A multi-period CAPM with heterogeneous beliefs

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Abstract

This paper introduces a simulation model extending the well known Capital Asset Pricing Model by Sharpe and Lintner. Investors are modeled as multi-period forward looking portfolio optimizers. However, the future is not known *a priori*, but has to be modeled and estimated. Optimal portfolio paths are derived analytically for different utility specifications. We allow agents to use past price information to forecast the future of asset returns, but with possibly different econometric forecasting techniques and different data sets. We use Microscopic Simulations to investigate the effects on equilibrium asset prices and on returns over an extended time period in a temporary equilibrium context. We show that models of this kind can reproduce key features of asset returns found in real life.

Keywords: multiperiod CAPM, heterogeneous agents, price dependent preferences, microscopic simulations

1. Introduction

The benchmark equilibrium model in finance is the single-period Capital Asset Pricing Model (CAPM, see [12, 15]). In this model investors are mean-variance efficient, and the mean and variance are assumed to be known and correctly anticipated (rational expectations).

The aim of this paper is to investigate economies in which economic agents (investors) choose their portfolios by solving a possibly multi-period investment problem. Contrary to the rational expectations situation, we investigate the case where economic agents have to model and estimate the future. In particular, we consider the case where economic agents trade-off expected return and risk of the wealth at the end of their time horizon, where the expected return and risk are not known in advance. We assume that the economic agents use econometric models and techniques to calculate the expected return and risk. Consequently, different agents will find different estimates, for instance, due to different sample sizes or sampling frequencies, or due to different econometric models and estimation techniques employed. We allow for single-period time horizons, as in the benchmark CAPM, and multi-period time horizons where investors might have different time horizons. We consider economic agents who assume *iid* returns, for which the optimal portfolio has been derived in [10] and [11] and we generalize this to economic agents who postulate general forms of time dependence. In the *iid* case, economic agents will use sample averages to estimate asset characteristics. In the non-*iid* case, more advanced econometric techniques might make more sense.

Since every time period new information becomes available, asset characteristics are re-estimated every period. So, the estimates will be time-varying and investor-specific. As a consequence, the equilibrium concept to be used cannot *a priori* be full rational equilibrium where present and future –correctly anticipated– prices are set such that the present and future markets clear. Instead, we use the concept of temporary equilibrium (see, for example, [2]): each period's prices are set such that the markets clear in that period.

Since we cannot solve the dynamics of the equilibrium prices analytically, we use microscopic simulation to simulate the economy over time. From that point of view, the set-up chosen in this paper can be seen as an alternative to the work by, among others, [1, 5–7, 9]. In these microscopic simulation models, economic agents are usually subdivided into various types, such as technical analysts or believers of the Efficient Market Hypothesis. In our economy, the agents are all of the same type, i.e., rational in the sense of utility maximization, but different in the way they quantify the future due to different models, different sample sizes, or different econometric techniques employed, etcetera.

The resulting asset prices and returns generated by the microscopic simulations are subsequently investigated empirically. For instance, we consider asset return predictability, we test whether the agents' assumption of *iid* returns holds, which is a requirement for the agents in this economy to have rational expectations, we model the volatility of the returns, and we test the Efficient Market Hypothesis. Moreover, assuming that the investors perform such an econometric analysis themselves, we can use the outcomes to improve the investors' estimation procedures.

The remainder of this paper is organized as follows. In section 2, we introduce some notation and present our economy. Section 3 contains some simulation results for a model with expected quadratic utility agents. We also describe an empirical analysis of these simulation results and we conclude in section 4 with a summary and some ideas for future research.

2. Model formulation

The time span of interest is $\{0, 1, 2, \ldots, T\}$ ($T \in \mathbb{N}_+$). A particular period will be denoted by τ. There are $J + 1 \in \mathbb{N}_+$ assets in the market. Asset 0 will be treated as the reference asset (with price normalized to 1, usually the riskfree asset). A particular asset will be denoted by j. The assets have return $r_\tau = (r_{0\tau}, \ldots, r_{J\tau})'$, where $r_{j\tau}$ denotes the gross return of asset j at time τ and is defined by $r_{j\tau} \equiv (p_{j,\tau+1} + d_{j,\tau+1})/p_{j\tau}$, where $p_{j\tau}$ denotes the price of asset j at time τ and $d_{j\tau}$ denotes the dividend paid out during the period between time $\tau - 1$ and τ in amounts of the reference asset.

