Graded Inference Theory in Fuzzy Logic

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ABSTRACT

The logical treatment of the theory of approximate reasoning is of increasing importance both in fuzzy logic and in artificial intelligence, and the most fundamental form of approximate reasoning seems to be the generalized modus ponens (briefly, GMP) where a major premise $A \to B$ and a minor premise A^* similar to A are given and a conclusion B^* is required to be deducted where A, A^*, B, B^* are abstract logic propositions. Zadeh first investigated a GMP-Like problem where A, A^* and B, B^* were supposed to be fuzzy subsets on universes X and Y respectively, and the compositional Rule of Inference (briefly, CRI) was proposed. But it seems that CRI is a method of numerical computing rather than logic deduction. The first author of the present paper tried to endow fuzzy reasoning with logic framework and the so called Triple I method was proposed based on the logical concept of tautologies. The aim of the present paper is to go a step further and propose a graded inference theory with the characteristic that logic deductions being graded by means of certain pseudo-metric on the set of propositions. The soundness and complete theorem is established.

Keywords: Łukasiewicz system, R_0 system, fuzzy logic, classical logic, graded inference, pseudo-metric.

1. PRELIMINARIES

Let $S = \{p_1, p_2, \cdots\}$ be a countable set, elements of S are atomic propositions (briefly, atoms). Let $F(S)$ be the free algebra of type $(\neg, \to)$ generated by S where $\neg$ and $\to$ are unary and binary operations respectively, elements of $F(S)$ are popositions (or formulae). Suppose that A, B are propositions of $F(S)$, the following deduction rule shown by Eq.(1) is called modus ponens (briefly, MP):

$$\begin{array}{ccc} A & \longrightarrow & B \\ A & & \\ \hline & & B \end{array} \tag{1}$$

Suppose that A^* is a proposition different from but similary to A, then the following Eq.(2) is called generalized modus ponens (briefly, GMP) [5]:

$$\begin{array}{ccc} A & \longrightarrow & B \\ A^* & & \\ \hline & & B^* \end{array} \tag{2}$$

Zadeh first investigated GMP-like problem where A, A^* and B, B^* were supposed to be fuzzy subsets $A(x), A^*(x)$ and $B(y), B^*(y)$ on universes X and Y respectively, and the Compositional Rule of Inference (briefly, CRI) was proposed and the conclusion B^* can be calculated by the following formula[9]:

$$B^*(y) = \sup\{A^*(x) \wedge R_z(A(x), B(y)) | x \in X\}, \quad y \in Y, \tag{3}$$

where R_z is Zadeh's implication operator such that

$$R_Z(a, b) = (1 - a) \vee (a \wedge b), \quad a, b \in [0, 1]. \tag{4}$$

The first author of the present paper improved, in certain sence, the CRI method by proposing the so called Triple I method of which the conclusion B^* is the smallest fuzzy subset of Y satisfying the condition that

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) = 1, \quad x \in X, y \in Y. \tag{5}$$

where $\to$ is any implication operator having an adjoint t-norm $\otimes$ (i.e., $a \otimes b \leq c$ if and only if $a \leq b \to c$), and B^* can then be computed by the formula [1-4]:

$$B^*(y) = \sup\{A^*(x) \otimes (A(x) \to B(y)) | x \in X\}, \quad y \in Y, \tag{6}$$

and it is pointed out that Eq.(6) is, in certain sense, better than Eq.(3). But both Eq.(3) and Eq.(6) are computation oriented rather than logical deduction oriented eventhough the latter can be brought into logic semantics[6]. The aim of the present paper is try to establish a graded inference theory in fuzzy logic as well as in classical propositional logic.

2.FUZZY LOGIC AND CLASSIC LOGIC

As mentioned above, a proposition (or formula) is an element of $F(S)$. Consider the unit inteval [0,1], define on [0,1] a unary operation $\neg : [0,1] \to [0,1]$ by letting $\neg\alpha = 1 - \alpha (0 \leq \alpha \leq 1)$, and a binary operation $\to : [0,1]^2 \to [0,1]$ by letting

$$a \to b = (1 - a + b) \wedge 1, \quad a, b \in [0,1] \tag{7}$$

or

$$a \to b = \begin{cases} 1, & a \leq b, \\ (1-a) \vee b, & a > b, \end{cases} \quad a, b \in [0,1], \tag{8}$$

