Invariance to scaling in formal concept analysis of data tables with fuzzy attributes

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Abstract

Filling a data table with fuzzy attributes (scaling) is subjective. What properties of scaling are important if we want to know whether two data tables filled by two experts lead to (almost) the same collections of extracted clusters in the sense of formal concept analysis? The paper provides two answers.

Keywords: formal concept analysis, fuzzy attribute, concept lattice, invariance to scaling

1. Introduction and problem setting

Tabular data describing objects and their attributes represents a basic form of data. Among the several methods for analysis of object-attribute data, formal concept analysis (FCA) is becoming increasingly popular, see [9, 8]. The main aim in FCA is to extract interesting clusters (called formal concepts) from tabular data. Formal concepts correspond to maximal rectangles in a data table.

Recently, formal concept analysis was extended to data with fuzzy attributes, see e.g. [1, 5, 11]. In this paper, we focus on a problem which is a common one in fuzzy logic modeling. Suppose we have two data tables, $\mathcal{T}_1$ and $\mathcal{T}_2$, with fuzzy attributes. $\mathcal{T}_1$ and $\mathcal{T}_2$ may be thought of as filled by expert 1 and expert 2. In filling the data table, expert i has to assign, for each object x and each attribute y, the truth degree $I_i(x,y)$ (table entry) to which x has attribute y ($i = 1,2$). The assignment of truth degrees (scaling) is subjective—while expert 1 may say, e.g. $I_1(x,y) = 0.7$, expert 2 may say $I_2(x,y) = 0.72$. Even a single expert, when filling a data table twice, need not be consistent in that he might assign two different, although very likely close, truth degrees when filling a table entry for the fist time and then for the second time. In FCA, we are interested in the clusters extracted from a data table. Therefore, it is a natural question to ask what is the relationship between the collections of clusters extracted from data tables filled by the two experts. In particular, we are interested if there are general rules of the form "if two input data tables are in such and such relationship then the corresponding collections of clusters are in such and such relationship".

The paper first recalls a result from [1, 2] saying that for particularly defined fuzzy similarity relations we have that the degree to which the input data tables are similar is less or equal to the degree to which the resulting collections of clusters are similar. Second, we introduce the notion of ordinal equivalence of data tables. Ordinal equivalence formalizes the intuitive notion of consistency between experts. We prove two theorems of a form "if two data tables are ordinally equivalent, then the resulting collections of clusters are (almost) isomorphic".

2. Preliminaries

Fuzzy logic and fuzzy sets As a set of truth degrees equipped with suitable operations (truth functions of logical connectives) we use a complete residuated lattice, i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". A common choice of $\mathbf{L}$ is a structure with $L = [0,1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$), Gödel: ($a \otimes b = \min(a,b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ else), Goguen (product): ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = \frac{b}{a}$ else). In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, we can take a finite subset $L \subseteq [0,1]$ that is closed under Łukasiewicz or Gödel operations. If we take $L = \{0,1\}$, we obtain this way the two-element Boolean algebra (structure of truth degrees of classical logic). Two boundary cases of hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization: $a^* = 1$ if $a = 1$, $a^* = 0$ else.

Having $\mathbf{L}$ as our structure of truth degrees, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, intersection of $\mathbf{L}$-sets $A,B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. For $a \in L$ and $A \in \mathbf{L}^U$, we define $\mathbf{L}$-sets $a \otimes A$ (a-multiple of A) and $a \rightarrow A$ (a-shift of A) by $(a \otimes A)(u) = a \otimes A(u)$, $(a \rightarrow A)(u) = a \rightarrow A(u)$ ($u \in U$). Given $A,B \in \mathbf{L}^U$, we define a subsethood degree $S(A,B) = \bigwedge_{u \in U} \big(A(u) \rightarrow B(u)\big)$, which generalizes the classical subsethood relation $\subseteq$. Described verbally, $S(A,B)$ represents the degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. Observe that $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. For $A \in \mathbf{L}^U$, we denote by ${}^1A$ the 1-cut of A, i.e. ${}^1A = \{u \in U \mid A(u) = 1\}$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in [3, 10].

Formal concept analysis of data with fuzzy attributes We suppose that we are given a complete residuated lattice $\mathbf{L}$. Let X and Y be sets of objects and attributes, respectively, I be a fuzzy relation between X and Y. That is, $I : X \times Y \rightarrow L$ assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x,y) \in L$ to which object x has attribute y. The triplet $\langle X,Y,I \rangle$ represents a data table with rows and columns corresponding to objects and attributes, and table entries containing degrees $I(x,y)$.

