Neural Networks for Extracting Implied Risk-Neutral Probability Density Surface of Stock Index Options

Ing-Chyuan Wu

Department of Informatics, Fo Guang University

Abstract

How to extract the implied risk-neutral probability density surface from market prices of stock index options is an important research topic. However, not only the observable option prices are discrete, insufficient, and noisy, but also the implied probability distribution is nonstationary. Therefore, extracting an implied probability distribution accurately from option prices is not an easy task. To overcome the difficulties coming from market reality, a multilayer feedforward neural network is applied. Based on a theory derived by Breeden and Litzenberger, the risk-neutral probability density surface implicit in option prices is then extracted. Market prices of S&P 500 Index options, SPX, are used to investigate performances of the proposed method. Empirical tests show that the extracted risk-neutral probability density surface is consistent with market data accurately and predicts future option price structure precisely. Thus, the proposed neural network technique is suitable for a variety of financial applications.

Keywords: Neural networks, Ito process, Risk-neutral probability density surface, Risk-neutral valuation method.

1. Introduction

Exchange-listed and over-the-counter (OTC) options have played an increasingly important role in the field of modern finance. Among different kinds of options product, the stock index options is conspicuous for its heavy trading volume and versatile role-playing in financial markets.

The price of a listed option comes from the competitive pricing system. Thus, demand and supply for the option governs its market price. The main factor in deciding the market price of a stock index option is the expectation of investors on future distribution of stock index level. Therefore, market prices of stock index options imply the probability distribution of future stock index level. The implied information is important for investors who use the options as a speculation or hedging tool.

In the modern OTC options market, when one investment bank issues some kind of exotic option or tailored plain vanilla option, the bank often hedges its risk exposure in listed options market. Thus, the price of an OTC option is usually under the influence of listed option prices. The probability distribution of future stock index level implicit in listed option prices is crucial for investment banks that develop OTC options on the stock index. Other representative applications of the implied information are calculating the value-at-risk of one's options position and exploring the sentiment of investors.

For the broad range of applications in financial world, how to extract the implied risk-neutral probability density surface from market prices of stock index options has been an important research topic. However, not only the observable option prices are discrete, insufficient, and noisy, but also the implied probability distribution is nonstationary. Therefore, it is not an easy task to extracting the implied probability distribution accurately from option prices. Although some approaches [1, 4, 7, 10] have been provided by researchers and practitioners, no one is accepted as the standard method for extracting the implied probability distribution. This article demonstrates an accurate and reliable method for extracting the implied probability distribution by the use of a neural network.

2. Ito process and governing partial differential equation

Options on a stock index are often modeled as options on an asset that pays continuous dividend [2, 9]. Assume that the movement of stock index level follows an Ito process. The dynamics of stock index level at time τ , S_τ, can be described as

$$dS_\tau = (\mu(S_\tau,\tau) - q_\tau)S_\tau d\tau + \sigma(S_\tau,\tau)S_\tau dW_\tau$$

Where $\mu(S_\tau,\tau)$ is the expected rate of return, q_τ is the dividend yield, $\sigma(S_\tau,\tau)$ is the volatility function, and W_τ is the Wiener process. A Wiener process can be expressed as

$$dW_\tau = \varepsilon\sqrt{d\tau}\,, \qquad \varepsilon \sim N(0,1)$$

Where ε is a random value sampled from a standard normal distribution.

A portfolio consisting of the stock index and the option can be set up in such a way that there is no uncertainty about the value of the portfolio at any time instant. In a world without arbitrage opportunities, riskless portfolios must earn the risk-free interest rate. This makes it possible to obtain a partial differential equation that relates the option price to stock index level, time to maturity, continuously compounded risk-free interest rate, dividend yield, and volatility function of the stock index level as

$$\frac{\partial V(S_\tau,\tau)}{\partial \tau} + \frac{1}{2}(\sigma(S_\tau,\tau))^2 S_\tau{}^2 \frac{\partial^2 V(S_\tau,\tau)}{\partial S_\tau{}^2}$$
$$+ (r_\tau - q_\tau)S_\tau \frac{\partial V(S_\tau,\tau)}{\partial S_\tau} - r_\tau V(S_\tau,\tau) = 0$$

Where r_τ is the continuously compounded risk-free interest rate and $V(S_\tau,\tau)$ is the option price. Since the volatility function $\sigma(S_\tau,\tau)$ can not be observed directly, there is no way to derive the theoretical price that fits market price well by solving the governing partial differential equation.

