Negations, Strict Monotonic t-norms and t-conorms for Finite Ordinal Scales

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Abstract

It is impossible to introduce strict monotonic t-norms and t-conorms on finite scales, e.g. on ordinal scales of linguistic evaluations of true values. Such operations are introduced in this work as result of embedding of initial scale in the set of multi-sets. These multi-sets are considered as memory for storing the results of operations. Several types of such memory are considered. On each of them, negation operations are also introduced. Possible applications of proposed technique in expert systems with qualitative plausibility evaluations are discussed.

Keywords: fuzzy logic, t-norm, negation, ordinal scale, expert system

1. Introduction

Finite ordinal scales arise, e.g., when true, possibility, plausibility, certainty or membership values are measured (evaluated) by man. The grades of such scales are linearly ordered and usually given linguistically, e.g. it may be considered the scale X_w= {*Impossible* < *Very Small Possibility* < *Small Possibility* < *Average Possibility* < *Large Possibility* < *Very Large Possibility* < *Sure*} or in abbreviated form: X_w= {*IMP* < *VSP* < *SP* < *AP* < *LP* < *VLP* < *SURE*}. Finite ordinal scales are often represented by numbers, e.g. the grades of the scale X_w may be represented as X_s= {0, 1, 2, 3, 4, 5, 6} or X_f= {0, 0.2, 0.4, 0.5, 0.6, 0.8, 1} where the ordering of numbers retains the ordering of linguistic grades. On such numeric scales the quantitative operations like multiplication and addition can not be used and only *min* and *max* operations of fuzzy logic [9] are adequate operations on ordinal scales. It is clear that *min* and *max* operations can be introduced directly on linguistic scale if the linear ordering on such scale in given.

Denote X_p ={x_0, x_1,..., x_{n+1}}, (n≥1), a finite ordinal scale such that $x_i < x_j$ for all $i < j$. The elements x_0 and x_{n+1} will be denoted as **0** and **1**. The operations *min* and *max* can be used, correspondingly, as conjunction $\wedge$ and disjunction $\vee$ operations on such scale. They satisfy on X_p axioms of *t*-norm and *t*-conorm [7], i.e. they are associative, commutative and satisfy the following boundary and monotonicity conditions:

$A\wedge 1 = A, \qquad A\vee 0 = A,$

$A\wedge B \le C\wedge D,$ and $A\vee B \le C\vee D,$ if $A \le C, B \le D.$

But *min* and *max* do not strict monotonic, i.e. they do not satisfy on X_p the following properties:

$A\wedge B < A\wedge C$ if $B<C$ and $A\neq$ **0,**

$A\vee B < A\vee C$ if $B<C$ and $A\neq$ **1.**

As result, we have $A\wedge B = A$ for all $A \le B$ and $A\vee B = A$ for all $B \le A,$ i.e. the results of operations do not changed if B is changing in some range defined by A. Non-strict monotonicity of *min* and *max* does not very important when these operations are used as a basis of intersection and union operations for fuzzy sets, moreover *min* an *max* show often their advantage in comparison with other fuzzy operations in many applications of fuzzy logic. But the strict monotonicity of considered operations is important, for example, for inference procedures of expert systems containing uncertainty values of facts and rules [1,3]. Suppose we have two rules:

If A and B then G,

If A and C then H.

It is quite reasonable to expect that if true, plausibility value of C is greater than corresponding value of B then the true, plausibility value of conclusion H should be grater than for conclusion G. Such supposition requires a strict monotonicity of conjunction operation used as connective *and*.

Operations of fuzzy logic on finite scales were considered, e.g., in [6,8]. It is clear, that it is impossible to introduce strict monotonic conjunction and disjunction operations on finite scales. The solution of this problem was proposed in [1-3] where the initial finite scale was embedded in the set of ordered strings of elements from initial scale. These strings called lexicographic valuations can be considered as memory for storing the true, plausibility values from initial finite scale. For example the result of conjunction VSP∧VLP∧LP∧VSP for min-lexicographic valuations will be presented as follows: (VSP, VSP, LP, VLP). Three classes of such lexicographic valuations were considered [1]. In two of these classes of strings only one of conjunction ($\wedge$) or disjunction ($\vee$) operations was strict monotonic. In the third, more complicated, class of "strings of strings" it was possible to introduce both strict monotonic operations. It was shown that in all of these three classes the introduced operations satisfy axioms of *t*-norms and *t*-conorms correspondingly [1].

