Conic Spiral Spline

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Abstract

We present results on the number and location of all possible curvature extrema of conic segments with fixed end points. These results are used to create spiral spline by joining conic spirals and/or arc/conic spirals. The use of spiral gives the designer excellent control over the shape of curves and surfaces in geometric modelling.

Key words: Bézier conic, Arc, Curvature, Spiral.

1 Introduction

Conic sections are widely used in industries due to its well-known properties and convenient implementations for the users. The curvature is one of the most important geometric concepts of curves and surfaces. Conics have no inflection points; however they do have curvature extrema. Therefore only well-chosen conic segments will have monotone curvature. Frey and Field [2] used *differentiation* of the curvature $\kappa(t)$ of conics whereas Ahn and Kim [1] used the *symmetry* of the conics. Our method also used the *differentiation* of the curvature $\kappa(t)$ but we simplify the derivation and the results of Frey et al. [2] and Ahn et al. [1] by using Descartes's rule of signs, i.e., easily calculating the number K of the curvature extrema on the position of $\boldsymbol{b}_1$ in the (ξ, μ)-plane.

Spirals have neither inflection points nor curvature extrema, and therefore have monotone curvature everywhere [3]. These spirals are desirable for applications such as the design of highway or railway routes and trajectories of mobile robots. Meek & Walton considered two-point G^1 planar Hermite interpolating spirals by joining T-cubic spirals and/or arc/T-cubic spirals in [5]. In this paper, the spiral segments will be created in two ways. The first is by taking a spiral segment from a conic, which gives a curve referred to as a conic spiral. The second is by joining a circular arc to a conic spiral in a G^3 manner, which gives a curve referred to as an arc/conic spiral. We find conic and arc/conic spirals computationally economical and more flexible than T-cubic and arc/T-cubic spirals in [5].

2 Preliminary Information on Conics

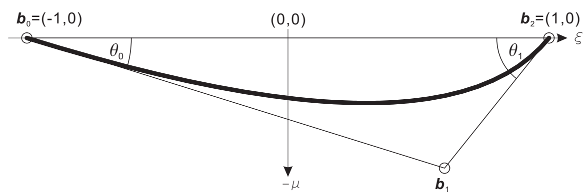

Figure 1. Bézier conic segment in normalized local coordinate system.

The standard form for a rational quadratic Bézier curve is

$$\boldsymbol{z}(t) = \frac{(1-t)^2\boldsymbol{b}_0 + 2wt(1-t)\boldsymbol{b}_1 + t^2\boldsymbol{b}_2}{(1-t)^2 + 2wt(1-t) + t^2}, \quad t \in [0, 1], \tag{1}$$

where $\boldsymbol{b}_0, \boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathbb{R}^2$ are non-collinear control points, $w \in \mathbb{R}$ is the weight associated with $\boldsymbol{b}_1$, $(1-t)^2 = B_{2,0}(t)$, $2t(1-t) = B_{2,1}(t)$, $t^2 = B_{2,2}(t)$ are Bernstein basis functions (see Figure 1).

3 Description of method

We restrict the end control points as follows: $\boldsymbol{b}_0 = (-1, 0), \boldsymbol{b}_2 = (1, 0)$ and denote $\boldsymbol{b}_1 = (\xi, \mu)$ with $\mu \neq 0$. Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{\boldsymbol{z}'(t) \times \boldsymbol{z}''(t)}{\|\boldsymbol{z}'(t)\|^3}, \tag{2}$$

where $\times$ stands for the two-dimensional cross product $(x_0, y_0) \times (x_1, y_1) = x_0y_1 - x_1y_0$ and $\|\bullet\|$ means the Euclidean norm.

