A Modified Model of Decision-Making Based on Lattice-Valued Logic LF(X)

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Abstract

In this paper, a modified model of decision-making with linguistic information is presented in the light of uncertainty reasoning based on lattice-valued logic. Firstly, the representation of linguistic terms is discussed, and the evaluation factors are analyzed from the viewpoint of logic. Secondly, the model is presented and illustrated by an typical example. Finally, the features and shortcomings are reviewed.

Keywords: uncertainty reasoning, decision making, lattice-valued logic, linguistic terms

1. Introduction

Decision-making is an important activity in human beings' life. In general, a typical process of decision-making has two features, *i.e.*, (1) decision is often made in an environment full of uncertainties, and (2) decision is often described by natural languages.

Due to the limitation of human beings' recognition, the information, which a decision-making is based on, is always interfered by a great amount of uncertainties existing not only in the objective things but also in the subjective recognitions. Moreover, in the course of processing such information, other uncertainties will be introduced also.

To treat with uncertain information, many approaches from different disciplines have been proposed and applied. Now, the linguistic approaches have drawn considerable attentions of researchers from various research fields, such as information retrieval, clinical diagnosis, market analyzing, software development, education, especially on decision making and evaluation [1–6, 8, 12]. Among them, the methods based on fuzzy set play an important role for the reason that fuzzy set is a kind of capable semantical model for natural language. Briefly speaking, there are two primary types of linguistic approaches, *i.e.*, the first one is on the basis of numeric computation, and the second on is on the basis of symbolic computation. We are of the opinion that a suitable linguistic approaches should be built in the framework of classical and non-classical logics for the reason that logics are the study of thinking and decision-making is an activity of thinking.

In [6], we have presented a simple model of handling linguistic terms, which can deal with not only comparable but also incomparable linguistic terms. In this paper, we shall refine it on the ground of a lattice-valued logic $LF(X)$. Concretely, in Section 2, some basic concepts of lattice-valued logic $LF(X)$ are presented; in Section 3, the representation of linguistic terms, transformation of evaluation factors, and the steps of the model are discussed, and an example is given to illustrate the model; in Section 4, the features and shortcomings of the model are reviewed.

2. Preliminary

Because "mathematical structures of a given set of truth values play an important role to study their respective logic [4]," in this section we only give the concept of lattice implication algebra, which is the corresponding logical algebra of the lattice-valued logic $LF(X)$.

Definition 1 [9] *Let* $(L, \vee, \wedge, ', O, I)$ *be a bounded lattice with universal boundaries* O *(the smallest element) and* I *(the greatest element) respectively, and "* $'$ *" an order-reversing involution. If a mapping* $\to$: $L \times L \to L$ *satisfies: for any* x, y, $z \in L$,

(I_1) $x \to (y \to z) = y \to (x \to z)$,

(I_2) $x \to x = I$,

(I_3) $x \to y = y' \to x'$,

(I_4) *if* $x \to y = y \to x = I$, *then* $x = y$,

(I_5) $(x \to y) \to y = (y \to x) \to x$,

(I_6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(I_7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

then $(L, \vee, \wedge, ', \to, O, I)$ *is called a lattice implication algebra.*

From Definition 1, it is easy to verify that the Boolean algebra, and the Łukasiewicz implication algebras on the interval $[0, 1]$ and on finite chains are special cases of lattice implication algebra.

By taking a lattice implication algebra L as the truth-value field, we can define the lattice-valued logic $LF(X)$. For the length of the paper, we refer readers to the reference [9] for details. Simply to say, a logical formula (φ, α) in $LF(X)$ is composed of two components, and the first part φ is designed for formal inference, the second part α is designed for truth-value transferring. Similar to classical logic, a logical consequence is also a deduction from the axioms set and antecedents, but the deduction is companied by truth-values transferring explicitly.