For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by $A^{\uparrow}(y) = \bigwedge_{x \in X}(A(x) \rightarrow I(x,y))$ and $B^{\downarrow}(x) = \bigwedge_{y \in Y}(B(y) \rightarrow I(x,y))$. Using basic rules of predicate fuzzy logic, $A^{\uparrow}(y)$ is the truth degree of "for each $x \in X$: if x belongs from A then x has y". Similarly for $B^{\downarrow}$. That is, $A^{\uparrow}$ is a fuzzy set of attributes common to all objects of A, and $B^{\downarrow}$ is a fuzzy set of objects sharing all attributes of B. The set

$$\mathcal{B}(X,Y,I) = \{\langle A,B \rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

of all fixpoints of $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ thus contains all pairs $\langle A,B \rangle$ such that A is the collection of all objects that have all the attributes of B, and B is the collection of all attributes that are shared by all the objects of A. Elements $\langle A,B \rangle \in \mathcal{B}(X,Y,I)$ will be called formal concepts of $\langle X,Y,I \rangle$; A and B are called the extent and intent of $\langle A,B \rangle$, respectively; $\mathcal{B}(X,Y,I)$ will be called a concept lattice of $\langle X,Y,I \rangle$. Both the extent A and the intent B are in general fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to intermediate degrees, not necessarily 0 and 1.

For $\langle A_1,B_1 \rangle, \langle A_2,B_2 \rangle \in \mathcal{B}(X,Y,I)$, put

$$\langle A_1,B_1 \rangle \leq \langle A_2,B_2 \rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1).$$

This defines a subconcept-superconcept hierarchy on $\mathcal{B}(X,Y,I)$. The structure of $\mathcal{B}(X,Y,I)$ is described by the so-called main theorem for fuzzy concept lattices (see [5, 11]). We only mention that $\mathcal{B}(X,Y,I)$ is under $\leq$ a complete lattice where infima and suprema are given by $\bigwedge_{j \in J} \langle A_j,B_j \rangle = \langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\downarrow\uparrow} \rangle$ and $\bigvee_{j \in J} \langle A_j,B_j \rangle = \langle (\bigcup_{j \in J} A_j)^{\uparrow\downarrow}, \bigcap_{j \in J} B_j \rangle$.

3. Invariance to scaling

We assume we have two data tables, $\langle X,Y,I_1 \rangle$ and $\langle X,Y,I_2 \rangle$ (for instance, filled with expert 1 and expert 2). As described in Section 1., we are interested in finding relationships between data tables which are likely to occur if two data tables are filled by two experts, and the corresponding relationships between the resulting concept lattices $\mathcal{B}(X,Y,I_1)$ and $\mathcal{B}(X,Y,I_2)$. If the corresponding entries of the two data tables are the same (expert 1 and expert 2 are fully consistent), i.e. $I_1(x,y) = I_2(x,y)$ for each $x \in X$ and $y \in Y$, then trivially, $\mathcal{B}(X,Y,I_1) = \mathcal{B}(X,Y,I_2)$. But what can we say about the relationship between $\mathcal{B}(X,Y,I_1)$ and $\mathcal{B}(X,Y,I_2)$ if expert 1 is not fully consistent with expert 2?

3.1. Data tables with close truth degrees

In [1], see [2] for extended version, we proved a result which gives a particular answer to our problem. Due to lack of space, we present the result without technical details. We introduced similarity relations (fuzzy equivalence relations) $\approx_{\mathcal{T}}$ on data tables and $\approx_{\mathcal{B}}$ on fuzzy concept lattices. The similarity relations are based on biresiduum $\leftrightarrow$ (defined for $a,b \in L$ by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$) and are based on the idea of close (logically equivalent) truth degrees. For example, for Łukasiewicz structure on $[0,1]$ we have $a \leftrightarrow b = 1 - |a-b|$, for product structure on $[0,1]$ we have $a \leftrightarrow b = \min(a/b, b/a)$ (where we put $a/0 = 0$), and for the Gödel structure on $[0,1]$ we have $a \leftrightarrow b = \min(a,b)$ pro $a \neq b$, $= 1$ pro $a = b$. $\langle X,Y,I_1 \rangle \approx_{\mathcal{T}} \langle X,Y,I_2 \rangle$ is the degree to which it is true that any two corresponding entries of $\langle X,Y,I_1 \rangle$ and $\langle X,Y,I_2 \rangle$ are close. $\mathcal{B}(X,Y,I_1) \approx_{\mathcal{B}} \mathcal{B}(X,Y,I_2)$ is the truth degree to which it is true that for each concept $\langle A_1,B_1 \rangle$ from $\mathcal{B}(X,Y,I_1)$ there is a similar concept $\langle A_2,B_2 \rangle$ from $\mathcal{B}(X,Y,I_2)$ and *vice versa*.