The first two classes of lexicographic valuations were realized in inference procedures of expert system shell LEXICO which was used for construction of several expert systems in chemical technology [5] based on experts and users qualitative evaluations of plausibility values of facts and rules.

Negation operation on X_p can be defined as follows: $N(x_i)=x_{n-i+1}$. This operation satisfies on X_p the negation axioms [2,7]:

$N(x) \le N(y)$ if $y \le x$,

$N(\mathbf{0}) = \mathbf{1}$, $N(\mathbf{1}) = \mathbf{0}$.

Generally, the following properties fulfilled on X_p define different types of negation operations:

$N(N(x))=x$	(involutive negation),
$x \le N(N(x))$	(weak negation),
$N(N(x)) \le x$	(usual negation).

For example, the negation $N(x_i)=x_{n-i+1}$ is involutive. The negation operations on the sets of lexicographic valuations were introduced as result of extension of negation operation given on X_p [2].

In [4] we have proposed more simple methods to introduce strict monotonic conjunction and disjunction operations for ordinal scales. These methods are based on multi-set memory for representation the results of operations.

Denote $X = \{x_1,\ldots, x_n\}$ the set of intermediate grades of X_p. A multi-set A over X is a string $(a_1, a_2, \ldots, a_n)$, where a_i is a number of appearance of element x_i in A. If a_j equals zero then A does not contain x_j. We will suppose that at least one element a_i in A is greater than 0. Denote F a set of all such multi-sets over X. Multi-sets will be used further as a form of memory for storing the operands of conjunction and disjunction operations. The analysis and program realization of multi-sets based memory is simpler than the approach based on lexicographic valuations [1,2,5]. In this work, we consider three classes of multi-sets over finite ordinal scales. In these classes of multi-sets strict monotonic t-norms or/and t-conorms are considered. On all of these sets of multi-sets we introduce and study negation operations.

The paper is organized as follows. In Section 2 we consider conjunctive multi-sets with strict monotonic cojunction operation. In Section 3 we consider disjunctive multi-sets with strict monotonic disjunction operation. In Section 4 we consider disjunctive-form (df-) multi-sets where both conjunction and disjunction operations are strict monotonic. In each of these sections we introduce negation operations. In Conclusion we discuss the possible applications of considered models.

2. Conjunctive multi-sets

Each multi-set from F is considered in this section as the result of conjunction of corresponding elements of X. For example for $X= \{VSP < SP < AP < LP < VLP\}$ the multi-set (1,0,2,1,0) denotes conjunction $VSP\wedge AP\wedge AP\wedge LP$ or any its commutative transformation. We can denote this multi-set as $VSP^1\wedge SP^0\wedge AP^2\wedge LP^1\wedge VLP^0$ or simply as $VSP^1\wedge AP^2\wedge LP^1$. Such multi-sets will be called conjunctive (c-) multi-sets. The conjunction operation is defined for any c-multi-sets $A = (a_1,\ldots, a_n)$, $B = (b_1,\ldots, b_n)$ from F as follows:

$(a_1,\ldots, a_n)\wedge(b_1,\ldots, b_n) = (a_1+ b_1,\ldots, a_n + b_n)$.

The definition of the ordering of c-multi-sets is based on the following property of conjunction operation: $A\wedge B \le A$, i.e. an adding of elements of *c*-multi-set B to elements of c-multi-set A decreases the plausibility of resulting *c*-multi-set $A\wedge B$. Introduce an ordering relation on F as follows:

$A = B$, if and only if $a_i = b_i$ for all $i = 1,\ldots, n;$

$A < B$, if it exists an index i, such that $a_i > b_i$ and $a_k = b_k$ for all $k < i.$

Suppose $F_p = F \cup \{\mathbf{0}\} \cup \{\mathbf{1}\}$ is an extended set of plausibility values. The extension on F_p of ordering relation from F will be defined as follows:

$\mathbf{0} < A$, $A < \mathbf{1}$, $\mathbf{0} < \mathbf{1}$,

for all *c*-multi-sets A from F. Further in this section A and B will denote elements from F_p. Denote $A \le B$, if $A = B$ or $A < B$. It is clear that $\le$ is a linear ordering on F_p and for all A, B from F_p it is fulfilled $A\le B$ or $B \le A$.

