# YinYang Bipolar Lattice and Bipolar L-Fuzzy Sets

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**Abstract.** The notions of bipolar poset, bipolar lattice, and bipolar strictness are proposed which lead to the notion of YinYang bipolar L-sets, bipolar L-fuzzy sets, and dynamic modus ponens. Notably, dynamic modus ponens builds a bridge from a linear, static, unipolar, and closed world to a non-linear, dynamic, bipolar, and open world of equilibria, quasi-equilibria, and non-equilibria for bipolar knowledge representation and open world inference.

**Keywords:** *Bipolar Poset; Bipolar Lattice; Dynamic Modus Ponens; Bipolar L-Fuzzy Sets*

## 1 Introduction

Classical lattice-based logics [1,2,3] are unipolar systems that can not be directly used for the representation and visualization of bipolar equilibria in micro- or macrocosms for bipolar information fusion in decision, optimization, and coordination [4-7]. Yinyang bipolar logics and relations [4,5,6] present a holistic approach to information and decision by focusing on the direct representation and visualization of bipolar equilibria and quasi- or fuzzy-equilibria.

This work extends the bipolar logical spaces in [5,6] and generalizes classical lattices to YinYang bipolar lattices and L-fuzzy sets [2,3] to bipolar L-fuzzy sets. It is shown that bipolar sets, bipolar lattices, bipolar logic, bipolar equilibrium relations, and bipolar modus ponens build a bridge from a linear, static, and closed world to a non-linear, dynamic, and open world of equilibria or quasi-equilibria, that provide a effective means for bipolar information fusion, visualization, and dynamic inference.

## 2 Bipolar Lattice and Bipolar L-Sets

### 2.1 Bipolar Poset

According to the ancient Chinese YinYang theory, every matter has a negative side and a positive side. The two sides form a bipolar equilibrium, non-equilibrium, or quasi-equilibrium (or fuzzy equilibrium). If we consider a positive pole (such as positive electric charge, action force, positive ion, cooperation, positive causality, or positive number, .., etc) as a **positive element**, a negative counterpart as a **negative element**, and an element with no polarity (e.g. 0) as **a neutral element**, we have

Definition 1. Let $e^+$ be any type of positive or neutral element and $e^- = -e^+$ be any type of negative or neutral element; we call a partially ordered set $\mathbf{P^+ = \{e^+\}}$ a **positive partially ordered set (positive poset)** and a partially ordered set $\mathbf{P^- = \{e^-\}}$ a **negative partially ordered set (negative poset).** Both positive posets and negative posets are called **unipolar posets**. Two unipolar posets $P^-$ and $P^+$ are called bijective **negations to each other if,** $\forall p^+ \in P^+$, we have $p^- = -p^+ \in P^-$, and $\forall p^- \in P^-$, we have $p^+ = -p^- \in P^+$. A unipolar poset is **crisp** if all its elements are numerical or linguistic crisp values; a unipolar poset is **fuzzy** if its elements are fuzzy values.

Definition 2. **A bipolar partially ordered set (bipolar poset) S** is any subset of the Cartesian product $P = P_1^- \times P_2^+$ or $P = P_2^+ \times P_1^-$, where $P_1^-$ is a negative poset and $P_2^+$ is a positive poset, and $\forall$ $(x,y),(u,v)\in P$, we have the **bipolar Comparison** defined on the absolute values of the poles as

$$(x,y)\geq\geq(u,v), \text{ iff } |x|\geq|u| \text{ and } |y|\geq|v|. \quad (2.1)$$

We follow the convention for unipolar and bipolar comparison: (1) if $\Theta$ is used as **a unipolar comparison operator,** $\Theta\in\{=, \leq, \geq, <, >\}$; and (2) if $\Theta$ is used as **a bipolar comparison operator,** $\Theta\in\{= =, \leq\leq, \geq\geq, \leq\geq, \geq\leq, <<, >>, <>, ><, <\leq, >\geq, <\geq, >\leq, \leq<, \geq>, \leq>, \geq>\}$.

