclusion and further study are discussed in Section 3. In this paper, we present some basic concepts, definitions and theorems that can also be found from our recent paper in [10-12].

2. Extended *K*th-Best Approach for Fuzzy Parameter Linear Bilevel Programming Problem

Consider the following fuzzy parameter linear bilevel programming (FPBLP) problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F : X \times Y \to F^*(R)$, and $f : X \times Y \to F^*(R)$,

$$\min_{x\in X} F(x,y)=\widetilde{c}_1 x+\widetilde{d}_1 y \tag{2.1a}$$

$$\text{subject to } \widetilde{A}_1 x+\widetilde{B}_1 y \precsim \widetilde{b}_1 \tag{2.1b}$$

$$\min_{y\in Y} f(x,y)=\widetilde{c}_2 x+\widetilde{d}_2 y \tag{2.1c}$$

$$\text{subject to } \widetilde{A}_2 x+\widetilde{B}_2 y \precsim \widetilde{b}_2 \tag{2.1d}$$

where $\widetilde{c}_1, \widetilde{c}_2 \in F^*(R^n)$, $\widetilde{d}_1, \widetilde{d}_2 \in F^*(R^m)$, $\widetilde{b}_1 \in F^*(R^p)$, $\widetilde{b}_2 \in F^*(R^q)$, $\widetilde{A}_1=(\widetilde{a}_{ij})_{p\times n}, \widetilde{a}_{ij}\in F^*(R)$, $\widetilde{B}_1=(\widetilde{b}_{ij})_{p\times m}$, $\widetilde{b}_{ij}\in F^*(R)$, $\widetilde{A}_2=(\widetilde{e}_{ij})_{q\times n}, \widetilde{e}_{ij}\in F^*(R)$, $\widetilde{B}_2=(\widetilde{s}_{ij})_{q\times m}$, $\widetilde{s}_{ij}\in F^*(R)$.

Associated with the FPBLP problem, we now consider the following linear multi-objective multi-follower bilevel programming (LMMBLP) problem:
For $x\in X\subset R^n$, $y\in Y\subset R^m$, $F: X\times Y\to F^*(R)$, and $f: X\times Y\to F^*(R)$,

$$\begin{aligned}&\min_{x\in X}\left(F(x,y)\right)_\lambda^L = c_{1\lambda}^L x + d_{1\lambda}^L y, \quad \lambda\in[0,1]\\ &\min_{x\in X}\left(F(x,y)\right)_\lambda^R = c_{1\lambda}^R x + d_{1\lambda}^R y, \quad \lambda\in[0,1]\end{aligned} \tag{2.2a}$$

subject to

$$A_{1\lambda}^L x+B_{1\lambda}^L y \leqq b_{1\lambda}^L,\ A_{1\lambda}^R x+B_{1\lambda}^R y\leqq b_{1\lambda}^R, \lambda\in[0,1] \tag{2.2b}$$

$$\begin{aligned}&\min_{y\in Y}\left(f(x,y)\right)_\lambda^L = c_{2\lambda}^L x + d_{2\lambda}^L y, \quad \lambda\in[0,1]\\ &\min_{y\in Y}\left(f(x,y)\right)_\lambda^R = c_{2\lambda}^R x + d_{2\lambda}^R y, \quad \lambda\in[0,1]\end{aligned} \tag{2.2c}$$

subject to

$$A_{2\lambda}^L x+B_{2\lambda}^L y \leqq b_{2\lambda}^L,\ A_{2\lambda}^R x+B_{2\lambda}^R y\leqq b_{2\lambda}^R, \lambda\in[0,1] \tag{2.2d}$$

where $c_{1\lambda}^L, c_{1\lambda}^R, c_{2\lambda}^L, c_{2\lambda}^R \in R^n$, $d_{1\lambda}^L, d_{1\lambda}^R$, $d_{2\lambda}^L, d_{2\lambda}^R \in R^m$, $b_{1\lambda}^L, b_{1\lambda}^R\in R^p$, , $b_{2\lambda}^L, b_{2\lambda}^R\in R^q$,

$$A_{1\lambda}^L=\left(a_{ij\lambda}^L\right), A_{1\lambda}^R=\left(a_{ij\lambda}^R\right)\in R^{p\times n},$$
$$B_{1\lambda}^L=\left(b_{ij\lambda}^L\right), B_{1\lambda}^R=\left(b_{ij\lambda}^R\right)\in R^{p\times m},$$
$$A_{2\lambda}^L=\left(e_{ij\lambda}^L\right), A_{2\lambda}^R=\left(e_{ij\lambda}^R\right)\in R^{q\times n},$$
$$B_{2\lambda}^L=\left(s_{ij\lambda}^L\right), B_{2\lambda}^R=\left(s_{ij\lambda}^R\right)\in R^{q\times m}.$$

Theorem 2.1 [12] *Let* (x^*, y^*) *be the solution of the LMMBLP problem* (2.2). *Then it is also a solution of the FPBLP problem defined by* (2.1).