Define an extension of conjunction operation from F on F_p as follows:

$A \wedge \mathbf{0} = \mathbf{0} \wedge A = \mathbf{0}$, $\quad A \wedge \mathbf{1} = \mathbf{1} \wedge A = A$.

Define disjunction operation $\vee$ on the set F_p:

$A \vee \mathbf{0} = \mathbf{0} \vee A= A$, $\quad A \vee \mathbf{1} = \mathbf{1} \vee A = \mathbf{1}$,

$(a_1,\ldots, a_n)\vee(b_1,\ldots, b_n) = max\{(a_1,\ldots, a_n), (b_1,\ldots, b_n)\}$,

where *max* is defined by linear ordering $\le$ on F_p.

Theorem 1. Operations $\wedge$ and $\vee$ are *t*-norm and *t*-conorm correspondingly on the set of c-multi-sets F_p and $\wedge$ is a strict monotonic on F_p.

Define the function N_c on F_p as follows. Let $A = (a_1,\ldots, a_n)$ is a c-multi-set from F and i is a minimal index such that $a_i > 0$. If $N(x_i) = x_j \in X$ then define $N_c(A)= (b_1,\ldots, b_n)$ such that $b_j= 1$ and $b_k = 0$ for all $k\neq j$. If $N(x_i) = \mathbf{0}$ or $N(x_i) = \mathbf{I}$ then define $N_c(A) = \mathbf{0}$ and $N_c(A) = \mathbf{I}$ correspondingly. Define $N_c(\mathbf{0})= \mathbf{1}$, $N_c(\mathbf{1})= \mathbf{0}$.

Theorem 2. The function N_c is a negation operation on F_p satisfying De Morgan low:

$N_c(A\wedge B)=N_c(A)\vee N_c(B)$.

If N is an involution or a weak negation on X_p then N_c is a weak negation on F_p.

3. Disjunctive multi-sets

Each multi-set from F is considered in this section as the result of disjunction of corresponding elements of X. For example for $X=\{VSP < SP < AP < LP < VLP\}$ the multi-set (1,0,2,1,0) denotes disjunction $VSP\vee AP\vee AP\vee LP$ or any its commutative transformation. We also can denote this multi-set as $VSP^1\vee SP^0\vee AP^2\vee LP^1\vee VLP^0$ or simply as $VSP^1\vee AP^2\vee LP^1$. Such multi-sets will be called disjunctive (d-) multi-sets. We will use brackets [1,0,2,1,0] instead of parenthesis to differ between disjunctive and conjunctive multi-sets. On the set of d-multi-sets introduce disjunction operation $\vee$:

$[a_1,\ldots, a_n]\vee[b_1,\ldots, b_n] = [a_1+b_1,\ldots, a_n+b_n]$.

The definition of the ordering of d-multi-sets is based on the following property of disjunction operation: $A\vee B \geq A$, i.e. an adding an elements of d-multi-set B to elements of d-multi-set A increase the plausibility of result. Introduce an ordering relation on F as follows:

$A = B$, if and only if $a_i = b_i$ for all $i = 1,\ldots, n$.

$A < B$, if it exists index i, such that $a_i < b_i$ and $a_k = b_k$ for all $k > i$.

The extension of ordering relation on the set of plausibility values $F_p = F \cup \{\mathbf{0}\} \cup \{\mathbf{1}\}$ will be defined as follows: $\mathbf{0}< A < \mathbf{1}$, $\mathbf{0}<\mathbf{1}$ for all d-multi-sets A from F.

Further in this section A and B will denote elements from F_p. Denote $A \leq B$, if $A = B$ or $A < B$. It is clear that $\leq$ is a linear ordering on F_p. Define an extension of disjunction operation on F_p:

$A \vee \mathbf{0} = \mathbf{0} \vee A = A$, $A \vee \mathbf{1} = \mathbf{1} \vee A = \mathbf{1}$.

Define operation $\wedge$ on the set F_p as follows:

$A \wedge \mathbf{0} = \mathbf{0} \wedge A= \mathbf{0}$, $A \wedge \mathbf{1} = \mathbf{1} \wedge A = A$,

$[a_1,\ldots, a_n]\wedge[b_1,\ldots, b_n] = min\{[a_1,\ldots, a_n], [b_1,\ldots, b_n]\}$,

where min is defined by linear ordering $\leq$ on F_p.