then $([0,1], \neg, \to)$ is an algebra of type $(\neg, \to)$. Operations $\to$ defined by Eq.(7) is called the Łukasiewicz implication operator and denoted $\to_L$ and that defined by Eq.(8) is called the R_0 implication operator and denoted $\to_0$[8]. A mapping $f : F(S) \to [0,1]$ is said to be a valuation of $F(S)$ is v is a homomorphism of type $(\neg, \to)$, i.e.,

$$v(\neg A) = \neg v(A), \qquad A \in F(S), \tag{9}$$

$$v(A \to B) = v(A) \to v(B), \qquad A, B \in F(S), \tag{10}$$

where $\to$ can be $\to_L$ or $\to_0$. The set consisting of all valuations of $F(S)$ will be deroted by Ω_L or Ω_0 whenever $\to_L$ or $\to_0$ are used respectively. Moreover, if we reduce $[0,1]$ to be $\{0,1\}$ and define on $\{0,1\}, \neg 0 = 1, \neg 1 = 0$ and $a \to b = 0$ if and only if $a = 1$ and $b = 0$, then $(\{0,1\}, \neg, \to)$ is also an algebra of type $(\neg, \to)$, and the set consisting of all valuations of $F(S)$ will be denoted by Ω^*. Assume in the following Ω is Ω_L, Ω_0 or Ω^*. Let A be a proposition of $F(S), A$ is said to be a tautology if $\forall v \in \Omega, v(A) = 1$. A is said to be a contradiction if $\forall v \in \Omega, v(A) = 0$. Suppose that $\Gamma \subseteq F(S)$ and $v \in \Omega$, if $v(A) = 1$ holds for every member A of Γ, then we say that v is a model of Γ. Suppose that $B \in F(S)$, if $v(B) = 1$ holds for every model v of Γ, then we say that Γ implies B semantically and denoted $\Gamma \models B$. Then B is a tautology means that $\emptyset \models B$, briefly, $\models$ B. These are logical semantics and it is classic logical semantics if $\Omega = \Omega^*$, and they are fuzzy logical semantics if $\Omega = \Omega_L$ or $\Omega = \Omega_0$. The following are corresponding logical syntactics.

For classical propsitional logic system L, there are three axiom schems:

$(L1)$ $A \to (B \to A)$

$(L2)$ $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

$(L3)$ $(\neg A \to \neg B) \to (B \to A)$.

For Łukasiewicz propositional logic system L_{uk}, there are four axiom schemes:

$(Lu1)$ $A \to (B \to A)$.

$(Lu2)$ $(A \to B) \to ((B \to C) \to (A \to c))$

$(Lu3)$ $((A \to B) \to B) \to ((B \to A) \to A)$

$(Lu4)$ $(\neg A \to \neg B) \to (B \to A)$.

And for R_0 propositional logic system $\mathcal{L}^*$, there are ten axiom schemes:

(L^*1) $A \to (B \to A \wedge B)$

(L^*2) $(\neg A \to \neg B) \to (B \to A)$

(L^*3) $(A \to (B \to C)) \to (B \to (A \to C))$

(L^*4) $(B \to C) \to ((A \to B) \to (A \to C))$

(L^*5) $A \to \neg\neg A$

(L^*6) $A \to A \vee B$

(L^*7) $A \vee B \to B \vee A$

(L^*8) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

(L^*9) $(A \wedge B \to C) \to (A \to C) \vee (B \to C)$

(L^*10) $(A \to B) \vee ((A \to B) \to \neg A \vee B)$

where $A \vee B = \neg((((A \to (A \to B)) \to A) \to A) \to \neg((A \to B) \to B))$. $A \wedge B = \neg(\neg A \vee \neg B)$

There is only one deduction rule, i.e., modus ponens as shown in Eq.(1). Suppose that $\Gamma \subseteq F(S), B \in F(S)$, we say that Γ syntactically implies B and denoted $\Gamma \vdash B$ if there is a sequence $A_1, \cdots, A_n$ with $A_n = B$ and $\forall i \leq n, A_i \in \Gamma$ or A_i is an axiom, or there exist $j, k < i$ such that A_i can be deducted from A_j and A_k by using MP. In case $\Gamma = \emptyset$ we say that B is a theorem if $\Gamma \vdash B$ holds, and denoted $\vdash B$.Γ is said to be consistent if $\Gamma \vdash B$ and $\Gamma \vdash \neg B$ do not hold simutaneously [11]. The following theorems have been proven:

Theorem A In classical propositional logic system L,

$$\Gamma \vdash A \text{ if and only if } \Gamma \models A. \tag{11}$$

Theorem B [1] In Łukasiewicz propositional logic system L_{uk},

$$\Gamma \vdash A \text{ if and only if } \Gamma \models A, \Gamma \text{ is finite.} \tag{12}$$