We proved that for for any data tables $\langle X,Y,I_1 \rangle$ and $\langle X,Y,I_2 \rangle$ we have

$$(\langle X,Y,I_1 \rangle \approx_{\mathcal{T}} \langle X,Y,I_2 \rangle) \leq (\mathcal{B}(X,Y,I_1) \approx_{\mathcal{B}} \mathcal{B}(X,Y,I_2)).$$

Put in words, this says that it is true (in degree 1) that if two data tables are similar then the corresponding concept lattices are similar as well. For a user, this is a desirable property. It says that the exact values of truth degrees do not matter (too much)—if the second data table is filled with truth degrees close to those of the first one, the resulting structures of extrated clusters are similar.

3.2. Ordinally equivalent data tables

First, a definition and some comments.

Definition Fuzzy sets A and B in U are said to be *ordinally equivalent*, in symbols $A \equiv B$, if for each $u, v \in U$ we have $A(u) \leq A(v)$ iff $B(u) \leq B(v)$. If, moreover, $A(u) = 1$ iff $B(u) = 1$ for each $u \in U$, we write $A \equiv_{\{1\}} B$.

Another way to express ordinal equivalence is the following. For a fuzzy set A in U, let $\leq_A$ be a binary relation on U defined by $u \leq_A v$ iff $A(u) \leq A(v)$. If A is interpreted as a fuzzy set of solutions representing the degree to which the particular solutions are good then $u \leq_A v$ means that v is preferred over u. Clearly, $\leq_A$ is a quasiorder and $A \equiv B$ iff $\leq_A$ coincides with $\leq_B$.

Particularly, for data tables $\langle X, Y, I_1\rangle$ and $\langle X, Y, I_2\rangle$ we have $I_1 \equiv_{\{1\}} I_2$ iff for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have

$$I_1(x_1, y_1) \leq I_1(x_2, y_2) \text{ iff } I_2(x_1, y_1) \leq I_2(x_2, y_2),$$

and $I_1(x, y) = 1$ iff $I_2(x, y) = 1$.

From the point of view of scaling (i.e. filling the data table by truth degrees), $I_1 \equiv I_2$ means that expert 1 and expert 2 agree on whether object x_1 has attribute y_1 to a higher degree than to which object x_2 has attribute y_2 (for each $x_1, x_2 \in X$, $y_1, y_2 \in Y$). $I_1 \equiv_{\{1\}} I_2$ means that, in addition to $I_1 \equiv I_2$, expert 1 and expert 2 agree on in which case an object x fully satisfies attribute y.

Example Consider the following data tables.

I_1	y_1	y_2	y_3	y_4
x_1	1	0.9	0.8	1
x_2	0	1	0.5	0.5
x_3	0.8	0.8	0.2	0.1

I_2	y_1	y_2	y_3	y_4
x_1	1	0.6	0.4	1
x_2	0	1	0.3	0.3
x_3	0.4	0.4	0.2	0.1

I_3	y_1	y_2	y_3	y_4
x_1	*0.9*	**0.8**	0.8	1
x_2	0	1	0.5	0.5
x_3	0.8	**0.9**	0.2	0.1

One can see that $I_1 \equiv_{\{1\}} I_2$. It is not true that $I_1 \equiv_{\{1\}} I_3$. Namely, $I_1(x_1, y_2) > I_1(x_3, y_2)$ but $I_3(x_1, y_2) < I_3(x_3, y_2)$ and so I_1 and I_3 are not ordinally equivalent. Moreover, $I_2(x_1, y_1) = 1$ and $I_3(x_1, y_1) = 0.9$, and so I_2 and I_3 do not agree on entries with 1's.

From now on, we assume that $\mathbf{L}$ is linearly ordered, that $a \otimes b = \min(a, b)$ for any $a, b \in L$, and X and Y are finite. Also, we use the following convention. For a mapping $h : L_1 \to L_2$ and a fuzzy set $A \in L_1^U$, we define a fuzzy set $h(A) \in L_2^U$ by $h(A)(u) = h(A(u))$.

