Optical Vector Logic Theorem-Proving

Jonathan Westphal,[1] John Caulfield,[2] Jim Hardy,[1] Lei Qian.[2]

[1]COLD Project, Business & Technology Center, 1651 Alvin Ricken Drive, ISU Research Park, Campus Box 8044, Idaho State University, Pocatello, ID 83201-8044
[2]Department of Physics, Fisk University, 1000 17th Ave. N., Nashville TN 37208 and COLD Project, Fisk University, Nashville TN 37208

1. Abstract

Vector Logic converts the processes of logical argument into geometrical transformations. There has been prior work in diagrammatic logic and the geometrical representations of logic [1]. Unlike some of these systems, Vector Logic has an optical implementation. By reducing theorem-proving to image translations, this implementation enables optical theorem-proving.

Keywords: “Vector Logic”, “optics”, “theorem-proving”.

2. Introduction

Given a set of premises, logic determines other propositions which the premises imply. These implications can be stated as theorems. Vector logic is a geometrical approach for finding and proving such implications. Some other and different geometrical representations have been previously discussed, but automatic optical theorem-proving has not been discussed in the past.

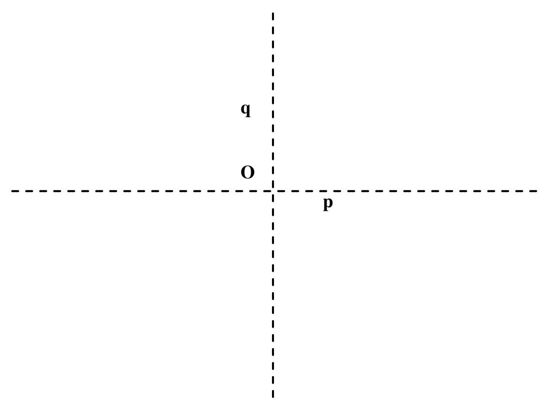
Vector Logic is a means of combining propositions represented by the premises of an argument into a new proposition which they jointly imply, in the same way that vectors add to form a resultant vector. To explain this, we will introduce the geometric representation and some of the basic concepts of Vector Logic. This will enable us to illustrate Vector Logic by proving a number of familiar theorems in the two-literal case. To show that Vector Logic allows the solution of much harder problems, we then show how it can be used to simplify logical functions in multiple variables. Finally, we offer some conclusions supported by this analysis.

This paper is intended only to introduce Vector Logic and its advantages, including the optical ones. It does not pretend to offer a detailed description of all of the procedures and rules or proofs of their soundness, nor of the completeness of the Vector Logic system. Papers to accomplish both tasks will be submitted subsequently [1].

3. The Geometry of Truth

For simplicity, let us assume that there are only two variables or literals: p and q. We can represent their truth in a 2D pq plane as shown in **Figure 1**.

Figure 1
The pq Plane



The origin, O, plays a very significant role in Vector Logic. It represents all truths, or “the True”, as Frege was prone to call it. A vector from O to p, for example, asserts that p is true. The truth-value of p can lie in either direction from the origin (1 or -1), however, so that the truth of p and the falsity of p are independently represented, the latter as a vector from O to –p.

Vector Logic theorem-proving amounts to the geometric *tracking* of the truth *via* vector operations as various premises are introduced.

The vector from O to (p, q) represents the assertion of p∨q. The vector from (p, q) to O represents the assertion of p∧q. In the same way, the dual of p (which is p itself) is represented as a vector from p to O.

Vector Logic theorem-proving can be viewed in either of two fully equivalent ways: summation of vectors in a fixed truth plane, or translation of the truth (the origin O) according to each premise. We will illustrate both with the following trivial problem:

p **v** q
-p
―――
?

Figure 2 uses vector summation to arrive at the theorem which is the conditional associated with the argument or sequent given above: [(p **v** q)**-**p] → q, or Disjunctive Syllogism as the corresponding rule of argument is known.

Figure 2
Disjunctive Syllogism (DS)

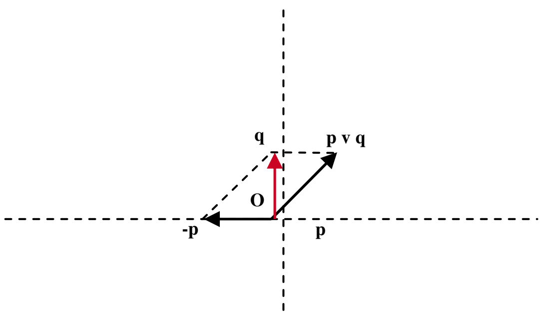
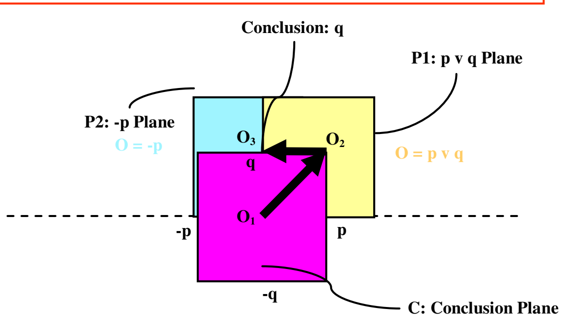


Figure 3 shows how successive translation from the origin of the conclusion plane to new origins after the translations, the final origin giving the conclusion on the original or "conclusion plane". It is as though (p **v** q) and (**-**p) move the truth (the origin) to the point q.

