Second Generation Wavelet Transforms of Yield Curve Shifts

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Abstract

Wavelets defined without recourse to the Fourier transform provide a refined econometric basis for yield curve modeling.

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1 Introduction

We apply wavelet analysis to term structure evolution. Previously, the barrier to a using wavelet analysis as described in [4] was basic; a key element in the construction of wavelets was a basis of functions that remained invariant under both translations and dilations. This made the techniques of Fourier analysis available, but it was inappropriate for term structure data, which derives from a set of fixed income securities where maturities are clustered toward the short term in no neat pattern, where all maturities are constrained to an interval, and where translation invariant measure has no intrinsic economic meaning.

These types of constraints are commonly encountered in many applications, and several authors have addressed them. A particularly promising line of development is due to Wim Sweldon, whose own work was inspired by Donoho [2] and Lounsbery [3].

"Second generation" wavelet analysis [5] obtains central elements of wavelet analysis without the Fourier transform and in such a way that the analysis can be extended easily to irregularly spaced observations on a finite interval equipped with any one of many possible metrics. Moreover, Sweldens's approach provides a way to exploit the latitude in the choice of wavelet basis so that the optimal wavelet basis for modeling yield curve shifts might be used.

We may choose a portfolio, the market portfolio, say, and construct wavelets which, as term structure shifts, have no effect on portfolio value. Thus, when wavelet analysis begins with the raw data of a yield curve shift and rewrites it as a sum of a simple smooth curve and wavelets, it follows that the smooth curve represents exactly the same impact on the value of the portfolio as the input data. One may then study the joint distribution of the curve parameters, or one may study the wavelets, but in either case, one maintains an economic connection to the original data.

Wavelet or "multiresolution" analysis, begins with a linear spline interpolating a raw zero-coupon yield curve. The analysis generates a cascade of splines that approximate an interest rate movement, each spline having progressively fewer nodes, and a smoother though coarser approximation of the original data. Our splines all retain the same portfolio weighted average change and duration as the input shift.

At each step of the cascade, the information that is lost in going from a finer to a coarser approximation is stored in a wavelet. Thus, at each step, the magnitude of the wavelet indicates the quality of the approximation at the next level of smoothness and the support of the wavelet indicates the scale at which the adjustment is being made.

At the end of the multiresolution process, one is left with a simple polynomial approximation of the yield curve and a collection of wavelets. At each level one has separated the yield curve shift into locally dis-associated movement, captured by the wavelet, and coherent movement, captured by the yield curve shift approximation at the next level, and one has repeated this process as long as it was possible to do so. No information has been lost, and input data can be recovered completely by reversing the process.

With wavelet analysis we have a way to model term structure shifts with changes restricted to segments of the term structure on a full gamut of scales of resolution. Simple order statistics of the wavelet coefficients indicate where introducing more detail into the yield curve shift approximations will most efficiently improve the model. Thus, the location of wavelet coefficients indicate where new nodes might be most effectively added when refining the multiresolution model.

These techniques are applied to the U. S. Treasury yield curve in [1].

2 Second Generation Wavelets for Yield Curve Data

Assume input data is a series of observed yield curve shifts. A given observation, y_0, is represented by a vector $\{y_{0,k}\}_{k=0}^{n}$ which records the values of y_0 at maturities $\{x_k\}_{k=0}^{n}$.

The first step is to split the signal y_0 into two separate signals with the "lazy" transform. The result of applying the lazy transform, L, to y_0 is the pair of signals, the "upper" signal of "forgotten" samples and the "lower" signal of "subsamples."

$$L : y_0 \mapsto (l_{-1}, u_{-1}),$$

where $l_{-1,k} = y_{0,2k}$ and $u_{-1,k} = y_{0,2k+1}$ for $k = 0, 1, 2, \ldots$. Note that y_0 may be recovered from l_{-1} and u_{-1} simply by interleaving the two sequences.