We call $S_1 \subseteq P_1^- \times P_2^+$ an **NP-bipolar poset** and $S_2 \subseteq P_2^+ \times P_1^-$ **a PN-bipolar poset**. Evidently, $S_1$ and $S_2$ can be obtained by rotating each other for 180 degrees. We denote this rotation with $S_1=-S_2$ or $S_2=-S_1$. If $S_1= -S_2$ we call the two bipolar posets **bipolar negations**.

It should be remarked that bipolar posets are proposed for characterizing bipolar equilibria and quasi- or non-equilibria (e.g. bipolar disorder). For some bipolar equilibria the only thing that matters is polarity and left or right makes no difference. Many equilibria in social sciences belong to this category. We say this kind of bipolar equilibria **LR-insensitive**. For some other bipolar equilibria not only does polarity matters, left or right position also matters. Many equilibria in physical sciences belong to this category. This kind of equilibria is said **LR-sensitive.** Evidently, bipolar mental disorder is **LR-insensitive.** This paper focuses on LR-insensitive NP-type bipolar sets.

### 2.2 Bipolar Lattice

Definition 3. **An NP bipolar lattice (blattice) B** is a quadruplet $(B, \oplus, \&, \otimes)$, where B is an NP bipolar poset and, $\forall(x,y),(u,v)\in B$, there is a **bipolar least upper bound (blub)**, a **bipolar greatest lower bound (bglb), and** a **cross-pole greatest lower bound (cglb)** as:

$$\mathbf{blub((x,y),(u,v))} \equiv (x,y) \oplus (u,v) \equiv (-(|x|\vee|u|),|y|\vee|v|) \equiv (-\max(|x|,|u|),\max(|y|,|v|)); \quad (2.2)$$

$$\mathbf{bglb((x,y),(u,v))} \equiv (x,y) \,\&\, (u,v) \equiv (-(|x|\wedge|u|),|y|\wedge|v|)); \quad (2.3)$$

$$\mathbf{cglb((x,y),(u,v))} \equiv (x,y)\otimes(u,v) \equiv (-(|x|\wedge|v|)\vee(|y|\wedge|u|),(|x|\wedge|u|\vee|y|\wedge|v|)). \quad (2.4)$$

In the above definition, $\oplus$ is a ***bipolar disjunctive or T-Conorm;*** & is a ***bipolar parallel conjunctive or T-norm;*** and $\otimes$ is a ***cross-pole serial conjunctive or T-norm.*** Similarly, **a PN-bipolar lattice** $B_2$ is the **bipolar negation of an NP-bipolar lattice** $B_1$ denoted as $B_2 = -B_1$. Since a bipolar variable is a vector instead of a scalar, it is natural to adapt vector operators for bipolar logical operations. The notational adaptations are necessary to avoid confusions on bipolar and unipolar connectives.

It can be shown that any bipolar lattice B is **a distributive structure** , That is, $\forall(x,y),(u,v)\in B$, we have

**Parallel Distributivity:**

$$(a,b)\&((c,d)\oplus(e,f))\equiv((a,b)\&(c,d))\oplus((a,b)\&(e,f)); \quad (2.5a)$$

$$(a,b)\oplus((c,d)\&(e,f))\equiv((a,b)\oplus(c,d))\&((b)\oplus(e,f)); \quad (2.5b)$$

**Serial Distributivity:**

$$(a,b)\otimes((c,d)\oplus(e,f))\equiv((a,b)\otimes(c,d))\oplus((a,b)\otimes(e,f)). \quad (2.6)$$

Definition 4. **A** bipolar lattice B (crisp or fuzzy) is **bounded** if it has both a unique minimal element denoted (0,0) and a unique maximal element denoted (-1,1). A bounded bipolar lattice B is **complemented** if, $\forall(x,y)\in B$, we have the **bipolar complement** $\neg(x,y)\in B$.

