Options with underlying asset driven by a fractional brownian motion: crossing barriers estimates

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Abstract: *The aim of this paper is to analyse the problem of an investor that holds options where the underlying asset obeys a fractional brownian motion. The task is to give an estimate about the probability that the asset crosses barriers because this can drive investment policies. The result can be extended also for hedging and for portfolio management.*

1 Introduction

The typical problem of an agent taking position in a financial market is joint to the evaluation of the price of the underlying assets of her options. Great oscillations can change dramatically the returns at the expiration dates. It is interesting to know the most probable time which either the lowest or the highest values are reached at. Excessive upwards or downwards can also give rise to the suspension of the asset from the market and also automatic trading systems, portfolio rebalancing problems and hedging strategies may be triggered when the option crosses some barrier. The study of the behavior of options relies on the study of the motion of the underlying assets. Fractional brownian motion (fBM) has been detected around a wide variety of financial data: stock market indexes, shares, foreign exchange currencies [1, 7, 12]. Moreover it was proven that the so called Ito type fractional Black-Scholes market has no arbitrage and is complete [10, 13]. The Black and Scholes formula has been extended in order to calculate the fair price of an option when the underlying asses is driven by a geometric fractional brownian motion (gfBM) [22, 23]. So the problem of the evaluation of an option over an asset that crosses some barrier can be studied through the problem of the crossing of thresholds by a fBM [6, 14]. This is an intriguing problem that relies over bounds that connect fBM to brownian motion, scaled brownian motion, filtered brownian motion [11, 21]. The aim of this paper is thus to explore the open problem about the case of non linear barriers [5] through the results about the transformation of the fractal degree of a process when filtered through a suitable function. The brownian motion is always included as a particular case.

2 Main theoretical tools

This section aims to provide estimates of the probability of the first passage time for fBM with respect to the level of the barrier and to the time.

2.1 Probability bounds for fBM

Let $B_H(t)$ be the fBM with parameter H. For $1/2 < H < 1$, $B_H(t)$ exhibits long term persistence and memory, whilst anti-persistence is characterized by $0 < H < 1/2$. The particular case $H = 1/2$ corresponds to brownian motion [16, 20]. Let $X(t)$ be either a a fBM $B_H(t)$ or a scaled brownian motion $S_H(t) = B(t^{2H})$, and define

$$A(X, c) := \sup\{X(t) - ct \,|\, t \geq 0\}.$$

The following results hold [3]:

Lemma 1 $\forall\ u \geq 0$ *it results*

$$P(A(B_H, c) > u) \leq P(A(S_H, c) > u),\ 1/2 \leq H < 1;$$

$$P(A(B_H, c) > u) \geq P(A(S_H, c) > u),\ 0 \leq H < 1/2.$$

The distribution of the suprema of a brownian motion with drift obeys the law [2]

$$P(A(B_{1/2}, c) > u) = \exp(-2cu), \qquad \forall\, u \geq 0.$$

An analogous result provides bounds for the probability that a fBM with drift crosses a fixed level u in the scaled brownian motion case [3]:

$$P(A(S_H, c) > u) \geq \exp(-2au^{2-2H}),$$
$$\forall\, u \geq 0,\ a = \frac{1}{2}(c/H)^{2H}\Big(\frac{1}{1-H}\Big)^{2-2H} \qquad (1)$$

2.2 First return time probability

Let T be the first return time of a fBM. Then

$$P(T) \sim T^{H-2} \qquad (2)$$

Strictly speaking, for a continuous-time fBM, the distribution of the first return time is not well defined in the sense that a finite time interval may contain infinite many crossing points with the zero axis and the above relation is valid in the limit of finite cutoffs that are always present in actual experiments due to the discrete nature of any measure apparatus [4, 17].

2.3 Nonlinear Transformations of Fractionally Integrated Processes

Long term memory properties depend strictly on the ordering of data. As an examples operations like shuffle of data destroy the correlation structure. However nonlinear transformations of long memory processes through functions for which the Hermite rank J can be calculated allow to give results over the fractal measure of the resulting process [5].

Theorem 2 *Let X_t an $I(d)$ process of order d ($H = d + 1/2$), let $g(\cdot) : R \to R$ a function with Hermite degree equal to J*

1. *if $-1/2 < d \leq 0$ then $g(X_t)$ is a short term memory process $I(0)$.*
2. *if $0 < d < 1/2$ then $g(X_t)$ is $I(\bar{d})$, $\bar{d} = max\{0, (d - .5)J + .5\}$*

We recall that $g(X_t) = X_t^3$ has Hermite rank $J = 1$; $g(X_t) = X_t^2$ and $g(X_t) = X_t^4$ have both Hermite rank $J = 2$.

2.4 Fractional option pricing

Let us consider an option which underlying asset follows a geometric fractional brownian motion (gfBM) evolution. We have to introduce the fractional Black Scholes model. In this setting, the discounted stock price is given by the Wick Ito Skorohod equation

$$\delta S_t = S_t \mu(t)\mathrm{d}t + S_t \sigma \mathrm{d}B_t \qquad (3)$$

where B is a fBM under the probability measure P and δ represents the divergence Malliavin operator. The solution of (3) is

$$S_t = S_0 \exp\Big(\int_0^t (\mu(s) - \sigma^2 H s^{2H-1})\mathrm{d}s + \sigma B_t\Big) \quad (4)$$

The Fractional Black-Scholes formula states that the price at every time $t \in [0, T]$ of an European call option with strike price K and maturity T is given by [13, 15]

$$C(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)}N(d2) \qquad (5)$$

where

$$d_{1,2} = \frac{ln(S_t/K) + r(T-t) \pm \sigma^2/2(T^{2H} - t^{2h})}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

Thus the mapping of S_t into $C(t, S_t)$ provides a direct relationship among the dynamics of S_t and $C(t, S_t)$.