Figure 3
DS

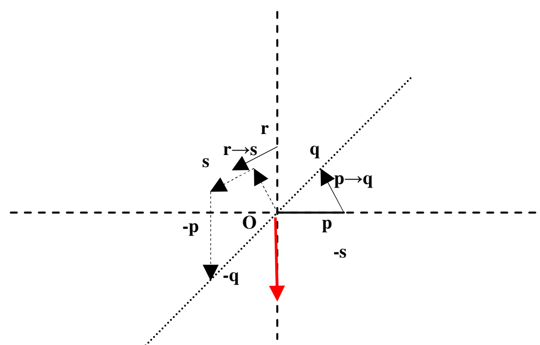


4. Proving a More and a Less Familiar Theorem by Means of Vector Logic

Vector Logic can prove theorems, and it is instructive to see how how simple and intuitive the proofs become when they are cast in Vector Logic form. The examples chosen so far for discussing and illustrating Vector Logic have been theorems that were either obvious or well-known. Vector Logic can be used to prove either new or nontrivial theorems as well. We offer here two representative examples. The first is a straight argument to a different conclusion. The second involves an argument to a redundancy (q **v** r), which can then be deleted.

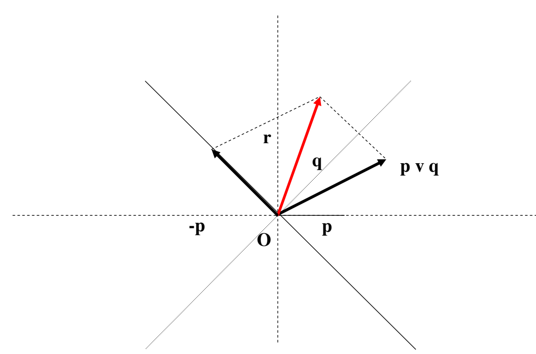
Example 1
Destructive Dilemma

Figure 4
Destructive Dilemma
{[(p→ q)(r→s)](-q **v** –s)} → (-p **v** -r)



Example 2
Consensus Theorem

Figure 5
Consensus Theorem
[(p **v** q)(q **v** r)(-p **v** r)] ↔ [(p **v** q)(-p **v** r)]

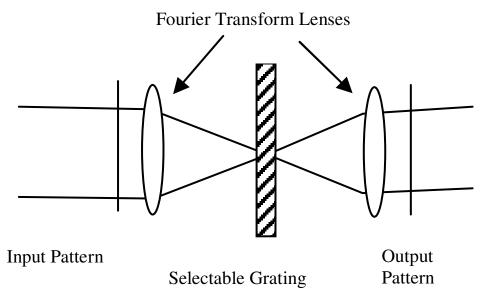


5. Optical Implementation

Once again, what we seek is a constructive existence-proof of the possibility of doing logic optically. The previous section showed that logic on literals and truth-functions of literals could be mapped into a problem of schema image-shifting. Fourier optics offers one way to do this.

Consider the traditional double Fourier transform system shown in **Figure 6**.

Figure 6
A selectable grating such as an electronically tuned hologram can deflect the input image to implement any selected logical operation.



But for a reversal of the X and Y axes, this is an imaging system. Let us redefine the coordinates in the output plane as $\xi = -x$ and $\eta = -y$. Then this system converts f (x, y) into f (ξ, η). In the intermediate plane there is a Fourier transform F (u, v).

Now suppose we insert a phase mask

$$M(u, v) = \exp[2\pi i(u\xi_0 + v\eta_0)]$$

in the u-v plane. The effect will be to shift f (ξ, η) to f ($\xi - \xi_0$, $\eta - \eta_0$).

It is possible to place both of the required masks for the two literals (p, and q) in the same plane provided both are within the depth of field of the Fourier transform. However, practical phase-only Spatial Light Modulators (SLMs) and their mounts are too thick to make that convenient. So it becomes necessary to cascade two of the Fourier imaging systems shown in **Figure 6** with one SLM in each Fourier plane.

The speed depends on the frame rate of the SLMs. Liquid crystal SLMs, for instance, can be pushed to millisecond frame rates. To work even faster we will require different and more expensive materials.

6. Conclusions

Optical Vector Logic offers at least the following advantages over other automated theorem-proving methods. It is simple, intuitive, and engages our visual reasoning capabilities. As a result it is an attractive way to represent logic. The analogy with the procedure of proofs by inspection with Venn diagrams is appropriate, but a better analogy might be with the method of the Karnaugh map. Unlike these methods, however, Vector Logic does not break down for more than five or six variables. A third analogy may be even more appropriate. In particle physics, keeping track of what interacts with what to produce what was very difficult until Feynman introduced his justly celebrated Feynman diagrams. These diagrams changed the way we think about physics and made particle physics visual and intuitive, allowing the solution of hard as well as simple problems. In all three cases (Venn diagrams, Karnaugh maps and Feynman diagrams) the diagrams make manifest facts which are already known but hard to visualize.

1. Translation of images is easy for optics, making optical theorem proving possible for the first time.
2. Vector Logic has already proved powerful in very difficult cases of theorem-proving.

Optical scientists, metamathematicians, metalogicians, and even metaphysicians may find Vector Logic and its optical implementation interesting.

7. Acknowledgements

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8. References

[1] Further information, along with a bibliography of related research is available at http://cold.isu.edu. See also:

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[2] Gottlob Frege, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Halle a. S.: Louis Nebert. Translated as *Concept Script, a formal language of pure thought modelled upon that of arithmetic*, by S. Bauer-Mengelberg in J. vanHeijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Cambridge, MA: Harvard University Press, 1967. It should be noted that Frege's wiring-like diagrams in the *Begriffschrift* introduce a diagrammatic element into his logic.