Although a bounded unipolar lattice is complemented, a bounded bipolar lattice is not necessarily complemented. A bounded and complemented NP bipolar lattice B (crisp or fuzzy) can be denoted as $(B,\equiv,\oplus,\otimes,\&,-,\neg,\Rightarrow)$ with $-$, $\neg$ and $\Rightarrow$ defined as:

***Negation:*** $-(x,y)\equiv(-1,0)\otimes(x,y)\equiv(-y,-x)$; (2.7a)

***Complement:*** $\neg(x,y)\equiv(\neg x,\neg y)\equiv(-1-x,1-y)$; (2.7b)

***Implication:*** $(x,y)\Rightarrow(u,v)\equiv(x\Rightarrow u,y\Rightarrow v)\equiv(\neg x\vee u,\neg y\vee v)$. (2.8)

It is evident that every bounded and complemented NP bipolar lattice B must be a subset of $B_F=[-1,0]\times[0,1]$. On bounded lattices, the non-linear conjunctive or bipolar T-norm $\otimes$ can be $\otimes_\wedge$, $\otimes_\times$, $\otimes_\Delta$, or any other T-norm with different granularity:

$$(x,y)\otimes_\wedge(u,v)\equiv(-\max(|x|\wedge|v|,|y|\wedge|u|),\max(|x|\wedge|u|,y\wedge v)); \quad (2.9a)$$

$$(x,y)\otimes_\times(u,v)\equiv(-\max(|x|\times|v|,|y|\times|u|),\max(|x|\times|u|,y\times v)); \quad (2.9b)$$

$$(x,y)\otimes_\Delta(u,v)\equiv(-\max(|x|\Delta|v|,|y|\Delta|u|),\max(|x|\Delta|u|,y\Delta v)). \quad (2.9c)$$

and the linear conjunctive & or T-norm can be either $\&_\wedge$, $\&_\times$, $\&_\Delta$, or any other linear T-norm.

Definition 5. A **bipolar L-set $B=(B^-,B^+)$ in X to a bipolar lattice $B_L$** is a bipolar equilibrium function or variable $B:X\Rightarrow B_L$. If $B_L$ is a bipolar crisp lattice we call B a **bipolar L-crisp set**; if $B_L$ is a bipolar fuzzy lattice we call B a **bipolar L-fuzzy set**; we use the terms "B-set," "B-fuzzy set," or "equilibrium variable" alternatively denoted as $B=(B^-,B^+)$, (x,y), or $(\phi^-,\phi^+)$.

### 2.3 Illustrations and Observations

**A bipolar lattice is different from a traditional lattice.** A bipolar lattice B can be considered as a combination of two lattices - a linear bipolar lattice $(B,\oplus,\&)$ and a non-linear bipolar lattice $(B,\oplus,\otimes)$. The linear part can be considered as an isomorphic structure of a traditional lattice $(B,\vee,\wedge)$. The non-linear part provides a crucial extension for modeling bipolar interaction among bipolar (equilibrium) sets and relations. Although we have blub $\geq\geq$ bglb for any pair of elements in a bipolar lattice, we do not have the relation blub $\geq\geq$ cglb. For instance, given (x,y)=(u,v)=(-1,0), we have blub((x,y),(u,v)) = (-1,0); bglb((x,y),(u,v)) = (-1,0); and cglb((x,y),(u,v)) = (0,1). Based on this observation, we can say that a bipolar lattice is not simply a traditional lattice due to the non-linear bipolar conjunctive $\otimes$.

To illustrate, the Hasse diagrams of some bounded and unbounded NP bipolar lattices are shown in Fig. 1. Where $B_1=\{-1,0\}\times\{0,1\}$ and $B_2=\{-2,-1,0\}\times\{0,1,2\}$ are crisp bipolar lattices; $B_F=[-1,0]\times[0,1]$ is a bipolar fuzzy lattice. $B_1$ can be considered as a polarization of the bivalent space {0,1} [1]; $B_2$ can be considered as a polarization of the 3-valued logical space {0,1,2}; and $B_F$ can be consi-dered a polarization of Zadeh's fuzzy space [0,1] [3].

A subset of a bipolar lattice might be a lattice but not a bipolar lattice and a bipolar lattice does not have to be balanced. It is interesting to examine the pair $S_2$ and $S_F$ in Fig. 2. Both can be verified perfect bipolar lattices but both are not balanced. The subset $S_1$, however, is a lattice but not a bipolar lattice because it does not have the element cglb((-1,0),(-1,0)) = (0,1).