At time $\tau = 0$, $I \in \mathbb{N}_+$ investors enter the market. At each time τ, these investors, $i = 1, \ldots, I$, are characterized by their asset holdings $h^i_{\tau-1} = (h^i_{0,\tau-1}, \ldots, h^i_{J,\tau-1})'$ (in numbers of assets), utility function U^i_τ (defined over different individual specific time horizons T^i), model for asset returns $\mathcal{M}^i_\tau$, data $\mathcal{D}^i_\tau$ and econometric technique $\mathcal{E}^i_\tau$.

The amount of wealth invested in asset j by a particular agent i at time τ will be denoted by $v^i_{j\tau}$ and is defined by $v^i_{j\tau} \equiv p_{j\tau} h^i_{j\tau}$. Define the total wealth of agent i at time τ by x^i_τ. We denote $u^i_\tau = (v^i_{1\tau}, \ldots, v^i_{J\tau})'$, $v^i_\tau = (v^i_{0\tau}, u^i_\tau)'$, $\tilde{\iota} = (1, \ldots, 1)' \in \mathbb{R}^J$ and $\iota = (1, \ldots, 1)' \in \mathbb{R}^{J+1}$. Define furthermore $R_\tau \equiv ((r_{1\tau} - r_{0\tau}), \ldots, (r_{J\tau} - r_{0\tau}))'$ which is the excess return of an asset over the return of the reference asset. Then $x^i_\tau = r'_{\tau-1} v^i_{\tau-1}$.

At each point in time $\tau \in \{0, 1, \ldots, T-1\}$, the agents are assumed to face a problem of the form[1] (with t and T^i relative to the current time τ)

$$\begin{aligned} \max \quad & U^i_\tau\left(\mathbb{P}_{x^i_{T^i}}; \mathcal{M}^i_\tau, \mathcal{D}^i_\tau, \mathcal{E}^i_\tau\right) \\ \text{s.t.} \quad & x^i_{t+1} = r'_t v^i_t \\ & \iota' v^i_t \leq x^i_t, \qquad t = 0, 1, \ldots, T^i - 1 \end{aligned}$$

where we consider the case where the investor is only interested in characteristics of the perceived distribution of final period wealth, $\mathbb{P}_{x^i_{T^i}}$, and uses the model $\mathcal{M}^i_\tau$ to predict these characteristics. The model is estimated using the data $\mathcal{D}^i_\tau$ and estimation technique $\mathcal{E}^i_\tau$. This problem will result in an optimal period τ choice h^{i*}_τ (related to $t = 0$).

[10] and [11] have solved this problem for mean-variance utility functions under the assumption that returns are *iid*. We have solved this problem while relaxing the *iid* assumption (see also [8]) and introducing the possibility to become satiated, but under the assumption that a riskfree asset, i.e., a reference asset that pays out a non-stochastic interest rate $r_{0\tau}$ at time τ, is available.

Prices are set by temporary equilibrium using the market clearing condition $\sum_{i=1}^I h^i_{j,\tau-1} = \sum_{i=1}^I h^{i*}_{j\tau}$ for all assets $j > 0$ at time τ. Note that, due to the set-up using the concept of temporary equilibria, at time τ, we only need the optimal portfolio of agent i for time $\tau + t$ for $t = 0$ and not for $t = 1, \ldots, T^i - 1$.

As an illustration, we discuss the case of expected quadratic utility using the *iid* assumption on returns for simplicity. A well-known drawback of expected quadratic utility is Increasing Absolute Risk Aversion (IARA). However, this drawback can be circumvented, for instance, by modeling the bliss level wealth dependent, see [13].

3. Expected quadratic utility

The expected quadratic utility problem as faced by economic agent i is formulated as follows.

$$\begin{aligned} \max \quad & U^i_\tau\left(\mathbb{P}_{x^i_{T^i}}; \mathcal{M}^i_\tau, \mathcal{D}^i_\tau, \mathcal{E}^i_\tau\right) = \widehat{\mathrm{E}}^i_\tau\left(-\frac{1}{2}(x^i_{T^i} - b^i)^2\right) \\ \text{s.t.} \quad & x^i_{t+1} = r'_t v^i_t \\ & \iota' v^i_t \leq x^i_t, \qquad t = 0, 1, \ldots, T^i - 1 \end{aligned}$$

Here b^i denotes the bliss level of agent i and $\widehat{\mathrm{E}}^i_\tau(\bullet)$ denotes the expectations operator according to model $\mathcal{M}^i_\tau$ using data $\mathcal{D}^i_\tau$ and econometric technique $\mathcal{E}^i_\tau$.