3. Risk-neutral valuation method

Risk-neutral valuation method [6] is another popular technique for pricing stock index option. In the risk-neutral world, the price of a European option price is equal to its expected payoff, discounted at the risk-free interest rate. The obtained option price is correct in the risk-neutral world as well as other risk preference worlds. That is

$$V(S_\tau,\tau) = e^{-r(T-\tau)}\int_0^\infty V(S_T,T)\, f_\tau(S_T,T)\, dS_T$$

Where $f_\tau(S_T,T)$ is the ending risk-neutral probability density function estimated at time τ and $V(S_T,T)$ is the payoff function of European option at maturity. The payoff function for European call option with strike price *K* is

$$V(S_T,T) = \max[S_T - K, 0]$$

The option price $V(S_\tau,\tau)$ can be observed from the market. The ending risk-neutral probability density function $f_\tau(S_T,T)$ is the implied probability distribution that matches the option price.

4. Theory for extracting implied risk-neutral probability density surface

If the asset price S_τ follows an Ito process, the ending risk-neutral probability density function $f_\tau(S_T,T)$ is not limited to the lognormal distribution. Analytic formula for the general $f_\tau(S_T,T)$ does not exist. Breeden and Litzenberger [3] provided the theoretical foundation for extracting the implied ending risk-neutral probability density function from observed market prices of European options.

Assume that the European option price $V(K,T)$ is a continuous and smooth function of the strike price and maturity. The formula for extracting implied risk-neutral probability density surface is

$$f_\tau(S_T,T)$$
$$= e^{r(T-\tau)} \frac{\partial^2 V(S_\tau,\tau,\sigma(S_\tau,\tau),q_\tau,r_\tau,K,T)}{\partial K^2}\Big|_{K=S_T}$$

The equation shows that $f_\tau(S_T,T)$ equals the second partial derivative of the European option price function with respect to the strike price at $K = S_T$, grown at the risk-free interest rate. The whole implied risk-neutral probability density surface, $f_\tau(S_T,T)$ for $\tau \le T \le T_{\max}$ and $0 < S_T < \infty$, can be obtained from the equation by varying independent variable T.

5. Neural networks for extracting implied risk-neutral probability density surface

Neural networks are renowned for the capability of learning nonlinear function mapping from noisy sampling data and the smooth generalization nature, making them a potential tool in options valuation [8]. A multilayer feedforward neural network can be trained to learn $\sigma(K,T)$. The neural network has the strike price and maturity as inputs and implied volatility of Black-Scholes-Merton formula [2] as the output. The trained neural network represents a continuous function mapping of implied volatility with respect to strike price and maturity.

After successful training, the neural network is able to provide an accurate implied volatility value given a strike price and a maturity. This neural network is used together with the Black-Scholes-Merton formula to evaluate an option price. For an option contract, the price is determined by the Black-Scholes-Merton formula with the implied volatility generated by the neural network. With the hybrid framework, the option price is a continuous and smooth function of the strike price and maturity.

The five-point interpolating polynomial method [5] for the second derivative is used in the study. The formula is

$$V_2'' = \frac{1}{24h^2}(-2V_0 + 32V_1 - 60V_2 + 32V_3 - 2V_4)$$

Where V is the option price and h is the step for the strike price.

6. Data sources

Quote prices of S&P 500 Index options, SPX, in 1996 are used to investigate performances of the proposed method. The SPX option contracts are European style and expire on the Saturday immediately following the third Friday of the expiration month. The last trading day is ordinarily on Thursday. The trading period of general trading day is 8:30 A.M. to 3:15 P.M. Chicago Time.

There are six expiration months for SPX, which are three near-term months followed by three additional months from the March quarterly cycle. The CBOE uses the opening reported sales price on Friday to calculate the exercise-set value. Therefore, when calculating the time to maturity, we use 8:30 A.M. of the third Friday as the maturity. The calendar day is used to calculate the time to maturity.

Market data of SPX comes from the Berkeley Options Database (BODB), which includes a complete record of trading activity on the CBOE. The weekly dividend yields for S&P 500 Index portfolio are collected from the *S&P 500 Information Bulletin*. The continuously compound interest rate of 3-month Treasury bill is used as the risk-free interest rate.

7. Empirical tests

Figure 1 plots the risk-neutral probability density functions for six expiration dates at 09:00:00 A.M. on July 22, 1996. Which are 25 days, 60 days, 88 days, 151 days, 242 days, and 333 days ahead of the observation time.

As a probability density function, each curve in Figure 1 must be nonnegative everywhere and the probability must sum to one. It is clear that each probability density function is positive everywhere. As shown in Table 1, the probability of each curve sums near to one for the given asset price range. The market expects the probability of 0.9988 that S&P 500 index level will fall between 510 and 700, 25 days later. Similarly, the market expects the probability of 0.9977 that S&P 500 index level will fall between 300 and 920, 333 days later. The information implied in option prices that are observed at 09:00:00 A.M. on July 22, 1996 is successfully extracted.