### 2.4 Bipolar Isomorphism and Strict Bipolar Lattices

Definion 5. A **bipolar isomorphism** is an order-preserving bipolar bijection from one bipolar lattice to another that also preserves blubs, bglb, and cglbs. We refer each smallest square in the Hasse diagram of a bipolar lattice B as **an inner square**. Each grid on the border of a bipolar lattice B is called **an outer grid**. A lattice formed with the four corners of an outer grid of B is called **a corner lattice of the grid.** A lattice formed with the corners of all outer grids of B is called **a corner lattice of B.** A balanced bipolar lattice B is called **a strict bipolar lattice** if the corner lattice of B is isomorphic to $B_1=\{-1,0\}\times\{0,1\}$. A bipolar lattice B is called **a strict bipolar chain** if every outer grid of B is strict.

## 3 Dynamic Modus Ponens

With the notions of bipolar sets and bipolar lattices, classical modus ponens have been generalized to a non-linear bipolar dynamic modus ponens for open-world inference [8,10].

Similar to unipolar case, we define a set of **bipolar axioms (BAs)** as a basic set of bipolar tautologies or **bipolar well-formed formulas (bwfs)** from which all bipolar tautologies can be derived**. A set of BAs is sound on a bounded and complemented bipolar lattice** $(B,\equiv,\oplus,\otimes,\&,-,\neg,\Rightarrow)$ iff it does not generate inconsistency on B. **A set of BAs is complete on** $(B,\equiv,\oplus,\otimes,\&,\neg,\Rightarrow)$ iff any other tautology must necessarily be a derivation from the set. **A set of BAs is sound and complete** on an isomorphic class of strict bipolar lattice.

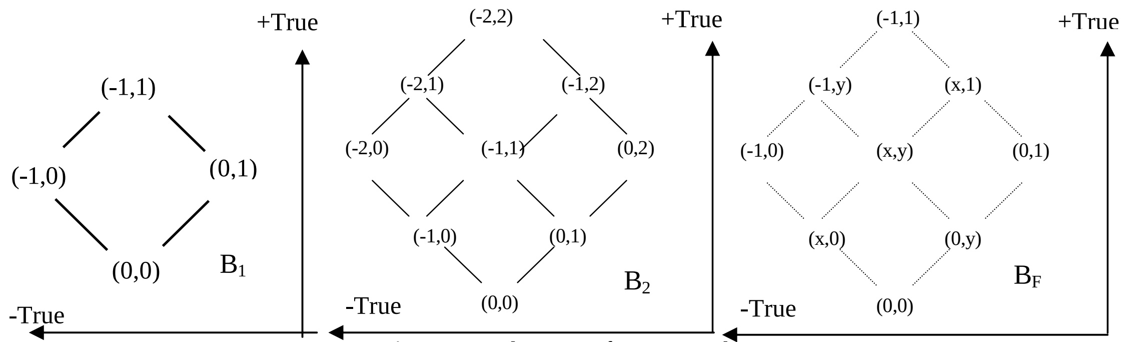


**Fig. 1.** Hasse diagrams of $B_1$ , $B_2$ , and $B_F$

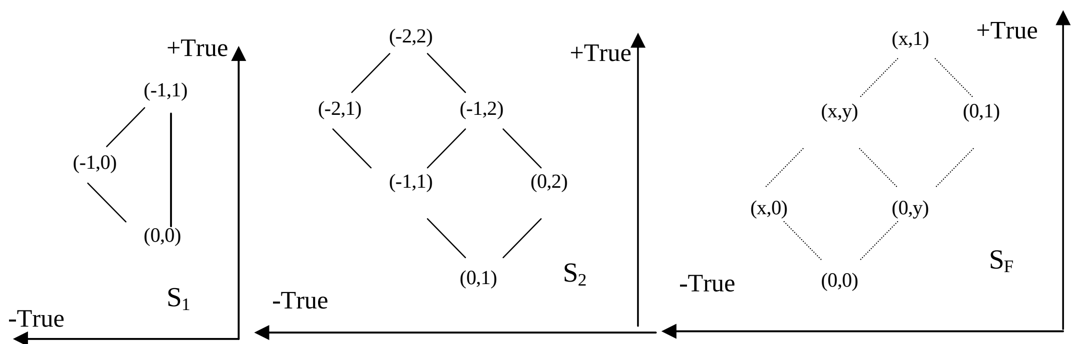


Fig. 2. Hasse diagrams of two bipolar lattices

It has been proved that (1) The 6-set axiomatization {BA1-BA5, BAR} in Fig. 3 is zero-order (propositional) sound and complete with respect to –,¬,&,and ⊗ on the corner lattice of any strict NP bipolar lattice B. (2) The 9-set axiomatization {BA1-BA7, BAR, BR} (see Fig. 3) is first-order (predicate) sound and complete with respect to ¬, &, and ⊗ on B. A set of derived linear and non-linear tautologies are listed in Fig.4.

