Shape Preserving Data Visualization with Cubic Splines

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Abstract

A piecewise cubic spline has been introduced to preserve the shape of the data when it is convex or monotone. The spline representation is interpolatory and applicable to the scalar valued data. The shape parameters, in the description of the cubic, have been constrained in such a way that they control the shape of the curve to avoid any noise. As far as visual smoothness is concerned, the curve scheme under discussion is GC^1.

Keywords: Visualization, Data, Spline, Interpolation, Shape.

1. Introduction

This paper examines the problem of shape preservation of data arose from some scientific phenomena or from some mathematical function. Many people have worked in the area of shape preservation [1 – 11]. For example, Fritsch & Carlson [5] and Fritsch & Butland [4] have discussed the piecewise cubic interpolation of monotonic data. Also, McAllister, Passow & Roulier [8] considered the piecewise polynomial interpolation of monotonic and convex data. In particular, an algorithm for quadratic spline interpolation is given in McAllister & Roulier [9]. An alternative to the use of polynomials, for the interpolation of montonic data, is the appliction of piecewise rational quadratic functions in [6]. Moreover rational cubics have also been considered by in [1, 3, 10]. They use piecewise rational functions having cubic numerator and quadratic denominator. Rational functions with cubic numerator and denominator have been discussed in [7].

Convexity and monotony are very important aspects of shapes. There are many physical situations where entities only have a meaning when their values appear in convex shape or monotonic shape. Therefore, it is very important to discuss convex / monotone interpolation problems to provide a computationally economical and visually pleasing solution to the problems of different scientific phenomena.

This work is also related to the convex / monotone interpolation. The interpolant used is a piecewise cubic function in a GC^1 form. The piecewise cubic functions used in this scheme have some freedom in the form of parameters, in each interval, which can control the shape of the curve. The subject of this paper is to search for appropriate constraints on the shape parameters so that the resultant curve is convex for convex data and monotone for monotone data. Moreover, it is also desired that the degree of smoothness is visually acceptable.

The method under consideration in this paper has the following important and advantageous features that no additional points (knots) need be supplied. In contrast, the cubic interpolation method of Brodlie and Butt [2] require the introduction of additional knots when used as shape preserving methods. Moreover, already existing algorithms like de Castlejau algorithm can be used for rapid computations.

The paper has been organized in such a way that Section 2 describes about the Hermite like Cubic interpolation. The problem of convexity is discussed in Section 3 where as the monotony problem has been explained in Section4. Section 5 concludes the paper.

2. Cubic Interpolant

Consider the set of data points as follows:

$$\left(x_i, f_i\right),\ i = 1, 2, ..., n, \text{ where } x_1 < x_2 < ... < x_n . \quad (2.1)$$

Let

$$h_i = x_{i+1} - x_i,\ \ \Delta_i = \frac{f_{i+1} - f_i}{h_i}, i = 1, 2, ..., n-1.$$

Let us have the following piecewise cubic function:

$$S\left(x\right) \equiv S_i\left(x\right) = U_i\left(1-\theta\right)^3 + 3V_i\theta\left(1-\theta\right)^2 + 3W_i\theta^2\left(1-\theta\right) + Z_i\theta^3 \quad (2.2)$$

where

$$\theta = \frac{x - x_i}{h_i} \quad (2.3)$$

To make the function (2.2) GC^1, one needs to impose the following interpolatory properties:

$$S(x_i) = f_i, \text{ and } S(x_{i+1}) = f_{i+1},$$
$$S^{(1)}(x_i) = \frac{d_i}{r_i}, \text{ and } S^{(1)}(x_{i+1}) = \frac{d_{i+1}}{r_i} \qquad (2.4)$$
which provide the following manipulations:
$$U_i = f_i, \; Z_i = f_{i+1}, \; V_i = f_i + \frac{h_i d_i}{3r_i},$$
and
$$W_i = f_{i+1} - \frac{h_i d_{i+1}}{3r_i}, \qquad (2.5)$$
where $S^{(1)}$ denotes derivative with respect to x and d_i denotes derivative value given at the knot x_i. This leads the piecewise cubic (2.2) to the following piecewise Hermite like interpolant $S \in C^1[x_1, x_n]$:
$$S(x) \equiv S_i(x), \qquad (2.6)$$
where
$$S_i(x) = f_i(1-\theta)^3 + 3V_i\theta(1-\theta)^2 + 3W_i\theta^2(1-\theta) + f_{i+1}\theta^3 \qquad (2.7)$$
The parameters r_i's, and the derivatives d_i's are to be chosen such that the convex shape is preserved by the interpolant (2.6) One can note that when $r_i = 1$, the cubic function obviously becomes the standard cubic Hermite polynomial. Variation for the values of r_i's control (tighten or loosen) the curve in different pieces of the curve. When $r_i \to 0$, it is simple to see the curve gets tightened in the corresponding interval. This interval shape control behavior is desired to constraint such that the interpolant automatically becomes convex to the convex data.

It should be noted that the shape control analysis is valid only if the bounded derivative values are assumed. In most applications, the derivative parameters $\{d_i\}$ are not given and hence must be determined either from the given data (x_i, f_i), $i = 1, 2, \ldots, n$, or by some other means. In this article, they are computed exactly from the arithmetic approximation scheme in [6]. The smoothness of the interpolant (2.6), hence would be GC^1.