The in-sample test has been performed to check the accuracy of the extracted risk-neutral probability distribution. The extracted risk-neutral probability distribution is used to price SPX call options that are observed at 09:00:00 A.M. on July 22, 1996. Each option price is obtained by risk-neutral valuation method with the corresponding extracted risk-neutral probability density function. Table 2 lists the obtained option prices and their differences with market option prices. The option price difference ranges between -$0.86 and $0.36. The average of absolute option price differences is $0.22.

To check how reliable the extracted risk-neutral probability density surface can do in dynamic hedging. The out-of-sample test has been performed. The risk-neutral probability density surface extracted on day t is used to price options on days $t+1$ and $t+7$. For the purpose, the risk-neutral probability density surface extracted on July 22, 1996 is used to price options on July 23, 1996 and July 29, 1996. On July 23, the time to maturities for expiration dates of August, September, October, December, March, and June are 24 days, 59 days, 87 days, 150 days, 241 days, and 332 days, respectively. On July 29, the time to maturities for expiration dates of August, September, October, December, March, and June are 18 days, 53 days, 81 days, 144 days, 235 days, and 326 days, respectively. The data are also shown in Table 2.

In-sample and out-of-sample results show that the price errors from neural network risk neutral probability density surface (NN-RNPDS) are far smaller than that from Black-Scholes-Merton model. The extracted risk-neutral probability density surface accurately matches market data and is capable of predicting option prices one-day and one-week ahead well.

8. Conclusions

The proposed neural network technique extracts the implied risk-neutral probability density surface accurately and has little reliance with the assumption of underlying asset price dynamics. Hence, the method is helpful in dynamic hedging, exotics pricing, value at risk, and other financial applications.

The extracted risk-neutral probability density surface can continuously vary with market conditions. Therefore, the technique is suitable for a real-time environment.

9. References

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Table 1 Asset price range and probability at 09:00:00 A.M. on July 22, 1996

Date	August	September	October	December	March	June
Range	[510, 700]	[470, 750]	[440, 780]	[390, 820]	[340, 880]	[300, 920]
Prob.	0.9988	0.9980	0.9982	0.9978	0.9978	0.9977

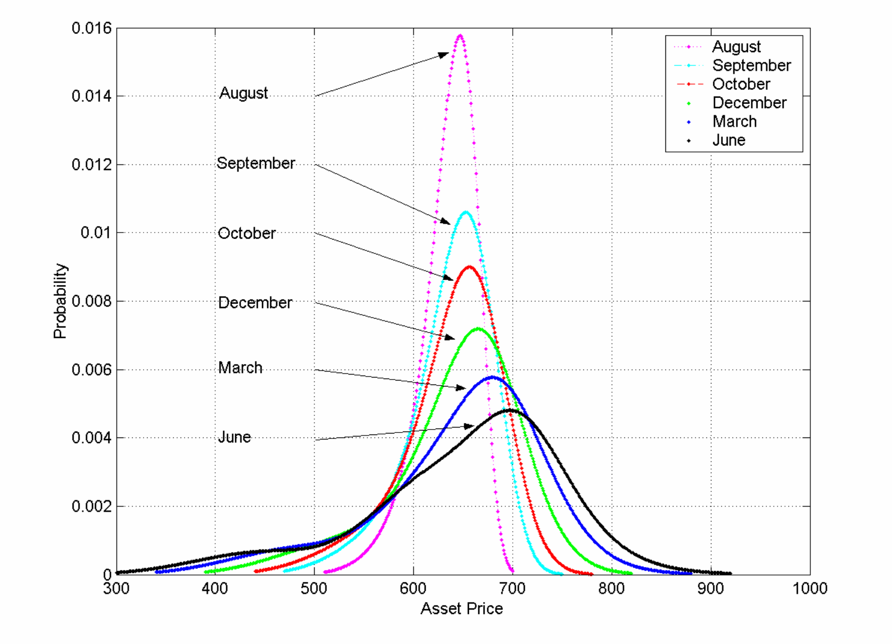

Figure 1 Risk-neutral probability distribution implied in SPX call at 09:00:00 A.M. on July 22, 1996

Table 2 In-sample and out-of-sample test results

Date	B-S-M Price Error Range $	B-S-M Average Price Error $	NN-RNPDS Price Error Range $	NN-RNPDS Average Price Error $
07/22	[-5.60, 5.84]	2.05	[-0.86, 0.36]	0.22
07/23	[-6.30, 5.37]	2.18	[-1.41, -0.13]	0.86
07/29	[-5.50, 4.23]	1.92	[-1.38, 0.85]	0.40