Lemma 2.1 [12] *If there is* (x^*, y^*) *such that* $cx+dy\geqq cx^*+dy^*$, $c_0^L x+d_0^L y\geqq c_0^L x^*+d_0^L y^*$ *and* $c_0^R x+d_0^R y\geqq c_0^R x^*+d_0^R y^*$, *for any* (x, y) *and isosceles triangle fuzzy numbers* $\widetilde{c}$ *and* $\widetilde{d}$, *then*
$c_\lambda^L x+d_\lambda^L y\geqq c_\lambda^L x^*+d_\lambda^L y^*$,
$c_\lambda^R x+d_\lambda^R y\geqq c_\lambda^R x^*+d_\lambda^R y^*$,
for any $\lambda\in(0,1)$, *where c and d are the centre of* $\widetilde{c}$ *and* $\widetilde{d}$ *respectively.*

Theorem 2.2 [12] *For* $x\in X\subset R^n$, $y\in Y\subset R^m$, *If all the fuzzy coefficients* $\widetilde{a}_{ij}, \widetilde{b}_{ij}, \widetilde{e}_{ij}, \widetilde{s}_{ij}, \widetilde{c}_i$ *and* $\widetilde{d}_i$ *have triangle membership functions of the FPBLP problem* (2.1).

$$\mu_{\widetilde{z}}(t)=\begin{cases}0 & t<z_0^L\\ \dfrac{t-z_0^L}{z-z_0^L} & z_0^L\leqq t<z\\ \dfrac{-t+z_0^R}{z_0^R-z} & z\leqq t<z_0^R\\ 0 & z_0^R\leqq t\end{cases}, \tag{2.3}$$

where $\widetilde{z}$ *denotes* $\widetilde{a}_{ij}, \widetilde{b}_{ij}, \widetilde{e}_{ij}, \widetilde{s}_{ij}, \widetilde{c}_i$ *and* $\widetilde{d}_i$ *and z are the centre of* $\widetilde{z}$ *respectively. Then, it is the solution of the problem* (2.1) *that* $(x^*, y^*)\in R^n\times R^m$ *satisfying*

$$\begin{aligned}&\min_{x\in X}\left(F(x,y)\right)_c = c_1 x+d_1 y,\\ &\min_{x\in X}\left(F(x,y)\right)_0^L = c_{10}^L x+d_{10}^L y,\\ &\min_{x\in X}\left(F(x,y)\right)_0^R = c_{10}^R x+d_{10}^R y,\end{aligned} \tag{2.4a}$$

$$\begin{aligned}\text{subject to } &A_1 x+B_1 y\leqq b_1,\\ &A_{10}^L x+B_{10}^L y\leqq b_{10}^L,\\ &A_{10}^R x+B_{10}^R y\leqq b_{10}^R,\end{aligned} \tag{2.4b}$$

$$\begin{aligned}&\min_{y\in Y}\left(f(x,y)\right)_c = c_2 x+d_2 y,\\ &\min_{y\in Y}\left(f(x,y)\right)_0^L = c_{20}^L x+d_{20}^L y,\\ &\min_{y\in Y}\left(f(x,y)\right)_\lambda^R = c_{20}^R x+d_{20}^R y,\end{aligned} \tag{2.4c}$$

$$\begin{aligned}\text{subject to } &A_2 x+B_2 y\leqq b_2,\\ &A_{20}^L x+B_{20}^L y\leqq b_{20}^L,\\ &A_{20}^R x+B_{20}^R y\leqq b_{20}^R.\end{aligned} \tag{2.4d}$$