Under the assumption of *iid* returns and the availability of a riskfree asset, the solution for $t = 0, 1, \ldots, T^i - 1$ for the problem agents are assumed to solve, is given by

$$\begin{aligned} u^{i*}_t &= (C^i_t + D^i_t x^i_t)\mathbb{1}_{(-\infty, b^i_t)}(x^i_t) \\ v^{i*}_{0t} &= \left(x^i_t - \tilde{\iota}' u^{i*}_{jt}\right)\mathbb{1}_{(-\infty, b^i_t)}(x^i_t) + b^i_t \mathbb{1}_{[b^i_t, \infty)}(x^i_t) \end{aligned}$$

with

$$\begin{aligned} C^i_t &= b^i_{t+1}\left(\widehat{\mathrm{E}}^i_\tau(R_t R'_t)\right)^{-1}\widehat{\mathrm{E}}^i_\tau(R_t) \\ D^i_t &= -r_{0t}\left(\widehat{\mathrm{E}}^i_\tau(R_t R'_t)\right)^{-1}\widehat{\mathrm{E}}^i_\tau(R_t) \\ b^i_t &= b^i \prod_{k=t}^{T^i-1} \frac{1}{r_{0k}}. \end{aligned}$$

[1] This states a multi-period investment problem with the possibility to become satiated at any point in time during the planning horizon. The intra-temporal budget restriction states that the investor does not need to invest the entire wealth at a particular point in time.

with $\mathbb{1}_{(\bullet)}(\bullet)$ the usual indicator function. The solution dictates an investor to follow a portfolio strategy that is the same as in a non-satiation context (that is, without allowing $\iota' v_t^i \leq x_t^i$) as long as she is not satiated. When she becomes satiated during her planning horizon, that is, there exists a time t for which it holds that $x_t^i \geq b_t^i$ then she invests exactly b_t^i in the riskfree asset at time t and zero in the risky assets, holds this portfolio until time T^i and she will have $x_{T^i}^i = b^i$ at time T^i. The actual portfolio choice at any time τ is given above for $t = 0$. Using the budget restrictions and the market clearing condition, temporary equilibrium prices at time τ of the J risky assets are determined by a linear system which allows for fast simulations. Prices of past periods appear in a non-linear way.

For simulations, we use the settings listed in table 1 for the model parameters. Here $\mathrm{U}(a,b)$ denotes a random number from the uniform distribution with parameters a and b and $\mathrm{N}(\mu,\sigma^2)$ denotes the normal distribution with parameters μ and σ^2. Here l^i denotes the length of the data set of agent i. The data set of agent i at time τ is the set defined by $\mathcal{D}_\tau^i = \{p_{\tau-t}, d_{\tau-t} \mid 1 \leq t \leq l^i\}$. The *iid* assumption for asset returns constitutes the model $\mathcal{M}_\tau^i$.

Param.	Setting	Param.	Setting
I	10	J	3
p_0	$(1,1,1,1)'$	h_0^i	$(0,1,1,1)'$ $(\forall i)$
l^i	$\mathrm{U}(40,80)$ $(\forall i)$	b^i	$\mathrm{N}(5,.05)$ $(\forall i)$
$d_{0\tau}$	0 $(\forall \tau)$	$d_{1\tau}$	$\mathrm{U}(0,.1)$ $(\forall \tau)$
$d_{2\tau}$	$\mathrm{U}(0,.15)$ $(\forall \tau)$	$d_{3\tau}$	$\mathrm{U}(0,.12)$ $(\forall \tau)$

Table 1: Parameter settings.

Agents use the following forecast functions for predicting the first and second moment of R_t at any point in time τ using data $\mathcal{D}_\tau^i$. These estimates are updated at every period.

$$\widehat{\mathrm{E}}_\tau^i(R_t) = \frac{1}{l^i}\sum_{k=\tau-1}^{\tau-l^i} R_k, \quad \widehat{\mathrm{E}}_\tau^i(R_t R_t') = \frac{1}{l^i}\sum_{k=\tau-1}^{\tau-l^i} R_k R_k'$$

Furthermore, agents recognize the fact that dividends increase total wealth in the market. To keep up with that development, agents apply the following rule after every period, that is, after prices and dividends have realized: $b_{\tau+1}^i = b_\tau^i + \iota' d_\tau$ with $b_0^i = b^i$. This also (generically) prevents agents from becoming satiated and leaving the market. Notice that this makes the bliss level (indirectly) wealth dependent.