Let $\mathcal{F}$ be the set of formal formulae in $LF(X)$, and $\mathcal{A} \subseteq \mathcal{F}$ the set of formal axioms of $LF(X)$. Suppose $A \subseteq \mathcal{F}$, and $\varphi \in \mathcal{F}$, a proof ω of φ from A is defined as follows [9]:

$$\omega\colon\ (\varphi_1, \alpha_1), (\varphi_2, \alpha_2), \cdots, (\varphi_n, \alpha_n), \tag{1}$$

where $\varphi_n = \varphi$, and for any $i \in \{1, \ldots, n\}$, $\varphi_i \in \mathcal{F}$, $\alpha_i \in L$, and

1. $\mathcal{A}(\varphi) = \alpha_i$, or
2. $A(\varphi) = \alpha_i$, or
3. there exist $j, k < i$, such that $\varphi_j = \varphi_k \to \varphi_i$ and $\alpha_i = \alpha_j \otimes \alpha_k = (\alpha_j \to \alpha'_k)'$, or
4. there exist $j < i$ and $\alpha \in L$ such that $\varphi_i = \alpha \to \varphi_j$ and $\alpha_i = \alpha \to \alpha_j$, or
5. there exists $j < i$ such that $\varphi_j = \psi_1 \to \psi_2$, $\varphi_i = \psi_1 \to (\forall x)\psi_2$, $\alpha_i = \alpha_j$, x is free in ψ_2 and bound in ψ_1, or
6. there exists $j < i$ such that $\varphi_j = \psi_1 \to \psi_2$, $\varphi_i = (\exists x)\psi_1 \to \psi_2$, $\alpha_i = \alpha_j$, x is free in ψ_1 and bound in ψ_2.

For more properties of the lattice-valued logic $LF(X)$, the readers can refer to [9].

3. Main Results

Notice that a typical decision-making, such as for example to select the best candidate(s) from all alternatives, can be seen as a goal-directed inference. Moreover, the inference is carried out in situation of uncertain information. This course can be treated as a deduction of a particular consequence (goal) from some antecedents (experts' comments) in hand from the viewpoint of logics. This section will give such a model.

3.1. Linguistic Terms

Suppose T is the set of linguistic terms the decision-maker and consultancy experts selected and used in decision-making. (In general, not all selected linguistic terms are the same with all experts and the decision-maker.) Let U be the universe of discourse and $\mathscr{F}(U)$ the set of all fuzzy sets on U. For each expert e, there is an associated mapping f_e from T to $\mathcal{P}(\mathscr{F}(U))$. For any $\mathscr{A} \in \mathscr{F}(U)$, if $\mathscr{A} \in f_e(t)$, then we shall say $\mathscr{A}$ is labelled by the linguistic term t. For two linguistic terms t_1 and t_2, if $f_e(t_1) = f_e(t_2)$, then we shall say t_1 and t_2 have the same meanings (also similar meanings, indistinguishable meanings). If $f_e(t_1) \subseteq f_e(t_2)$, then we shall say t_1 has more clear (strength) meanings than t_2 (or t_2 has less clear meanings than t_1), and denoted by $t_2 \leqslant t_1$. Therefore T is a partial-ordered set *w.r.t.* "$\leqslant$". (Note: the partial-ordered set $(T, \leqslant)$ does not always be a lattice in general.)

It is easy to show that $\mathscr{F}(U)$ is a complete and distributive lattice according to the partial order induced by the inclusion of fuzzy set. In [9], the following conclusion holds:

Theorem 1 *In $\mathscr{F}(U)$, for any $\mathscr{A}$, $\mathscr{B} \in \mathscr{F}(U)$, if operations $\vee$, $\wedge$, $'$, and $\to$ are defined as follows:*

$$(\mathscr{A} \vee \mathscr{B})(x) = \max\{\mathscr{A}(x), \mathscr{B}(x)\}, \tag{2}$$
$$(\mathscr{A} \wedge \mathscr{B})(x) = \min\{\mathscr{A}(x), \mathscr{B}(x)\}, \tag{3}$$
$$(\mathscr{A})'(x) = 1 - \mathscr{A}(x), \tag{4}$$
$$(\mathscr{A} \to \mathscr{B})(x) = \min\{1, 1 - \mathscr{A}(x) + \mathscr{B}(x)\}, \tag{5}$$

then $(\mathscr{F}(U), \vee, \wedge, ', \to, \mathbf{O}, \mathbf{I})$ is a lattice implication algebra, where $\mathbf{O}$ and $\mathbf{I}$ are the fuzzy sets taking identic value 0 and 1 respectively.