## 4. L-Equilibrium Relations

### 4.1 Bipolar L-Relations

A **bipolar (binary) L-relation** R from X to Y on a bipolar lattice B, where X = $\{x_i\}$, $0<i\le m$, and Y = $\{y_j\}$, $0<j\le n$, is a collection of ordered pairs or subsets of X × Y characterized by a bipolar set or membership function $\mu_R(x_i,y_j)$ which maps each ordered pair $(x_i,y_j)$ to B. Formally, R is the set $\{\mu_R(x_i,y_j)|\forall i,j,\ 0<i\le m,\ 0<j\le n,\ \mu_R:(x_i,y_j)\Rightarrow B\}$. When B is finite-valued R is called **a strict bipolar crisp relation**; when B is infinite-valued we say R is **a strict bipolar fuzzy relation**.

Given two bipolar relations (crisp/fuzzy) $R_1$ and $R_2$ in X, where X = $\{x_i\}$, if $\forall i,j$, $\mu_{R1}(x_i,x_j) \oplus \mu_{R2}(x_i,x_j) = \mu_{R2}(x_i,x_j)$, we say that $R_1$ is **contained in (or smaller than or equal to)** $R_2$ denoted by $R_1 \subset R_2$ or $R_1 \le\le R_2$. The **⊕-⊗ or max-⊗ composition** of an m×n bipolar L-relation (crisp or fuzzy) R in X×Y and an n×k bipolar L-relation Q in Y×Z, denoted by R⊗Q, is defined as

$$\mu_{R\otimes Q}(x,z)= \max_{y} (\mu_R(x,y)\otimes\mu_Q(y,z)), \forall x,y,z, x\in X, y\in Y, z\in Z,$$

where max is equivalent to ⊕. The n-fold composition of R in X with itself is denoted as $R^n$.

If Ω is used as a binding operator for a NP bipolar relation R, $\forall i,j$, we have

$R \equiv (R^-,R^+) = R^-\Omega R^+ = [r_{ij}] = [r_{ij}^-]\Omega[r_{ij}^+] = R^+\Omega R^- = [r_{ij}^+]\Omega[r_{ij}^-] = [(r_{ij}^-,r_{ij}^+)]$;

$(\neg R^-,R^+) = [(\neg r_{ij}^-,r_{ij}^+)]$, $(R^-,\neg R^+) = [(r_{ij}^-,\neg r_{ij}^+)]$, $\neg R = (\neg R^-,\neg R^+) = [(\neg r_{ij}^-,\neg r_{ij}^+)]$.

### 4.2 Bipolar L-Equilibrium Relations

Definition 4.1 A bipolar L-relation R in X to a bounded NP bipolar lattice B, where X = $\{x_i\}$, $0<i\le n$, is **bipolar symmetric** if, $\forall i,k$, $1\le i,k\le n$, we have

$\mu_R(x_i,x_k) = \mu_R(x_k,x_i)$.

It is **positive pole reflexive** if, $\forall i$, $1\le i\le n$, we have

$\mu_R(x_i,x_i) = (n,1)$.

It is **negative pole reflexive** if, $\forall i$, $1\le i\le n$, we have

$\mu_R(x_i,x_i) = (-1,p)$.

It is **bipolar reflexive** if, $\forall i$, $1\le i\le n$, we have

$\mu_R(x_i,x_i) = (-1,1)$.