Figure 1 presents a graphical representation of the resulting price processes over the last 1,000 periods of a 100,000 period simulation. We observe that there is no evidence of returning patterns in asset prices and the figure also seems to indicate that returns are not *iid*, suggesting that there are no rational expectations.

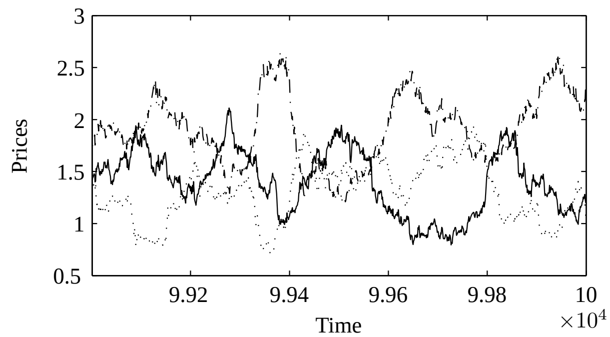

Figure 1: Price processes.

Table 2 presents some statistics of the last 20,000 net asset returns (that is, $r_{1\tau} - 1$). We find excess kurtosis, a skewed distribution and a significant deviation from normality for all of the asset returns.

Asset	Mean	Covariance matrix		
1	.0442	.0019	-.0004	-.0003
2	.0462	-.0004	.0013	-.0003
3	.0423	-.0003	-.0003	.0013
Asset	**Skewness**	**Kurtosis**	**Jarque-Bera**	
1	1.3296	14.8839	$1.2355 \times 10^{5*}$	
2	.7679	5.4242	$6.8603 \times 10^{3*}$	
3	.5576	4.9697	$4.2677 \times 10^{3*}$	

Table 2: Return statistics, * significant at the 1% level.

For the absolute net returns of asset 1 (that is, $|r_{1\tau} - 1|$), we conducted a classical R/S analysis (see, for example, [14]). The Hurst parameter, estimated from the R/S analysis, has a value of $\widehat{H} = .86809$. It has been shown that $H = d + \frac{1}{2}$ with d the parameter of fractional integration in time series models (see, for instance, [16]). This results in an estimate of $\widehat{d} = .36809$ which is less than $\frac{1}{2}$ which in turn indicates that the time series of absolute net returns of asset 1 does not display long run dependence.

More interesting though is to test whether the assumption on *iid*-ness of asset returns makes sense. To test this, we formulate the following model for asset returns.

$$r_{\tau+1} = c + \rho r_\tau + \varepsilon_{\tau+1}, \quad \mathrm{E}(\varepsilon_{\tau+1} \mid I_\tau) = 0$$

We have estimated this model using OLS for asset 1 for 99 consecutive non-overlapping time windows of 1000 periods length each. The consecutive estimates of $\widehat{\rho}$ are depicted in figure 2 by the topmost line. Its standard error is depicted in the same figure with the bottom line.

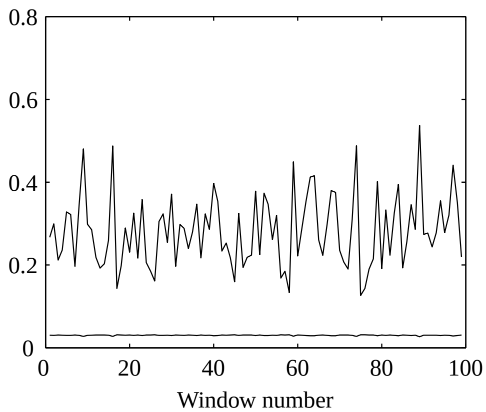

Figure 2: Estimate of ρ and its standard error over non-overlapping windows.

We can see that ρ deviates significantly from zero for all time windows and that there is no convergence from or to *iid*-ness. This leads us to conclude that the returns of asset 1 do indeed not satisfy the *iid* assumption, implying also rejection of the hypothesis agents having rational expectations in this economy.

4. Summary and further research

The conclusion in the previous section that returns are not *iid* calls for changes to the model. We consider several directions. First of all, we have derived closed form solutions for optimal portfolio paths in a non-*iid* context. We use these to improve the model. In addition, variations on $(\mathcal{M}^i_\tau, \mathcal{D}^i_\tau, \mathcal{E}^i_\tau)$ are investigated.

Another direction that is of interest is choosing other forms for the utility function. The method to derive closed form solutions to multi-period investment problems as presented by [10] and [11] allows to use very different forms for the utility function. It also allows for a general mean-variance analysis.

We can introduce consumption in the model and establish a link with, for instance, [3]. Another possible extension is the direction of prospect theory (see, for instance, [4] and [17]).

5. References

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