Theorem 3. Operations $\wedge$ and $\vee$ are t-norm and t-conorm correspondingly on the set of d-multi-sets F_p, and $\vee$ is a strict monotonic operation on F_p.

Define the function N_d on F_p as follows. Let $A = [a_1,\ldots, a_n]$ is a d-multi-set from F and i is a maximal index such that $a_i > 0$. If $N(x_i) = x_j \in X$ then define $N_d(A)= [b_1,\ldots,b_n]$ such that $b_j= 1$ and $b_k = 0$ for all $k\neq j$. If $N(x_i) = \mathbf{0}$ or $N(x_i) = \mathbf{I}$ then define $N_d(A) = \mathbf{0}$ and $N_d(A)= \mathbf{I}$ correspondingly. Define $N_d(\mathbf{0})= \mathbf{1}$, $N_d(\mathbf{1})= \mathbf{0}$.

Theorem 4. The function N_d is a negation operation on F_p satisfying De Morgan low:

$N_c(A\vee B)=N_c(A)\wedge N_c(B)$.

If N is an involution or usual negation on X_p then N_c is a usual negation on F_p.

4. Disjunctive form multi-sets

On the sets of conjunctive and disjunctive multi-sets only one of conjunction and disjunction operations is strict monotonic. Consider a set of ordered strings of multi-sets. This set is constructed as analogue of disjunctive forms given by disjunction of conjunctions. As conjunctions consider c-multi-sets and as a disjunction consider the list of c-multi-sets ordered in descending order with the ordering relation defined on the set of c-multi-sets. Disjunctive form (df-) multi-set may be represented as a column of c-multi-sets $A= [A_1,\ldots, A_p]$ from F such that $A_k \geq A_{k+1}$:

$$A = \begin{bmatrix} A_1 \\ \ldots \\ A_p \end{bmatrix} = \begin{bmatrix} (a_{11},\ldots,a_{1n}) \\ \ldots \\ (a_{p1},\ldots,a_{pn}) \end{bmatrix}. \qquad (1)$$

We suppose that df-multi-set can contain repeated c-multi-sets. Denote the set of df-multi-sets as G. The number of c-multi-sets in df-multi-set $A = [A_1,\ldots, A_p]$ will be called a length $|A|= p$ of A. We will write it also explicitly as $A(p) = [A_1,\ldots, A_p]$. Suppose $A(p) = [A_1,\ldots, A_p]$ and $B(q)= [B_1,\ldots, B_q]$ are two df-multi-sets with length p and q respectively. The ordering of df-multi-sets from G is defined as follows:

$A(p) \leq B(q)$ if $p\leq q$ and $A_i= B_i$ for all $i=1,\ldots, p$ or

if $A_1 < B_1$ or

if it exists $r >1$, $r \leq min(p, q)$, such that

$A_r < B_r$ and $A_i = B_i$ for all $i =1,\ldots, r-1$.

This relation is based on the property of disjunction operation $A \leq A \vee B$ and determined by the maximal c-multi-set where two df-multi-sets differ. As usually, we write $A = B$ if $A \leq B$ and $B \leq A$. We write $A < B$ if $A \leq B$, and $A\neq B$. The extension of ordering relation on an extended set of plausibility values $G_p = G \cup \{\mathbf{0}\} \cup \{\mathbf{1}\}$ is defined as follows: $\mathbf{0}< A$, $A< \mathbf{1}$, $\mathbf{0}< \mathbf{1}$, for all df-multi-sets A from G. In general case A and B will denote elements from G_p.

On the set G of df-multi-sets introduce disjunction operation $\vee_G$ as follows. Suppose $A= [A_1,\ldots, A_p]$ and $B=[B_1,\ldots, B_q]$ are to df-multi-sets. The df-multi-set $C= A(p)\vee B(q)$ will contain all $p+ q$ c-multi-sets from A and B sorted in descending order.

From the definition of $\vee_G$ operation we can write:

$A = [A_1,\ldots, A_p] = [A_1] \vee_G \ldots\vee_G [A_p]$.