3 Barriers

We are going to study the probability that S_t crosses some barrier. The corresponding conclusions over C must be deduced numerically, due to its nonlinearity. In the stationary case (4) becomes

$$S_t = S_0 exp(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H})$$

Let us consider a barrier $b(t)$. We are interested in determining the probability that $S(t) > b(t)$, w.r.t. the time and without. Let us consider the following cases:

1. If $b(t) = exp((\sigma c + \mu)t - .5\sigma^2 t^{2H})$ and $u = (-ln(S_0))/\sigma)$ then the problem is reduced to the results on the crossing of barriers by a fBM and the estimates provided in Lemma 1 and section 2.2 can be used.

2. Let J be the Hermite rank of $g(\cdot)$. Keeping the due attention to the monotone region of $g(\cdot)$ the problem of $B_H > b(t)$ can be solved through the study of $g(B_H) > g(b(t))$. If $1/2 < H < 1$ and $\bar{d} = (d - .5)J + .5 > 0$ then $g(B_H)$ is a fBM with $\bar{H} = \bar{d} + 1/2$. The results of Lemma 1 can be used if $g(b(t)) = ct + u$. The result (2) always applies.

   (a) $g(X_t) = X_t^3$. Then $b(t)$ must satisfy the equation $(-log(b(t)) + \mu t - \frac{1}{2}\sigma^2 t^{2H} + u)^3 = -ct + v$, i.e. $b(t) = exp(\mu t - \frac{1}{2}\sigma^2 t^{2H} + u - (-ct + v)^{1/3})$. This transformation can keep the long memory property for each fBM B_H with $H > 1/2$.

   (b) $g(X_t) = X_t^2$, $g(X_t) = X_t^4$, $J = 2$. In order to get a long term memory process $H > 3/4$.

   (c) $g(X_t)$ has an Herminte rank $J \geq 3$. In order to get a long term memory process $H > 1/2 + 1/3$. The highiest the Hermite rank, the highiest H.

4 Numerical results

The above methods can be used in order to provide the probability of crossing barriers if the underlying asset obeys a gfBM. Thus the first step towards the application of such analysis is the detection of gfBM. The theory about the speculative bubbles due to endogenous causes can provide a useful hint [8]. First of all the logarithm of data must be considered for the analysis in the case in which the magnitude of the crash is proportional to the price, as it happens for the NASDAQ crash of April 2000 [9]. Moreover a polynomial approximation w.r.t. the time to crash $y(t) = exp(A + B(t_c - t)^m)$ has been well assessed by the theory and tested on data. The values for the parameters $A = 9.36$, $B = -1.60$, $m = 0.26$ and t_c corresponding to April 11th, 2000 can be estimated by the minimum least squares method [18]. After having detrended the logarithm of time series it results that $H = 0.51$ since Jan. 1st, 1997 till Dec. 15th, 1998, then $H = 0.46$ till Dec. 8th, 1999 and $H = 0.43$ till the crash. A numerical example of the use of probability bounds for the crossing of linear barriers can be found in [19]. If the underlying S_t is compatible

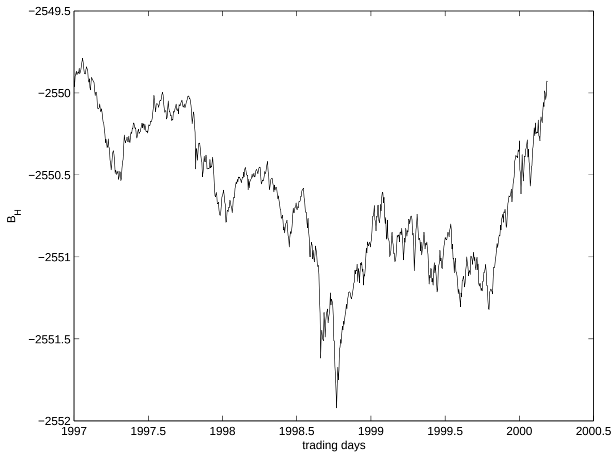

Figure 1: Plot of B_H given by (6). S_t is the NASDAQ index since Jan. 1st, 1997 till April 11th, 2000

with the hypothesis of gfBM then

$$log(S_t) - log(S_0) - \mu t + \frac{1}{2}\sigma^2 t^{2H} \simeq \sigma B_H(t) \quad (6)$$

This hypothesis is satisfied for S_t given by the NASDAQ index during the raise of the bubble that started since the beginnings of 1997 and ended in the large crash of April 2000. Equation (6) holds for $H \simeq 0.44$, that belongs to the range of the H estimated on the detrended $log(S_t)$. The change of the value can be explained also through the effect of the adding of the function $-log(S_0) - \mu t + \frac{1}{2}\sigma^2 t^{2H}$ [16]. The detection of the gfBM allows to perform numerical estimates of the probability to cross barriers listed on case 1.

5 Conclusions

The paper explores the problem of an option to cross some barriers. The analysis is performed through the study of the underlying. This paper shows how to use the results over the supremum of fractional brownian motion and shifted brownian motion with drift in order to perform a first step towards the probability description for the crossing of linear and non linear barriers. Moreover other theoretical results on fractional brownian motion are used and lead to results that give an estimate of the probability of the first return. The new task of non linear barrier can also give interesting insights over the structure of the market and take into account investor's set of preferences. The obtained results can be used in order to drive investments decisions and fund management and for the setting of automatic trading softwares.

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