It is **⊕-⊗ or max-⊗ bipolar transitive or serial** if, $\forall i,j,k$, $1\le i,j,k\le n$, we have

$$\mu_R(x_i,x_k) \ge\ge \max_{x_j} (\mu_R(x_i,x_j)\otimes\mu_R(x_j,x_k)).$$

Definition 4.2 A bipolar L-relation R in X to a bounded bipolar lattice B, where X = $\{x_i\}$, $1\le i\le n$, is called a **bipolar L-equilibrium relation** if it is (1) **bipolar symmetric**; (2) **positive pole or bipolar reflexive**, and (3) **bipolar ⊕-⊗ transitive**. R is a **L-crisp equilibrium relation** if B is crisp or R is a **L-fuzzy equilibrium relation** if B is fuzzy.

**Bipolar Linear Axioms:**
**BA1:** $(\phi^-,\phi^+)\Rightarrow((\varphi^-,\varphi^+)\Rightarrow(\phi^-,\phi^+))$; **BA2:** $((\phi^-,\phi^+)\Rightarrow((\varphi^-,\varphi^+)\Rightarrow(\chi^-,\chi^+)))\Rightarrow(((\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+))\Rightarrow((\phi^-,\phi^+)\Rightarrow(\chi^-,\chi^+)))$;
**BA3:** $(\neg(\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+))\Rightarrow((\neg(\phi^-,\phi^+)\Rightarrow\neg(\varphi^-,\varphi^+))\Rightarrow(\phi^-,\phi^+))$;
**BA4:** **(a)** $(\phi^-,\phi^+)\&(\varphi^-,\varphi^+)\Rightarrow(\phi^-,\phi^+)$; **(b)** $(\phi^-,\phi^+)\&(\varphi^-,\varphi^+)\Rightarrow(\varphi^-,\varphi^+)$; **BA5**: $(\phi^-,\phi^+)\Rightarrow((\varphi^-,\varphi^+)\Rightarrow((\phi^-,\phi^+)\&(\varphi^-,\varphi^+)))$;

**Non-Linear Bipolar Augmentation Rule (BAR) or Bipolar Dynamic Modus Ponens**
**BR1:** $\forall(\phi^-,\phi^+),(\psi^-,\psi^+),(\varphi^-,\varphi^+),(\chi^-,\chi^+)$, Serializable$((\phi^-,\phi^+),(\psi^-,\psi^+))$, and Serializable$((\varphi^-,\varphi^+),(\chi^-,\chi^+))$,
$((\phi^-,\phi^+)\otimes(\psi^-,\psi^+))\&[((\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+))\&((\psi^-,\psi^+)\Rightarrow(\chi^-,\chi^+))]\Rightarrow((\varphi^-,\varphi^+)\otimes(\chi^-,\chi^+))$;

**BA6:** $\forall x,(\phi^-(x),\phi^+(x))\Rightarrow(\phi^-(t),\phi^+(t))$; **BA7:** $\forall x,((\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+))\Rightarrow((\phi^-,\phi^+)\Rightarrow\forall x,(\varphi^-,\varphi^+)$;
**BR2-Generalization:** $(\phi^-,\phi^+)\Rightarrow\forall x,(\phi^-(x),\phi^+(x))$