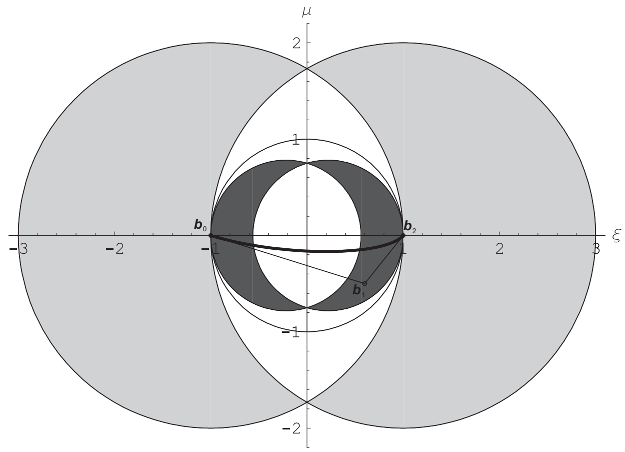

Figure 2. Distribution of the number of curvature extrema, Black: $K = 0$**, White:** $K = 1$**, Grey:** $K = 2$**.**

Theorem 3.1. *Suppose K is the number of curvature extrema, then in Figure 2, if the control point $\boldsymbol{b}_1$ is on*

(i) *the black region then $K = 0$ for $1/2 \le w^2 \le 1$.*

(ii) *the white region then $K = 1$ for $0 < w \le 1$.*

(iii) *the grey region then $K = 2$ for $0 < w^2 < 1/2$.*

(iv) *the boundaries then K is the smaller one of K for the regions beside the boundaries.*

Proof. A simple calculation for $t = 1/(1+s)$, $s > 0$, gives

$$\kappa'(t) = \frac{-6w\mu(1+s)^2(1+2sw+s^2)^2 N(s)}{\left[(1-s^2)^2w^2\mu^2+\{2s+w(1-\xi)+s^2w(1+\xi)\}^2\right]^{5/2}}, \quad (3)$$

where

$$N(s) = \sum_{i=0}^{4} a_i s^i \quad (4)$$

and $p(t)(= p_\mu(t)) = 1 + t - w^2\left\{\mu^2 + (1+t)^2\right\}$ with

$$a_4 = -wp(\xi), \quad a_3 = -2\left\{p(\xi) - \xi\right\}, \quad a_2 = 6w\xi,$$
$$a_1 = 2\left\{p(-\xi) + \xi\right\}, \quad a_0 = wp(-\xi). \quad (5)$$

Depending on the signs of a_4 and a_0, we consider the following four cases where " $+$ " and " $-$ " include "0", "?" means either " $+$ " or " $-$ ", and $q(t) = p_0(t)$.

Case 1 ($a_4 \ge 0$, $a_0 \le 0$): The signs of $S(a)(= (a_4, a_3, a_2, a_1, a_0))$ are equal to $(+,+,+,?,-)$ for $\xi \ge 0$ and $(+,?,-,-,-)$ for $\xi < 0$. Therefore, using the Descartes's rule of signs and the intermediate value theorem, we can show the existence of a unique curvature extremum (relative minimum) if $w^2\mu^2 \ge \mathrm{Max}[q(\xi), q(-\xi)]$.

Case 2 ($a_4 \ge 0$, $a_0 \ge 0$): Note $q(\xi) \le w^2\mu^2 \le q(-\xi)$ which implies $(w^2 - 1/2)\xi \ge 0$. If $w^2 > 1/2$, then $\xi \ge 0$ from which the signs of $S(a)$ are all nonnegative. If $w^2 = 1/2$, i.e., $\xi^2 + \mu^2 = 1$, then $S(a) = \xi(0, 2, 3\sqrt{2}, 2, 0)$. Therefore, use Descartes's rule of signs to show no curvature extrema, i.e., monotone decreasing if $w^2 \ge 1/2$ and $q(\xi) \le w^2\mu^2 \le q(-\xi)$.

Next assume $0 < w^2 < 1/2$ and $a_4 > 0$. Then, $\xi \le 0$ or $a_2 \le 0$. From (4), we have

$$N(s) = \left(\frac{4a_4 s + a_3}{16a_4}\right) N'(s) + \quad (6)$$
$$C\left(s + w + \sqrt{\frac{1-w^2}{3}}\right)\left(s + w - \sqrt{\frac{1-w^2}{3}}\right),$$

where

$$C\left(= \frac{8a_4a_2 - 3a_3^2}{16a_4}\right) = \quad (7)$$
$$\frac{-3[4w^4\mu^2\xi^2 + \left\{-1 + w^2(1 + \mu^2 - \xi^2)\right\}^2]}{4a_4} \quad (< 0).$$