We note

$$\overline{A}_1 x+\overline{B}_1 y \leqq \overline{b}_1 \tag{2.4b'}$$

$$\overline{A}_2 x+\overline{B}_2 y \leqq \overline{b}_2 \tag{2.4d'}$$

where

$$\overline{A}_1 = \begin{pmatrix} A_1 \\ A_{10}^L \\ A_{10}^R \end{pmatrix}, \overline{A}_2 = \begin{pmatrix} A_2 \\ A_{20}^L \\ A_{20}^R \end{pmatrix}, \overline{B}_1 = \begin{pmatrix} B_1 \\ B_{10}^L \\ B_{10}^R \end{pmatrix}, \overline{B}_2 = \begin{pmatrix} B_2 \\ B_{20}^L \\ B_{20}^R \end{pmatrix},$$

$$\overline{b}_1 = \begin{pmatrix} b_1 \\ b_{10}^L \\ b_{10}^R \end{pmatrix}, \overline{b}_2 = \begin{pmatrix} b_2 \\ b_{20}^L \\ b_{20}^R \end{pmatrix}.$$

Then we can re-write (2.4) by using

$$\min_{x\in X} \left(F(x,y)\right)_c = c_1 x + d_1 y,$$
$$\min_{x\in X} \left(F(x,y)\right)_0^L = c_{10}^L x + d_{10}^L y, \qquad (2.4a')$$
$$\min_{x\in X} \left(F(x,y)\right)_0^R = c_{10}^R x + d_{10}^R y,$$

subject to $\overline{A}_1 x + \overline{B}_1 y \leqq \overline{b}_1$, (2.4b')

$$\min_{y\in Y} \left(f(x,y)\right)_c = c_2 x + d_2 y,$$
$$\min_{y\in Y} \left(f(x,y)\right)_0^L = c_{20}^L x + d_{20}^L y, \qquad (2.4c')$$
$$\min_{y\in Y} \left(f(x,y)\right)_\lambda^R = c_{20}^R x + d_{20}^R y,$$

subject to $\overline{A}_2 x + \overline{B}_2 y \leqq \overline{b}_2$, (2.4d')

To solve the problem (2.4'), we can use the method of weighting to this problem, such that it is the following problem:

$$\min_{x\in X}\left(F(x,y)\right) = \left(c_1 x + d_1 y\right) + \left(c_{10}^L x + d_{10}^L y\right) + \left(c_{10}^R x + d_{10}^R y\right) \qquad (2.5a)$$

subject to $A_1 x + B_1 y \leqq b_1$,

$$A_{10}^L x + B_{10}^L y \leqq b_{10}^L, \qquad (2.5b)$$
$$A_{10}^R x + B_{10}^R y \leqq b_{10}^R,$$

$$\min_{y\in Y}\left(f(x,y)\right) = \left(c_2 x + d_2 y\right) + \left(c_{20}^L x + d_{20}^L y\right) + \left(c_{20}^R x + d_{20}^R y\right) \qquad (2.5c)$$

subject to $A_2 x + B_2 y \leqq b_2$,

$$A_{20}^L x + B_{20}^L y \leqq b_{20}^L, \qquad (2.5d)$$
$$A_{20}^R x + B_{20}^R y \leqq b_{20}^R.$$

Definition 2.1

(a) Constraint region of the linear BLP problem:

$S = \{(x,y) : x \in X, y \in Y, \overline{A}_1 x + \overline{B}_1 y \leqq \overline{b}_1, \overline{A}_2 x + \overline{B}_2 y \leqq \overline{b}_2\}$

Feasible set for the follower for each fixed $x \in X$:

$S(x) = \{y \in Y : \overline{B}_2 y \leqq \overline{b}_2 - \overline{A}_2 x\}$

(c) Projection of S onto the leader's decision space:

$S(X) = \{x \in X : \exists y \in Y, \overline{A}_1 x + \overline{B}_1 y \leqq \overline{b}_1, \overline{A}_2 x + \overline{B}_2 y \leqq \overline{b}_2\}$

Follower's rational reaction set for $x \in S(X)$:

$P(x) = \{y \in Y : y \in \arg\min[\left(f(x,\hat{y})\right) : \hat{y} \in S(x)]\}$

where $\arg\min[f(x,\hat{y}) : \hat{y} \in S(x)]$

$= \{y \in S(x) : \left(f(x,y)\right) \leqq \left(f(x,\hat{y})\right), \hat{y} \in S(x)\}$

Inducible region:

$IR = \{(x,y) : (x,y) \in S, y \in P(x)\}$

The rational reaction set $P(x)$ defines the response while the inducible region IR represents the set over which the leader may optimize his objective. Thus in terms of the above notations, the linear BLP problem can be written as

$$\min\{F(x,y) : (x,y) \in IR\}. \qquad (2.6)$$

Theorem 2.3 *The inducible region can be written equivalently as piecewise linear equality constraint comprised of supporting hyperplance of constraint region S.*

Corollary 2.1 *The linear BLP problem* (2.5) *is equivalent to minimizing F over a feasible region comprised of a piecewise linear equality constraint.*

Corollary 2.2 *A solution for the linear BLP problem occurs at a vertex of IR.*

Theorem 2.3 *The solution* (x^*, y^*) *of the linear BLP problem occurs at a vertex of S.*

Corollary 2.3 *If* x *is an extreme point of* IR *, it is an extreme point of* S .