The second, or "prediction", step operates on the upper signal, u_{-1}. The subsamples in l_{-1} are used to predict the data at the points x_{2j+1}. The differences between these predictions and the values of u_{-1} are then recorded as d_{-1}, the difference coefficients. To the extent that yields at the various maturities move together, the data $\{l_{-1}\}$ will form the basis for an accurate prediction and the coefficients $\{d_{-1}\}$ will be small.

There are several ways of formulating a prediction [6]. The method is chosen that best reflects the structure of the data. We will assume only that the yield curve is continuous. Therefore we expect to obtain an efficient encoding of yield curve fluctuation by using linear interpolation of the even values to predict the odd. [1] Thus

$$d_{-1,k} = u_{-1,k} - \frac{(x_{2k+2} - x_{2k+1})l_{-1,k} + (x_{2k+1} - x_{2k})l_{-1,k+1}}{x_{2k+2} - x_{2k}}.$$

and we write

$$P : (l_{-1}, u_{-1}) \mapsto d_{-1}.$$

The last operator, the "update" step, is applied to the lower signal, l_{-1}. If yield fluctuations correlate strongly across maturities which are close together, the difference, or detail, coefficients d_{-1} will be small, and the lower signal, though coarser, will remain a good estimator of the entire yield curve shift. On the other hand, there are global properties of the observed signal that an approximation should retain. The update step is designed to recoup information from the detail coefficients in such a way that, by minimal changes to $l_{-1,k}$ induced by neighboring $d_{-1,k}$, the updated signal retains specified global properties of y_0. These changes will be effected by an invertible linear transformation as explained below (Section 3.1). We symbolize the update by

$$U : (l_{-1}, d_{-1}) \mapsto y_{-1}.$$

The wavelet transform is schematized in Figure 1. The inversion may be diagramed by reversing the right pointing arrows and toggling the $+$ and $-$ signs. The final result is that the original signal y_0 can be recovered from the coarse approximation y_{-1} and the details d_{-1} that the coarse approximation leaves out.

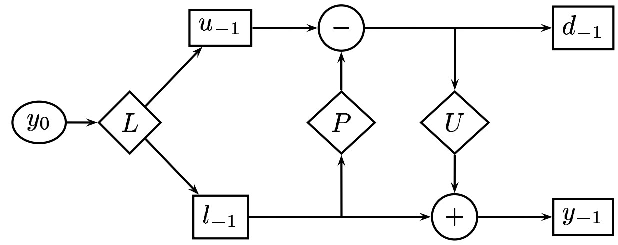

Figure 1: Scheme of the wavelet transform through lifting.

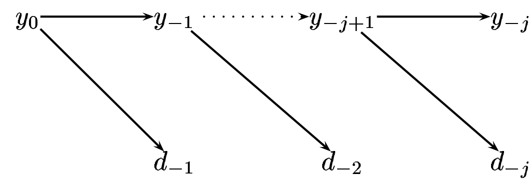

Figure 2: Multiresolution analysis of y_0.

The process that takes the input sequence y_0 to the pair of output sequences (d_{-1}, y_{-1}) can, in turn, be applied to the sequence y_{-1} itself. That is, the same process, suitably modified, would take y_{-1} as input and produce a new pair of sequences d_{-2}, y_{-2} as output. Thus, with our specific prediction and update routines, we would begin with a sequence y_0 defined on a mesh of points $\{x_k\}_{k=0}^{n}$ where n would be a perfect power of 2, say $n = 2^j$. We would then iterate our process j times and from y_0 produce a series of detail coefficients, $\{d_{-1}, d_{-2}, \ldots, d_{-j}\}$ and one last approximation y_{-j}, defined only at points x_0 and x_n. We view the totality of these iterations as a single transformation, calling it the wavelet transform or multiresolution analysis. We have illustrated the multiresolution process in Figure 2.

Since each of the steps $y_{-j} \mapsto (d_{-j-1}, y_{-j-1})$ is invertible, it follows that perfect recovery of the orginal sequence y_0 is possible if one knows only the details of the transforms, the nodes x_k, the detail coefficients d_{-k} and the final approximation y_{-j}.