Theorem C [11]In R_0 porpositional logic system $\mathcal{L}^*$,

$$\Gamma \vdash A \text{ if and only if } \Gamma \models A, \Gamma \text{ is consistant.} \tag{13}$$

Theorem A is a well known result in classical propositional logic system. Theorem C first appeared in [2] and [11] proved it in a more elaborately way. These theorems feature the soundness and adequacy of corresponding logic systems.
In the following we use the symbol $A \sim B$ to denote the fact that both $\vdash A \to B$ and $\vdash B \to A$ hold, and $A \sim B$ means that A and B are provably equivalent. Similarly, $A \approx B$ means both $A \to B$ and $B \to A$ are tautologies and we say in this case A and B are logically equivalent.

3. PSEUDO-METRICS ON $F(S)$

Let $A = A(p_1, \cdots, p_n)$ be any proposition of $F(S)$, where $p_1, \cdots, p_n$ are the atoms appeared in A, then A induces a mapping $\overline{A} : [0,1]^n \to [0,1]$ as follows: $\overline{A}(x_1, \cdots, x_n)$ is an expression obtained by connecting $x_1, \cdots, x_n$ using $\neg$ and $\to$ in the same way as $p_1, \cdots, p_n$ being connected using $\neg$ and $\to$ to form $A(p_1, \cdots, p_n)$ e.g., if $A(p_1, p_2, p_3) = (p_1 \to \neg p_2) \to p_3$, then $\overline{A}(x_1, x_2, x_3) = (x_1 \to (1 - x_2)) \to x_3$ where $\to$ can be chosen to be $\to_L$ or $\to_0$ corresponding to which one of the systems L_{uk} or $\mathcal{L}^*$ is considered. In [10] a pseudo-metric was defined on $F(S)$ in L_{uk} as follows:

$$\rho(A, B) = \int_0^1 \cdots \int_0^1 \mid \overline{A}(x_1, \cdots, x_n) - \overline{B}(x_1, \cdots, x_n) \mid dx_1 \cdots dx_n, \tag{14}$$

where $p_1, \cdots, p_n$ are the atoms simutaneously appeared in A and B, and it was pointed out in [10] that Eq.(14) can be used to compute $\rho(A, B)$ even A and B contains different atoms because one can, in provably equivalent sense, use extensions A^* and B^* instead of A and B correspondingly such that A^* and B^* contain one and the same group of atoms. e.g., if $A = p_1, B = p_2$, then let $A^* = (p_1 \vee p_2 \to p_1 \vee p_2) \to p_1, B^* = (p_1 \vee p_2 \to p_1 \vee p_2) \to p_2$ and we see that $A^* \sim A, B^* \sim B$. Hence it is not necessary to list all the variables and Eq.(14) can be abbreviated as

$$\rho(A, B) = \int_\Delta |\overline{A} - \overline{B}| dw, \tag{15}$$

where $dw = dx_1 \cdots dx_n$ and $\Delta = [0,1]^n$. Moreover, it was proved in [10] that

$$\rho(A, B) = 1 - \int_\Delta (\overline{A} \to \overline{B}) \wedge (\overline{B} \to \overline{A}) dw. \tag{16}$$

And it is exactly in this way, one can define pseudo-metric by using Eq.(16) on $F(S)$ in the logic system $\mathcal{L}^*$.

Lastly, the concept of truth degrees of formulae of $F(S)$ was introduced in the classical propositional system L in [9], and a pseudo-metric was defined as follows:

$$d(A, B) = 1 - \tau(A \to B) \wedge \tau(B \to A), \quad A, B \in F(S). \tag{17}$$

Now we see that a pseudo-metric can always be defined on $F(S)$ no matter which of the logic systems $L_{uk}, \mathcal{L}^*$ or L is under consideration.

4. A GRADED INFERENCE THEORY

Assume in the subseguent that $F(S)$ is the free algebra of type $(\neg, \to)$ generated by $S = \{p_1, p_2, \cdots\}$ and ρ is a pseudo-metric on $F(S)$. Let $w = w_1 \cdots w_n$ be a sequence in $F(S)$, then we use $l(w)$ to denote the length n of w.