For any t_1, $t_2 \in T$, we shall define the following operations $\vee_T$, $\wedge_T$, $'_T$, and $\to_T$ as follows:

$$t_1 \vee_T t_2 = \{t \in T \mid f_e(t) = f_e(t_1) \cap f_e(t_2)\}, \tag{6}$$
$$t_1 \wedge_T t_2 = \{t \in T \mid f_e(t) = f_e(t_1) \cup f_e(t_2)\}, \tag{7}$$
$$(t_1)'^T = \{t \in T \mid f_e(t) = \mathscr{F}(U) \setminus f_e(t_1)\}, \tag{8}$$
$$t_1 \to_T t_2 = \{t \in T \mid f_e(t) = f_e(t_1) \to f_e(t_2)\}, \tag{9}$$

where $f_e(t_1) \to f_e(t_2)$ means $\{\mathscr{A} \to \mathscr{B} \mid \mathscr{A} \in f_e(t_1)$, and $\mathscr{B} \in f_e(t_2)\}$.

3.2. Evaluation Factors

Suppose A is the set of alternatives, E is the set of consultancy experts, F is the set of evaluation factors, and G is the final decision goal.

To the decision-maker, a decision-making is indeed a process of selecting an alternative a, such that the following expression has the biggest reliability:

$$F(a) \to G(a). \tag{10}$$

Because F is a set of evaluation factors, Eq. (10) can be seen as

$$f_1(a) \wedge f_2(a) \wedge \cdots \wedge f_n(a) \to G(a), \tag{11}$$

where f_1, f_2, …, $f_n \in F$. In fact, the Eq. (11) is a formula by replacing x with a in $(\forall x)F(x) \to G(x)$. To the expert $e \in E$, whose assessment on an alternative $a \in A$ can be seen as a set of antecedents of form

$$(f_i^e(a) \to G(a), t_i^e), f_i \in F, t_i^e \in T. \tag{12}$$

Therefore, the decision-making is the deduction of $F(a) \to G(a)$ from $f_1(a) \to G(a)$, $f_2(a) \to G(a)$, $\cdots$, $f_n(a) \to G(a)$, and this can be down in the framework of logic.

3.3. Model

The model includes the following steps:

1. Construct possible deduction of $F(x) \to G(x)$ from $f_1(x) \to G(x)$, $\cdots$, $f_n(x) \to G(x)$. In $LF(X)$, this can be done by using following sequence:

   $$(\varphi \wedge \psi \to \psi, I)$$
   $$(\psi \to \varphi \vee \psi, I)$$
   $$((\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma)), I)$$
   $$((\varphi \to \gamma) \vee (\psi \to \gamma) \to ((\varphi \wedge \psi) \to \gamma), I).$$

2. Compute synthesized assessment in the follow equation: for all deductions Ω, let

   $$\alpha(x) = \bigvee_{\omega \in \Omega} \bigotimes_{f_i \in F} f_i(x) \to G(x). \tag{13}$$

3. Select linguistic terms as follows: let $T_a \subseteq T$, such that for any $t \in T_e$,

   $$f_e(t)(x) = f_e(\alpha(x)). \tag{14}$$

4. Select the best alternative(s) by $f_e(t)(x)$.

3.4. Example

To illustrate the model, we shall consider the following example [5]: "assume that a manager needs to decide which of four available candidates, a_1, a_2, a_3, a_4, for an open job to hire. The goal is to hire a candidate with long experience relevant to the job description, under the constraints that the required salary is low and that the candidate is not much older than 30." Suppose the year of relevant experience, required salary, and age of the four candidates are shown in Table 1.

	a_1	a_2	a_3	a_4
experience (years)	7.5	1.3	4.5	5.6
required salary (k$)	62	54	68	58
age (years)	38	26	32	35

Table 1: Information of candidates.

For this example, we can let G be "hire a candidate, " F be compose of three evaluation factors, *i.e.*, f_1 is "long relevant experience," f_2 is "low required salary," and f_3 is "not much older that 30."