3 The Wavelet Transform of Yield Curve Shifts

The wavelet transform analyzes the yield curve shift into two components; one is localized to maturities in a short interval while the other involves a broader range of maturities and preserves specified global properties of the input shift. These global properites are defined in terms of the effect the shift has on the value of a porfolio. Specifically, we represent a portfolio as a linear combination of delta functions

$$\sum_i b_i \delta_i \tag{1}$$

corresponding to a payment schedule $\{x_i\}$. The function δ_i is the point mass concentrated at x_i, and b_i is the current market value of the payment expected at time x_i. Thus the value of the entire portfolio may be expressed

$$\int_0^T \sum_i b_i \delta_i(x) dx,$$

where T is an upper bound for all maturities x_i. Thinking of expression (1) as a weight function for the integral inner product and letting $y(x)$ denote the yield curve shift at maturity x, we regard

$$\int_0^T y(x) \sum_i b_i \delta_i(x) dx \tag{2}$$

as (T times) the average shift. Suitably normalized, the first moment of this same function,

$$\int_0^T xy(x) \sum_i b_i \delta_i(x) dx \tag{3}$$

is the duration, or coefficient of the best linear approximation to the portfolio change in value brought on by y. In terms of these measures, then, a yield curve shift will be resolved into a shift that is localized and a shift that is smoother and effects the same average (2) and duration (3) change as the original.

3.1 Update details

The estimation of the global component of a given yield curve fluctuation is obtained first by subsampling and second by updating the subsamples. We begin with a basis of fundamental functions $\{\phi_{i,j}\}$, where $i = 0, -1, \ldots, -n$ and $j = 0, 1, \ldots, 2^{n+i}$. The finest level of nodes is chosen comensurate with the level of observation. For the first level, level $i = 0$, the functions $\phi_{0,j}$ are piecewise linear functions interpolating the delta funtions on the points of the observation mesh. For succeeding levels, the $\phi_{i,j}$ are linear interpolations of the delta function on the subsampled meshes.

In general, when the observed values are $\{s_j\}$, then the original (level 0) linearly interpolated approximation is given by the function

$$\sum_{j=0}^{2^n} s_{0,j} \phi_{0,j}. \tag{4}$$

The lazy transform provides the basis for the coarser (level -1) estimate simply by subsampling at the even indexed points. Interpolation predicts values at the omitted points.

$$\sum_{j=0}^{2^{n-1}} s_{-1,j} \phi_{-1,j}.$$

The integral of the yield curve shift against the weight of market portfolio cash flows is obtained by substituting (4) for y in (2).

$$\int_0^T \sum_{j=0}^{2^n} s_{0,j} \phi_{0,j} \sum_i b_i \delta_i(x) dx. \tag{5}$$

Computing (5) reduces to computing the integrals of the functions $\phi_{i,j}$ alone. Thus, set

$$M_{i,j}^{(0)} = \int_0^T \phi_{i,j}(x) \sum_k b_k \delta_k(x) dx,$$

and (5) becomes

$$\sum_j s_{0,j} M_{0,j}^{(0)}.$$

Similarly, the first moment of a yield curve's movement (3) may be expressed as a linear combination of the moments of the basis functions,

$$\sum_j s_{0,j} M_{0,j}^{(1)}.$$

where

$$M_{i,j}^{(1)} = \int_0^T x\phi_{i,j}(x) \sum_k b_k \delta_k(x) dx.$$

In order to retain the integral and moment for the coarser levels, it is required that the new coefficients $s_{-1,j}$ satisfy

$$\sum_j s_{-1,j} M_{-1,j}^{(i)} = \sum_j s_{0,j} M_{0,j}^{(i)}, \ i = 1, 2, \tag{6}$$

and we want the method used to derive the the $s_{-1,j}$ to be the same, regardless of particular shift at hand.