Since

$$N'\left(-w \mp \sqrt{\frac{1-w^2}{3}}\right) = \frac{\pm 32a_4(1-w^2)}{3}\sqrt{\frac{1-w^2}{3}}, \quad (8)$$

cubic $N'(s)$ has three zeros α, β, γ ($\alpha < \beta < \gamma, \alpha < 0$) and then by (6), quartic $N(s)$ has four zeros $p_i, 1 \le i \le 4$ ($p_1 < \alpha < p_2 < \beta < p_3 < \gamma < p_4$). Use a relation of the three zeros and the coefficients of the cubic $N'(s)$: $\alpha\beta + \beta\gamma + \gamma\alpha = a_2/(2a_4)(\le 0)$ to obtain $\gamma > 0$. In addition, $N(0)(= a_0) \ge 0$ from which $p_2 \le 0 \le p_3$. Therefore, quartic $N(s)$ with the positive leading term has a negative minimum at γ on $(0, \infty)$. Thus, if $N(0)(= a_0) = (>)\ 0$, the segment has one (two) curvature extremum (extrema). If $a_4 = 0, a_0 > 0$, the similar argument can also be applied. Therefore, the segment has two curvature extrema (first, relative minimum and next, relative maximum as t increases from 0 to 1) if $w^2 < 1/2$ and $q(\xi) < w^2\mu^2 < q(-\xi)$ and one curvature extremum (relative maximum) if $w^2 < 1/2$ and in addition, $q(\xi) = w^2\mu^2 < q(-\xi)$ or $q(\xi) < w^2\mu^2 = q(-\xi)$.

Case 3 ($a_4 \le 0$, $a_0 \ge 0$): Replace ξ with $-\xi$ and use the argument in Case 1 to show an existence of a unique curvature extremum (relative maximum) if $w^2\mu^2 \le \mathrm{Min}[q(\xi), q(-\xi)]$ and $w \le 1$ where this case can not occur for $w > 1$.

Case 4 ($a_4 \le 0$, $a_0 \le 0$): Note $q(-\xi) \le w^2\mu^2 \le q(\xi)$ which implies $(w^2 - 1/2)\xi \le 0$. Use the argument in Case 2 to obtain the similar results with ξ replaced by $-\xi$. $\square$

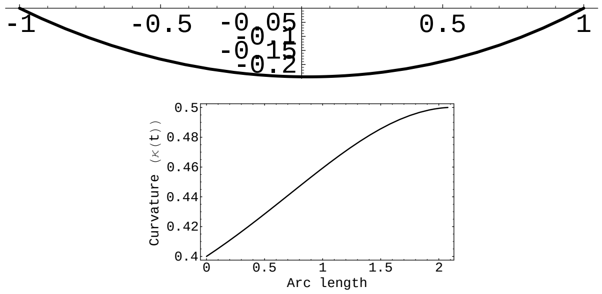

Figure 3. Conic spiral with plot of $\kappa(t)$.

Remark 3.1. *A change in weight $w(\geq 1/\sqrt{2})$ makes the black region in Figure 2 covering the whole unit circle in the (ξ, μ)-plane. Therefore, if $\boldsymbol{b}_1$ is in the unit disk: $\xi^2 + \mu^2 \leq 1$, a choice of the weight brings the segment without the curvature extrema. Therefore the sufficient conditions for spiral are $0 < \theta_0 < \theta_1$ and $\theta_0 + \theta_1 \leq \pi/2$.*

Corollary 3.1. *The segment (1) for $w^2 \geq 1/2$ has no curvature extrema (i.e., conic spiral) if*

$$\left\{w^2\mu^2 - q(\xi)\right\}\left\{w^2\mu^2 - q(-\xi)\right\} \leq 0. \tag{9}$$

Proof. See first part of Case 2 and similarly Case 4. □

We have a simple example in Figure 3 for $\kappa(0) = 0.4$ and $\kappa(1) = 0.5$ and $w = 0.9$ which can not be covered by T-cubic spiral [5].