Define an extension of this operation on G_p:

$A\vee_G\mathbf{0} = \mathbf{0}\vee_G A = A$, $A\vee_G\mathbf{1} = \mathbf{1}\vee_G A = \mathbf{1}$.

The conjunction operation $\wedge_G$ on the set G is defined as follows. If $A = [A_1]= [(a_{11},\ldots, a_{1n})]$ and $B = [B_1] = [(b_{11},\ldots, b_{1n})]$ then $A\wedge_G B = [A_1]\wedge_G[B_1]= [A_1\wedge B_1]= [(a_{11}+ b_{11},\ldots, a_{1n}+ b_{1n})]$ where $\wedge$ is the conjunction operation on the set of c-multi-sets F. Generally for $A = [A_1,\ldots, A_p]$, $B = [B_1,\ldots, B_p]$ a conjunction $A\wedge_G B$ is defined by the distributive low:

$A \wedge_G B = ([A_1] \vee_G \ldots \vee_G [A_p]) \wedge ([B_1] \vee_G \ldots \vee_G [B_q]) = [A_1 \wedge B_1] \vee_G \ldots \vee_G [A_1 \wedge B_q] \vee_G [A_2 \wedge B_1] \vee_G \ldots \vee_G [A_2 \wedge B_q] \vee_G \ldots \vee_G [A_p \wedge B_1] \vee_G \ldots \vee_G [A_p \wedge B_q]$.

$A \wedge_G B$ consists of $p \times q$ c-multi-sets $(a_{i1}+b_{j1}, \ldots, a_{in}+b_{jn})$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, ordered in descending order.

Define conjunction operation $\wedge$ on the set G_p :

$A \wedge_G \mathbf{0} = \mathbf{0} \wedge_G A = \mathbf{0}$, $A \wedge_G \mathbf{1} = \mathbf{1} \wedge_G A = A$.

Theorem 5. Operations $\wedge_G$ and $\vee_G$ are strict monotonic t-norm and t-conorm correspondingly on the set of df-multi-sets G_p.

Negation operation on the set of df-multi-sets may be introduced by several ways. Consider any elements x_j of X_p as a c-multi-set $(a_1, \ldots, a_n)$ such that $a_j = 1$, $a_k = 0$ for all $k \neq j$. Suppose N is an involutive negation $N(x_i) = x_{n-i+1}$ on X_p. For df-multi-set (1) define a function N_1 as follows:

$N_1(A) = [N_c(A_1)]$,

where N_c is a negation on the set of c-multi-sets.

Suppose $a_{1k1}, \ldots, a_{1km}$, $(k_1 < \ldots < k_m)$, is a sequence of positive elements of A_1 in (1). Define a function N_2 as a df-multi-set as follows:

$$N_2(A) = \begin{bmatrix} N(x_{1k_1}) \\ \ldots \\ N(x_{1k_m}) \end{bmatrix}$$

where N is a negation operation on X_p and $N(x_{1kj})$ is repeated a_{1kj} times, $j = 1, \ldots, m$.

Define a function M_3 as follows: $M_3(A) = (N(x_{1k1}), \ldots, N(x_{skm}))$, where $s = p$ if $|A_j| = 1$, $j = 1, \ldots, p$, else s is the minimal index in $\{1, \ldots, p\}$ such that $|A_s| > 1$. Define $N_3(A) = [(a_1, \ldots, a_n)]$, where a_1 is a number of values x_1 in $M_3(A)$, a_2 is a number of values x_2 in $M_3(A)$ etc.

Theorem 6. The functions N_1-N_3 are negations on the set of df-multi-sets.

5. Conclusions

The methods to introduce strict monotonic conjunction and disjunction operations for finite ordinal scales are considered. These methods use multi-set memory for storing the values of operands. Three types of such memory were considered and on all of them negation operations are introduced. The proposed models are simpler than the similar models based on lexicographic operations [1,2]. The proposed methods can be used in development of inference procedures of expert systems with qualitative experts and users evaluations. Linguistic scales usually have 5-9 grades and differentiation of more than 10 solutions obtained on the output of decision making procedure or expert system with qualitative grades of uncertainties is difficult. The proposed approach to representation and processing of uncertainties can refine true values of solutions obtained on the output of such systems. The proposed methods can be used also for processing linguistic information and computing with words [10] when membership values of fuzzy models are given qualitatively.

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7. References

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