Fig. 3. Bipolar axioms and rules of inferences

| | |
|---|---|
| Non-contradiction | $\neg((\varphi^-,\varphi^+)\&\neg(\varphi^-,\varphi^+))$ |
| Excluded middle | $(\varphi^-,\varphi^+)\oplus\neg(\varphi^-,\varphi^+)$ |
| Double complement | $\neg(\neg(\varphi^-,\varphi^+))\Rightarrow(\varphi^-,\varphi^+)$ |
| Double negation | $-(-(\varphi^-,\varphi^+))\Rightarrow(\varphi^-,\varphi^+)$ |
| DeMorgan's laws | $\neg((\varphi^-,\varphi^+)\&(\phi^-,\phi^+))\Leftrightarrow\neg(\varphi^-,\varphi^+)\oplus\neg(\phi^-,\phi^+))$; $\neg((\varphi^-,\varphi^+)\oplus(\phi^-,\phi^+))\Leftrightarrow\neg(\varphi^-,\varphi^+)\&\neg(\phi^-,\phi^+))$ |
| Law of Separation | $(\varphi-,\varphi+)\Rightarrow((\varphi-,0)\oplus(0,\varphi+))$ |
| Modus Tollens | $(\neg(\varphi^-,\varphi^+)\&((\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+)))\Rightarrow\neg(\phi^-,\phi^+)$; |
| Modus Tollens | $\{\neg((\varphi^-,\varphi^+)\otimes(\psi^-,\psi^+))\&[((\varphi^-,\varphi^+)\otimes(\psi^-,\psi^+))\Rightarrow((\phi^-,\phi^+)\otimes(\psi^-,\psi^+))]\}\Rightarrow\neg((\varphi^-,\varphi^+)\otimes(\psi^-,\psi^+))$ |
| Laws of Negation* | $(\varphi^-,\varphi^+)\otimes(\phi^-,0)\Rightarrow(-\varphi^+,-\varphi^-)$; $(\phi^-,0)\Rightarrow((\varphi^-,\varphi^+)\Rightarrow((\phi^-,0)\otimes(-\varphi^+,-\varphi^-)))$; |
| Linear BMP | $[(\phi^-,\phi^+)\&((\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+))]\Rightarrow(\varphi^-,\varphi^+)$; |
| Non-Linear Bipolar Dynamic Modus Ponens | $((\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+))\Rightarrow[((\phi^-,\phi^+)\otimes(\chi^-,\chi^+))\Rightarrow((\varphi^-,\varphi^+)\otimes(\chi^-,\chi^+))]$;<br>$((\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+))\Leftrightarrow[((\phi^-,\phi^+)\otimes(\phi^-,\phi^+))\Rightarrow((\varphi^-,\varphi^+)\otimes(\varphi^-,\varphi^+))]$;<br>$[((\phi^-,\phi^+)\Rightarrow(\varphi^-,\varphi^+))\&((\psi^-,\psi^+)\Rightarrow(\chi^-,\chi^+))]\Rightarrow[((\varphi^-,\varphi^+)\otimes(\psi^-,\psi^+))\Rightarrow((\phi^-,\phi^+)\otimes(\chi^-,\chi^+))]$;<br>$\{((\varphi^-,\varphi^+)\otimes(\psi^-,\psi^+))\&[((\varphi^-,\varphi^+)\otimes(\psi^-,\psi^+))\Rightarrow((\phi^-,\phi^+)\otimes(\chi^-,\chi^+))]\}\Rightarrow((\phi^-,\phi^+)\otimes(\chi^-,\chi^+))$;<br>$\{((\varphi^-,\varphi^+)\otimes(\psi^-,\psi^+))\&[((\varphi^-,\varphi^+)\otimes(\psi^-,\psi^+))\Rightarrow((\phi^-,\phi^+)\otimes(\psi^-,\psi^+))]\}\Rightarrow((\phi^-,\phi^+)\otimes(\psi^-,\psi^+))$ |

Fig. 4. Derived linear and non-linear bipolar tautologies

It has been proved that, given a bipolar crisp/fuzzy L-relation R in a set X to an NP bounded crisp/fuzzy bipolar lattice B, where $X = \{x_i\}$, $1\le i\le n$, the following conditions are **necessary** and **sufficient** for R to be a bipolar L-equilibrium relation:

(1) $R^+$ is an equivalence/similarity relation;
(2) $|R^-|\cup R^+$ is an equivalence/similarity relation;
(3) If $(R^+\cap|R^-|)$ is not null it must be a local equivalence/similarity relation;
(4) If $(|R^-|\cup R^+)-(R^+\cap|R^-|)$ is not null it must be a local equivalence/similarity) relation;
(5) If $(|R^-|\cup R^+)-|R^-|$ is not null it must be a local equivalence/similarity relation; and
(6) $R^+-(R^+\cap|R^-|)\equiv(|R^-|\cup R^+)-|R^-|$.

The above properties lead to bipolar partitioning or fuzzy clustering [4,7,9].

## 5. Conclusions

The notions of bipolar poset, bipolar lattice, bipolar strictness have been proposed which lead to the notion of bipolar L-sets, L-fuzzy sets, and a bipolar dynamic modus ponens. This work presents a theoretical framework for bipolar holistic knowledge representation, visualization, and open world reasoning in different applications [4,7,8,9].

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