For the first factor f_1. Suppose $U_1 = [0, 10]$ is the number of years of relevant experience, and suppose the linguistic terms set T_1 includes the terms "extreme experienced (EE)," "very experienced (VE)," "more experienced (ME)," "experienced (E)," "rather experienced (RE)," "poor experienced (PE)," "non-experienced (NE)." Then we have: a_1 is "very experienced," a_2 is "poor experienced," a_3 is "experienced," and a_4 is "more experienced."

For the second factor f_2. Suppose $U_2 = [50, 70]$ is the range of required salary, and suppose the linguistic terms set T_2 includes the terms "very high (VH)," "high (H)," "middle (M)," "low (L)," "very low (VL)." Then we have: a_1's requirement is "high," a_2's requirement is "low," a_3's requirement is "very high," and a_4's requirement is "middle."

For the third factor f_3. Suppose $U_3 = [20, 40]$ is the range of age, and suppose the linguistic terms set T_3 includes the terms "very young (VY)," "young (Y)," "middle (M)," "old (O)," "very old (VO)." Then we have: a_1 is "very old," a_2 is "young," a_3 is "middle," and a_4 is "old."

Now let T include the linguistic terms "very suitable (VS)," "more suitable (MS)," "suitable (S)," "rather suitable (RS)," and "non suitable (NS)." Suppose mappings $g_1\colon\ T \to T_1$, $g_2\colon\ T \to T_2$, and $g_3\colon\ T \to T_3$ are defined as Tables 2, 3, and 4, respectively. Therefore we have,

$$(f_1(a_1) \to G(a_1), \text{VS}), \quad (f_1(a_2) \to G(a_2), \text{RS}),$$
$$(f_1(a_3) \to G(a_3), \text{S}), \quad (f_1(a_4) \to G(a_4), \text{MS}),$$
$$(f_2(a_1) \to G(a_1), \{\text{RS, NS}\}),$$
$$(f_2(a_2) \to G(a_2), \text{VS}), \quad (f_2(a_3) \to G(a_3), \text{NS}),$$
$$(f_2(a_4) \to G(a_4), \{\text{MS, S, RS}\}),$$
$$(f_3(a_1) \to G(a_1), \{\text{S, RS, NS}\}),$$
$$(f_3(a_2) \to G(a_2), \text{VS}), \quad (f_3(a_3) \to G(a_3), \text{VS}),$$
$$(f_3(a_4) \to G(a_4), \text{MS}).$$

Thus, deductions of $F(a_i) \to G(a_i)$ can be built in

$LF(X)$, $i = 1, 2, 3, 4$. Then, we get

$$(F(a_1) \to G(a_1), RS)$$
$$(F(a_2) \to G(a_2), RS)$$
$$(F(a_3) \to G(a_3), NS)$$
$$(F(a_4) \to G(a_4), MS)$$

by Eq. (13). Hence, the best candidate is a_4. (This is similar to the conclusion in [5])

	EE	VE	ME	E	RE	PE	NE
g_1(VS)	1	1	0	0	0	0	0
g_1(MS)	1	1	1	0	0	0	0
g_1(S)	1	1	1	1	0	0	0
g_1(RS)	0	0	0	1	1	1	0
g_1(NS)	0	0	0	0	1	1	1

Table 2: Linguistic terms for experience (g_1).

	VH	H	M	L	VL
g_2(VS)	0	0	0	1	1
g_2(MS)	0	0	1	1	1
g_2(S)	0	0	1	1	1
g_2(RS)	0	1	1	0	0
g_2(NS)	1	1	0	0	0

Table 3: Linguistic terms for required salary (g_2).

	VY	Y	M	O	VO
g_3(VS)	0	1	1	0	0
g_3(MS)	0	1	1	1	0
g_3(S)	0	1	1	1	1
g_3(RS)	1	0	0	0	1
g_3(NS)	1	0	0	0	1

Table 4: Linguistic terms for age (g_3).

4. Conclusion

In this paper, we discussed a modified decision-making model with linguistic terms in the framework of logic. The presented work can deal with comparable and incomparable linguistic terms, and is embedded in the guidance of logic. Due to the complexity of human being's natural language, there still some problems unsolved, such as for example, how to select the best deduction, how to represent and aggregate experts' assessments efficiently. Some work is on the way.

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