4 The Arc/Conic Spiral

The arc/conic spiral that matches given geometric Hermite data is described below. If the point of intersection $\boldsymbol{b}_1$ of the tangent lines is outside the unit disk: $\xi^2 + \mu^2 \leq 1$ (i.e., $\theta_0 + \theta_1 > \pi/2$), as shown in Figure 2, i.e., $\left\{w^2\mu^2 - q(\xi)\right\}\left\{w^2\mu^2 - q(-\xi)\right\} > 0$, then the arc/conic is formed by joining a circular arc to the point of extreme curvature of a conic spiral in such a way that the unit tangents match at the join. The curvature of the circle is chosen to match the curvature of the conic spiral at the join point making the circle a circle of curvature, and making the composite curve a spiral. This join is G^3 since the derivative of the curvature of the circular arc and the derivative of the curvature of the conic spiral at the join point are both zero.

For $0 < \theta_0 < \theta_1 < \pi$, $w^2 \geq 1/2$ and referring to Figure 4, let the conic part of the arc/conic start at the point $\boldsymbol{b}_0(-1, 0)$ with tangent vector at angle $-\theta_0$ with parameter $t = 0$, and end at the joining point

$$\begin{aligned} \boldsymbol{Q} &= \{1 + r(\sin(\theta_1 - \theta) - \sin\theta_1), \\ &\quad r(\cos\theta_1 - \cos(\theta_1 - \theta))\}, \end{aligned} \tag{10}$$

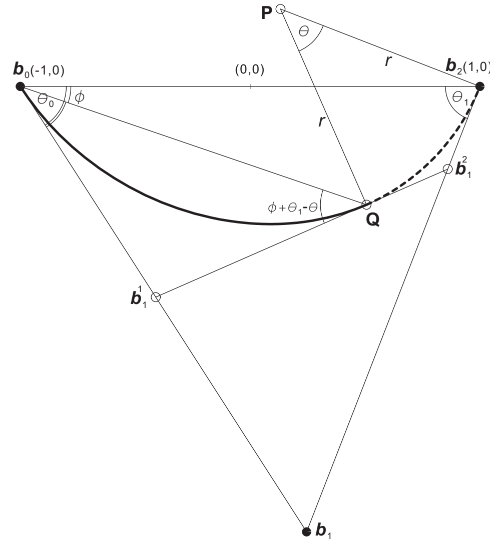

Figure 4. Arc/Conic spiral.

with tangent vector at angle $\theta_1 - \theta$ with parameter value $t = 1$. Both tangent vectors intersect at $\boldsymbol{b}_1^1(\xi_1, \mu_1)$ which is the unique solution of $m_0 = -\tan\theta_0$ and $m_1 = \tan(\theta_1 - \theta)$, where m_0 and m_1 are the slopes at $t = 0$ and $t = 1$, respectively, of the quadratic Bézier curve (1) for $\boldsymbol{b}_1 = \boldsymbol{b}_1^1$, $\boldsymbol{b}_2 = \boldsymbol{Q}$:

$$\begin{aligned} (\xi_1, \mu_1) = \frac{1}{\sin(\theta - \theta_0 - \theta_1)} &\{r\cos\theta_0(\cos\theta - 1) \\ &+ \sin(\theta + \theta_0 - \theta_1), \sin\theta_0(r(1 - \cos\theta) \\ &-2\sin(\theta - \theta_1))\}, \end{aligned} \tag{11}$$

Let the arc part of the arc/conic have radius r, start at the point $\boldsymbol{Q}$ with tangent vector at angle $\theta_1 - \theta$, sweep through an angle θ and end at the point $\boldsymbol{b}_2(1, 0)$ with tangent vector at angle $\pi + \theta_1$. Both tangent vectors intersect at $\boldsymbol{b}_1^2$. Then quadratic Bézier curve (1) is the desired circular arc for $\boldsymbol{b}_1 = \boldsymbol{b}_1^2$ and $w = \cos(\theta/2)$. The radius r can easily be derived from $f(r) (= \kappa(1) - 1/r) = 0$. Since $f(0) < 0$ and $f(r_0) > 0$ for $\theta_0 + \theta_1 - \pi/2 \leq \theta \leq \theta_1 - \theta_0$, the intermediate value theorem implies that $f(r) = 0$ has its solution belonging to $(0, r_0)$, where $r_0 = \sin\theta_0 \csc(\theta/2) \csc(\theta_0 + \theta_1 - \theta/2)$.