Theorem 1 *Under the above assumptions, if $I_1 \equiv_{\{1\}} I_2$ then $\mathcal{B}(X, Y, I_1)$ and $\mathcal{B}(X, Y, I_2)$ are isomorphic under an isomorphism g such that for the corresponding formal fuzzy concepts $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle = g(A_1, B_1)$ we have $A_1 \equiv_{\{1\}} A_2$ and $B_1 \equiv_{\{1\}} B_2$.*

Proof. Denote $L_1 = I_1(X, Y) \cup \{1\}$, i.e. $a \in L_1$ if $a = I_1(x, y)$ for some $x \in X$ and $y \in Y$, or $a = 1$. Similarly, denote $L_2 = I_2(X, Y) \cup \{1\}$.

First, observe that for any $A \in L^X$ we have $A^{\uparrow_{I_1}}(y) \in L_1$ and $A^{\uparrow_{I_2}}(y) \in L_2$ for each $y \in Y$. Indeed, for any $x \in X$ we have either $A(x) \leq I_1(x, y)$ or $A(x) > I_1(x, y)$. In the former case, $A(x) \to I_1(x, y) = 1 \in L_1$, in the latter case, $A(x) \to I_1(x, y) = I_1(x, y) \in L_1$. Due to finiteness of X we have

$$A^{\uparrow_{I_1}}(y) = \bigwedge_{x \in X} A(x) \to I_1(x, y) = \min_{x \in X} A(x) \to I_1(x, y) \in L_1.$$

Similarly we get $A^{\uparrow_{I_2}}(y) \in L_2$. Moreover, one show analogously that for any $B \in L^Y$ we have $B^{\downarrow_{I_1}}(y) \in L_1$ and $B^{\downarrow_{I_2}}(y) \in L_2$ for each $x \in X$.

Therefore, we have both $\mathcal{B}(X, Y, I_1) \subseteq 2^{L_1^X \times L_1^Y}$ and $\mathcal{B}(X, Y, I_2) \subseteq 2^{L_2^X \times L_2^Y}$.

Let us define a mapping $h : L_1 \to L_2$ by $h(1) = 1$ and $h(I_1(x, y)) = I_2(x, y)$ for each $x \in X$, $y \in Y$. This is a correct definition: First, if $I_1(x, y) = 1$ then $I_1 \equiv_{\{1\}} I_2$ implies $I_2(x, y) = 1$. Second, if $I_1(x, y) = I_1(x', y')$ then again, $I_1 \equiv_{\{1\}} I_2$ implies $I_2(x, y) = I_2(x', y')$.

Now, both L_1 and L_2 are closed under the operations of $\mathbf{L}$ with the possible exception of 0 from L not necessarily belonging to L_1 or L_2 (easy to check). Equipped with the restrictions of the operations from $\mathbf{L}$ and having the least element of L_i as the nullary operation 0_i, L_i is a complete residuated lattice $\mathbf{L}_i$ ($i = 1, 2$). Furthermore, h is an isomorphism between $\mathbf{L}_1$ and $\mathbf{L}_2$. Namely, we easily see that $h(a \vee b) = h(\max(a, b)) = \max(h(a), h(b)) = h(a) \vee h(b)$ and similarly for $\wedge$. For $\to$, we distinguish two cases. Either $a \leq b$ and then $h(a) \leq h(b)$ from which we get $h(a \to b) = h(1) = 1 = h(a) \to h(b)$, or $a > b$ and then $h(a \to b) = h(b) = h(a) \to h(b)$. Since h is an isomorphism, it is a complete morphism (that is, h preserves arbitrary infima and suprema).

Now we have $I_2 = h(I_1)$. Using Theorem 3 of [4], we get that the mapping $\bar{h}(A, B) = \langle h(A), h(B)\rangle$ is a surjective homomorphism of $\mathcal{B}(X, Y, I_1)$ to $\mathcal{B}(X, Y, I_2)$. Since h is injective, we easily see that $\bar{h}$ is injective as well.

which is moreover injective. To be accurate, $\bar{h}$ is a morphism of $\mathcal{B}(X, Y, I_1)$ to $\mathcal{B}(X, Y, I_2)$ if we consider $\mathcal{B}(X, Y, I_i)$ as a fuzzy concept lattice over the structure $\mathbf{L}_i$ of truth degrees. However, we have seen above that $\mathcal{B}(X, Y, I_i)$ is the same no matter if we consider it over $\mathbf{L}_i$ or over $\mathbf{L}$.