DEFINITION 1. Let Γ be a subset of $F(S), \mathcal{A}$ be the set consisting of all axioms, $\Gamma^* = \Gamma \cup \mathcal{A}$, and $w = w_1 \cdots w_n$ be a sequence in $F(S)$. Define $d(\Gamma, w)$, the degree to which w deviates from Γ, by induction on $l(w)$ as follows:
(i) If $l(w) = 1, d(\Gamma, w) = \rho(w, \Gamma^*)$.
(ii) If $w = w_1 \cdots w_n$ $(n > 1)$, then

$$\begin{aligned} d(\Gamma, w) \quad = \quad & \rho(w_n, \Gamma^*) \wedge (\bigvee\{d(\Gamma, w_1 \cdots w_k) | k < n\} \vee \\ & \bigvee\{\rho(w_i, \Gamma^*) \vee \rho(w_j, \Gamma^*) | i, j < n, \\ & \{w_i, w_j) \vdash w_n\}). \end{aligned}$$

DEFINITION 2. Let Γ be a subset of $F(S), B \in F(S)$. Then the error $\text{Eded}(\Gamma, B)$ with which Γ Syntactically implies B is defined as follows:

$$\begin{aligned} Eded(\Gamma, B) \quad = \quad & \bigwedge\{d(\Gamma, w) | w = w_1 \cdots w_n \\ & \text{is a consistent sequence in} \\ & F(S), w_n = B\} \end{aligned} \tag{18}$$

DEFINITION 3. Let Γ be a subset of $F(S), B \in F(S)$. Then the error $\text{Econ}(\Gamma, B)$ with which Γ semantically implies B is defined as follows:

$$\begin{aligned} Econ(\Gamma, B) \quad = \quad & \bigwedge\{\rho(\Sigma, \Gamma^*) | \Sigma \\ & \text{is a finite consistent subset of} F(S), \\ & \Sigma \models B\}, \end{aligned} \tag{19}$$

where $\Gamma^* = \Gamma \cup \mathcal{A}, \rho(\Sigma, \Gamma^*) = \vee\{\rho(A, \Gamma^*) | A \in \Sigma\}$.

THEOREM (The Soundness nad Adequacy Theorem) Let Γ be a subset of $F(S), B \in F(S)$, then

$$Eded(\Gamma, B) = Econ(\Gamma, B). \tag{20}$$

PROOF. First prove

$$Eded(\Gamma, B) \leq Econ(\Gamma, B). \tag{21}$$

In fact, suppose that $Econ(\Gamma, B) < \varepsilon$, then it follows form Eq.(19) that there is a finite consistent subset Σ of $F(S)$ such that

$$\rho(\Sigma, \Gamma^*) < \varepsilon \quad \text{and} \quad \Sigma \models B. \tag{22}$$

Since Σ is finite and consistent it follows from Theorem A-Theorem C that $\Sigma \vdash B$. Hence there exists a proof sequence $w = w_1 \cdots w_n$ such that $w_k \in \Sigma(k \leq n-1)$ and $w_n = B$. Now there exist $i, j < n$ such that $\{w_i, w_j\} \vdash w_n$, and it follows from $\rho(\Sigma, \Gamma^*) < \varepsilon$ and Definition 1 that $d(\Gamma, w) < \varepsilon$. Hence it follows from Eq.(18) that $Eded(\Gamma, B) < \varepsilon$. This proves Eq.(21). Conversely, it remains to prove

$$Econ(\Gamma, B) \leq Eded(\Gamma, B). \tag{23}$$

In fact, suppose that $Eded(\Gamma, B) < \varepsilon$, then it follows from Eq.(18) that there exists a consistet sequence $w = w_1 \cdots w_n$ such that

$$d(\Gamma, w) < \varepsilon \quad \text{and} \quad w_n = B. \tag{24}$$

It follows from Eq.(24) and Definition 1 that
(i) $\rho(B, \Gamma^*) = \rho(w_n, \Gamma^*) < \varepsilon$, choose Σ to be $\{B\}$, then $\Sigma \models B, \Sigma$ is consistent and hence it follows from Eq.(19) that $Econ(\Gamma, B) < \varepsilon$, or
(ii) there are $i, j < n$ such that $\{w_i, w_j\} \vdash B$(hence $\{w_i, w_j\} \models B$) and $\rho(w_i, \Gamma^*) < \varepsilon, \rho(w_j, \Gamma^*) < \varepsilon$, choose Σ to be $\{w_i, w_j\}$, then $\Sigma \models B, \Sigma$ is consistent and it follows from Eq.(19) that $Econ(\Gamma, B) < \varepsilon$. This proves Eq.(23) and the proof of the main Theorem is completed.

5.CONCLUSIONS

A unified graded inference theory is established which is suitable for Łukasiewicz fuzzy logic, R_0 fuzzy logic as well as for classical propositional logic. The method of the present paper can be used to propose a graded inference theory in several predicate logics, and it will be completed in forthcoming papers.

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