Theorem 4.1. *The conic part of an arc/conic has no curvature extrema (i.e., spiral) If $\theta_0 + \theta_1 - \pi/2 \leq \theta \leq \theta_1 - \theta_0$, $0 < \theta_0 < \theta_1 < \pi$ and $w^2 \geq 1/2$.*

Proof. With reference to Remark 3.1, the sufficient spiral conditions for the conic part of arc/conic are $\theta_0 + \theta_1 - \pi/2 \leq \theta \leq \theta_1 - \theta_0 + 2\phi$ and $0 < \phi < \theta_0$ for some suitable choice of $w(\geq 1/\sqrt{2})$. □

An example of arc/conic spiral is given in Figure 5 with $\theta_0 = \pi/5$, $\theta_1 = \pi/3$, $\theta = \theta_0 + \theta_1 - \pi/2$, $w = 1/\sqrt{2}$

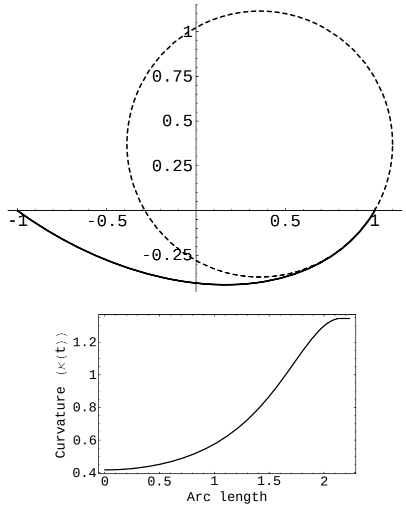

Figure 5. Arc/Conic spiral with plot of $\kappa(t)$.

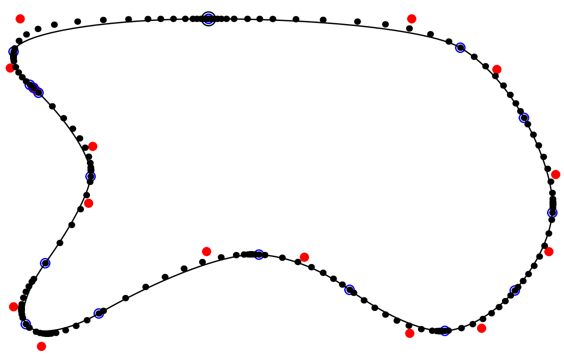

Figure 6. A G^1 closed curve made of conic and arc/conic spirals fitting given points.

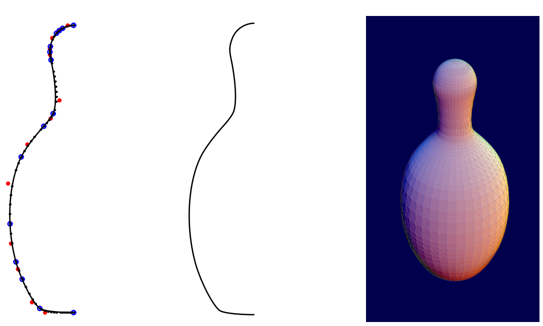

Figure 7. A bowling pin made of conic spirals fitting given points.

5 Examples and Conclusion

Figures 6 and 7 are data fitting example where the data are a set of points taken from a smooth curve. The critical points are chosen automatically by calculating the approximate curvature at each given point, and those values were searched for local maxima, local minima, near zero values. Approximations to the unit tangents to the curve at the critical points were calculated from the given points. Formula for approximating the tangents can be found in [4]. The problem is now to find Hermite interpolating spiral between each pair of critical points. The given points are shown as dots, start point is indicated by the double circles around the given point, each given point that is a critical point is indicated by a circle around the given point, and control points are shown as disk. Conic and arc/conic spiral splines are shown as solid line matching the given points well in these simple examples.

In this paper, we characterized the conditions for the curvature of a quadratic rational Bézier curve to be monotone in $[0, 1]$, to have a unique curvature extremum, or to have two curvature extrema in $(0, 1)$ and visualized in Figure 2. Conic spiral segments are more flexible and computationally less expensive than T-cubic segments, hence more comfortable for use in practical applications. Our future work directions are to investigate the possibilities of G^2 conic spiral spline and develop efficient algorithm for implementation on web server.

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