Theorem 2.3 and Corollary 2.3 have provided theoretical foundation for our new algorithm. It means that by searching extreme points on the constraint region S , we can efficiently find an optimal solution for a linear BLP problem. The basic idea of our extended properties algorithm is that according to the objective function of the upper level, we descendent order all the extreme points on S , and select the first extreme point to check if it is on the inducible region IR . If yes, the current extreme point is the optimal solution. If not, select the next one and check.

More specifically, let $(x_{[1]}, y_{[1]}), \ldots, (x_{[N]}, y_{[N]})$ denote the N ordered extreme points to the linear programming problem

$$\min\{\overline{c}_1 x + \overline{d}_1 y : (x,y) \in S\}, \qquad (2.6)$$

such that

$\overline{c}_1 x_{[i]} + \overline{d}_1 y_{[i]} \leqq \overline{c}_1 x_{[i+1]} + \overline{d}_1 y_{[i+1]}, i = 1, \ldots, N-1.$

Let $\widetilde{y}$ denote the optimal solution to the following problem

$$\min(f(x_{[i]}, y) : y \in S(x_{[i]})). \qquad (2.7)$$

We only need to find the smallest i ($i \in \{1, \ldots, N\}$) under which $y_{[i]} = \tilde{y}$.

Let write (2.7) as follows

$$\min f(x, y)$$
$$\text{subject to } y \in S(x)$$
$$x = x_{[i]}.$$

From Definition 2.1(a) and (c), we have

$$\min f(x, y) = \bar{c}_2 x + \bar{d}_2 y \quad (2.8a)$$
$$\text{subject to } \bar{A}_1 x + \bar{B}_1 y \underline{\prec} \bar{b}_1 \quad (2.8b)$$
$$\bar{A}_2 x + \bar{B}_2 y \underline{\prec} \bar{b}_2 \quad (2.8c)$$
$$x = x_{[i]} \quad (2.8d)$$
$$y \underline{\succ} 0. \quad (2.8e)$$

The solving is equivalent to selecting one ordered extreme point $(x_{[i]}, y_{[i]})$, then solving (2.8) to obtain the optimal solution $\tilde{y}$. If $\tilde{y} = y_{[i]}$, $(x_{[i]}, y_{[i]})$ is the global optimum to (2.5). Otherwise, check the next extreme point.

Based on these definitions and theorems, an extended kth-best approach for FPBLP problems is described as follows:

Step 1 Put $i \leftarrow 1$. Solve (2.6) with the simplex method to obtain the optimal solution $(x_{[1]}, y_{[1]})$. Let $W = \{(x_{[1]}, y_{[1]})\}$ and $T = \phi$. Go to Step 2.

Step 2 Solve (2.8) with the bounded simplex method. Let $\tilde{y}$ denote the optimal solution to (2.8). If $\tilde{y} = y_{[i]}$, stop; $(x_{[i]}, y_{[i]})$ is the global optimum to (2.8) with $K^* = i$. Otherwise, go to Step 3.

Step 3 Let $W_{[i]}$ denote the set of adjacent extreme points of $(x_{[i]}, y_{[i]})$ such that $(x, y) \in W_{[i]}$ implies $\bar{c}_1 x + \bar{d}_1 y \underline{\succ} \bar{c}_1 x_{[i]} + \bar{d}_1 y_{[i]}$. Let $T = T \cup \{(x_{[i]}, y_{[i]})\}$ and $W = (W \cup W_{[i]}) \setminus T$. Go to Step 4.

Step 4 Set $i \leftarrow i + 1$ and choose $(x_{[i]}, y_{[i]})$ so that $f x_{[i]} + g y_{[i]} = \min\{\bar{c}_1 x + \bar{d}_1 y : (x, y) \in W\}$. Go to Step 2.

3. Conclusion and Further Study

Following our previous research [5-7, 12], this paper proposes a fuzzy number based extended *K*th-best approach to solve the proposed FPBLP problem. Further study will include the development of fuzzy parameter based multi-follower and multi-objective bilevel programming problems.

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