Since h^{-1} is an isomorphism of $\mathbf{L}_2$ to $\mathbf{L}_1$, the corresponding $\overline{h^{-1}} : \mathcal{B}(X,Y,I_2) \to \mathcal{B}(X,Y,I_1)$ is again an injective morphism. Therefore, as $\overline{h}^{-1} = \overline{h^{-1}}$, $\overline{h}$ is in fact an isomorphism of $\mathcal{B}(X,Y,I_1)$ to $\mathcal{B}(X,Y,I_2)$. That fact that for the corresponding $\langle A_1,B_1\rangle \in \mathcal{B}(X,Y,I_1)$ and $\langle A_2,B_2\rangle = \overline{h}(A_1,B_1) \in \mathcal{B}(X,Y,I_2)$ we have $A_1 \equiv_{\{1\}} A_2$ and $B_1 \equiv_{\{1\}} B_2$ follows by a moment's reflection. Therefore, $g = \overline{h}$ is the required isomorphism. □

From the point of view of a user, Theorem 1 says that if expert 1 and expert 2 supply two data tables which are ordinally equivalent and agree on entires filled with 1's, the resulting collections of extracted clusters are, in fact, almost the same.

Next, we are going to consider what happens if we weaken the assumptions of Theorem 1 so that we assume only $I_1 \equiv I_2$. Requiring $I_1 \equiv_{\{1\}} I_2$ as above means that the two experts agree on what it means that an object x fully has an attribute y (i.e. $I_1(x,y) = 1$ iff $I_2(x,y) = 1$). However appealing such a requirement, it might be argued that it is not warranted. We need the following lemma.

Lemma 2 *For a data table $\langle X,Y,I\rangle$ with $I(x',y') = 1$ for some x',y', let*

$$I'(x,y) = \begin{cases} I(x,y) & \text{for } I(x,y) < 1 \\ a & \text{for } I(x,y) \end{cases}$$

for some $a < 1$ such that $I(x,y) < a$ for all x,y with $I(x,y) \neq 1$. Then there is a complete surjective homomorphism g of $\mathcal{B}(X,Y,I')$ to $\mathcal{B}(X,Y,I)$ such that each formal concept of $\mathcal{B}(X,Y,I)$ has at most two preimages from $\mathcal{B}(X,Y,I')$. In more detail, if ${}^1A \neq \emptyset$ and ${}^1B \neq \emptyset$ then $|g^{-1}(A,B)| = 2$, otherwise $|g^{-1}(A,B)| = 1$.

Proof. The proof is technically involved and thus omitted due to lack of space. □

Using Lemma 2, we can prove the following theorem in a way similar to the proof of Theorem 1 (proof omitted due to lack of space).

Theorem 3 *Under the above assumptions, suppose $I_1 \equiv I_2$, $I_1 \not\equiv_{\{1\}} I_2$, and let $I_1(x,y) = 1$ for some x,y. Then there is a complete morphism g of $\mathcal{B}(X,Y,I_1)$ to $\mathcal{B}(X,Y,I_2)$ such that*

(1) for $\langle A_1,B_1\rangle \in \mathcal{B}(X,Y,I_1)$ and $\langle A_2,B_2\rangle = g(A_1,B_1)$ we have $A_1 \equiv A_2$ and $B_1 \equiv B_2$;

(2) for each $\langle A_2,B_2\rangle \in \mathcal{B}(X,Y,I_2)$, if ${}^1A_2 \neq \emptyset$ and ${}^1B_2 \neq \emptyset$ then $|g^{-1}(A_2,B_2)| = 2$, and if ${}^1A_2 = \emptyset$ or ${}^1B_2 = \emptyset$ then $|g^{-1}(A_2,B_2)| = 1$.

From the point of view of a user, Theorem 3 says that if expert 1 and expert 2 supply two data tables which are ordinally equivalent (and need not agree on entires filled with 1's), the resulting collections of extracted clusters are, in fact, again very similar.

Note that for other structure of truth degrees, Theorems 1 and 3 do not hold.

4. Future research

Future research will be directed to results on scaling invariance w.r.t. so-called concept lattices with hedges [7] and w.r.t. attribute implications extracted from data tables with fuzzy attributes [6].

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