Since the nodes for level -1 form a subset of the level 0 nodes, the functions $\phi_{1,j}$ lie in the span of the functions $\phi_{0,j}$, and there exist coefficients $h_{-1,j,k}$ satisfying

$$\phi_{-1,j} = \sum_k h_{-1,j,k} \phi_{0,k}.$$

Thus

$$M_{-1,j}^{(i)} = \sum_k h_{-1,j,k} M_{0,k}^{(i)} \quad \text{for} \quad i = 0, 1,$$

and (6) is satisfied. It follows that we can define a wavelet transform with a (banded) linear update step

$$s_{-1,j} = s_{0,2j} + A_{-1,j-1} d_{-1,j-1} + B_{-1,j} d_{-1,j} \tag{7}$$

that satisfies (6) by choosing the kernel of the transform appropriately.

Suppose the wavelet transform maps a sequence $\{s_{0,j}\}$ to a sequence $s_{-1,j} = 0$ for all j, and suppose only the single detail coefficient $d_{-1,k} = 1$, the other detail coefficients equaling 0. It follows from (7) that

$$s_{0,2k} = -B_{-1,k} \text{ and } s_{0,2k+2} = -A_{-1,k}.$$

The sequence satisfying this requirement is obtained by evaluating

$$-B_{-1,k}\phi_{-1,k} + \phi_{0,2k+1} - A_{-1,k}\phi_{-1,k+1} \tag{8}$$

at the level 0 nodes. Since (8) maps to zero at level -1, its integral and first moment are also required to be 0. This leads to the linear equation

$$\begin{bmatrix} M^{(0)}_{-1,k} & M^{(0)}_{-1,k+1} \\ M^{(1)}_{-1,k} & M^{(1)}_{-1,k+1} \end{bmatrix} \begin{bmatrix} B_{-1,k} \\ A_{-1,k} \end{bmatrix} = \begin{bmatrix} M^{(0)}_{0,2k+1} \\ M^{(1)}_{0,2k+1} \end{bmatrix} \tag{9}$$

for each k. Thus the integrals and moments of the fundamental functions determine the coefficients $B_{-1,k}$ and $A_{-1,k}$, and these coefficients are used to calculate the update transformation of any yield curve shift.

Thus, since (9) is consistent, the update map may be taken in the banded form (7). The function (8) is known as a wavelet.

3.2 Multiresolution

The wavelet transform maps the yield curve described at $2^n + 1$ points to a coarser representation based on $2^{n-1} + 1$ points. Each successive yield curve representation is transformed until a representation based only on the original end points of the maturity interval is obtained. The final representation together with the wavelets generated along the way is called a “multiresolution” analysis of our finest scale representation.

That is, let $\psi_{i,j}$ be the wavelet (8) with coefficients A and B satisfying (9), notation suitably adapted. Each step in the multiresolution process amounts to rewriting the approximation at level $i+1$ as the sum of the approximation at level i and wavelets:

$$\sum s_{i+1,j}\phi_{i+1,j} = \sum s_{i,j}\phi_{i,j} + \sum d_{i,j}\psi_{i,j}.$$

Thus, our fine scale function at level 0 (4) is resolved to the sum

$$s_{-n,0}\phi_{-n,0} + s_{-n,1}\phi_{-n,1} + \sum_{j=1}^{n} \sum_{i=0}^{2^{n-j}-1} d_{-j,i}\psi_{-j,i}. \tag{10}$$

The first two terms of (10) constitute the linear curve shift having the same mean and duration change as the observed shift. That is, measured in terms of the mean and duration change described in (2) and (3), these two terms describe the best linear approximation to the yield curve shift. For example, if the yield curve shift were the paradigmatic parallel shift, then all the detail coefficients would be zero, and the shift would be entirely captured by the two coefficients $s_{-n,0}$ and $s_{-n,1}$ of (10).

The detail or wavelet coefficients indicate not only the extent to which the yield curve shift fails to be linear, but it also hints at the nature of the failure. In particular if a simple piecewise linear model accurately depicts yield curve movements, we expect most detail coefficients to be essentially zero.

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Notes

[1] The particular wavelet transfrom used here is an adaptation of the biorthogonal (2,2) Cohen-Daubechies-Feauveau transform. This adaptation uses ”second generation wavelets” or the ”lifting scheme”.