2.1. Demonstration

For the demonstration of this GC^1 Hermite like cubic spline scheme, we will choose $r_i = 1$ as default choice of shape parameters. However, other values of shape parameters can also be allocated for the achievement of a controlled curve. The Figure 1 is the default curve to the convex data in Table 1. It can be seen that the ordinary spline curve does not guarantee to preserve the shape. Some odd behavior (noise) can be seen in the presentation of the curve.

3. Convex Interpolation

The proposed way, which is effective, useful and is the target of this article, is the automated generation of shape preserving curve. This requires an automated computation of suitable shape parameters. To proceed with this strategy, some mathematical treatment is needed.

For given data (2.1), let us assume convex set of data so that
$$\Delta_1 \le \Delta_2 \le \ldots \le \Delta_{n-1}. \qquad (3.1)$$
Similarly, one can assume concave data so that
$$\Delta_1 \ge \Delta_2 \ge \ldots \ge \Delta_{n-1}.$$
In this paper we shall develop necessary and sufficient conditions on piecewise cubics under which GC^1 convex interpolation is preserved. We describe the cubic spline S on the grid $x_1 < x_2 < \ldots < x_n$. The key idea to preserve convexity using $S(x)$ is to assign suitable values to r_i in each interval. But, first of all, we determine conditions for r_i, which guarantee convexity.

For a convex interpolant $S(x)$, it is then necessary that the derivative parameters should be such that
$$d_1 \le \Delta_1 \le d_2 \le \Delta_2 \le \ldots \le \Delta_{n-1} \le d_n. \qquad (3.2)$$
($d_1 \ge \Delta_1 \le d_2 \ge \Delta_2 \ge \ldots \ge \Delta_{n-1} \ge d_n$, for concave data).
Now $S(x)$ is convex if and only if
$$S^{(2)}(x) \ge 0, \quad x_1 \le x \le x_n. \qquad (3.3)$$
For $x \in [x_1, x_n]$, it can be shown, after some simplification, that
$$S^{(1)}(x) = \sum_{j=1}^{3} A_{j,i}(1-\theta)^{3-j}\theta^{j-1},$$
where
$$A_{1,i} = \frac{d_i}{r_i}, \; A_{2,i} = 3\Delta_i - \left(\frac{d_i}{r_i} + \frac{d_{i+1}}{r_i}\right), \; A_{3,i} = \frac{d_{i+1}}{r_i}. \qquad (3.4)$$
and
$$S^{(2)}(x) = \sum_{j=1}^{2} B_{j,i}(1-\theta)^{2-j}\theta^{j-1},$$
where
$$B_{1,i} = \left[6\Delta_i - 2\left(\frac{2d_i}{r_i} + \frac{d_{i+1}}{r_i}\right)\right]/h_i, \qquad (3.5)$$
$$B_{2,i} = \left[2\left(\frac{d_i}{r_i} + \frac{2d_{i+1}}{r_i}\right) - 6\Delta_i\right]/h_i. \qquad (3.6)$$
The sufficient conditions for convexity on $[x_1, x_n]$ are:

$$B_{j,i} \geq 0, \quad j = 1, 2.$$

where the necessary conditions (3.2) are assumed. If $\Delta_i > 0$ (strict inequality) then following are sufficient conditions for (3.5) and (3.6):

$$r_i = \frac{d_i + d_{i+1}}{2\Delta_i}, \tag{3.7}$$

We will consider this as the default automatic choice. This choice satisfies (3.3) and produces pleasing results. It should be noted that if $\Delta_i = 0$, then it is necessary to set $d_i = 0 = d_{i+1}$, and thus $S(x) = f_i = f_{i+1}$ is a constant on $[x_1, x_n]$. Hence the interpolant (2.7) is convex together with the conditions (3.2) and (3.7). For the case where the data is convex but not strictly convex (i.e. when some $\Delta_i = 0$) it would be necessary to divide the data into strictly convex parts. If we set $d_i = 0 = d_{i+1}$ whenever $\Delta_i = 0$, then the resulting interpolant will be C^0 at break points. This leads to the following:

Proposition 1: The cubic polynomial (2.7) preserves convexity if and only if (3.7) is satisfied.

Remark: This method can be used in both cases when either d_i's are particularly specified or estimated by some method. We propose the Geometric Mean approximation method [1] for the practical implementation in this paper.

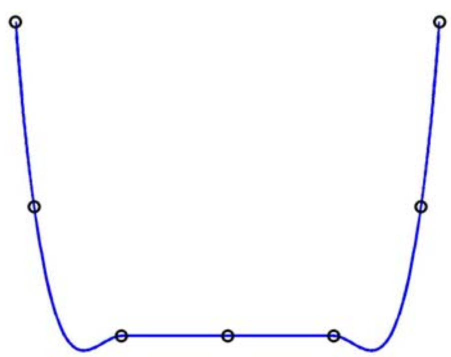

Figure 1: Cubic Hermite Spline curve to the data in Table 1.

Table 1

x	-4	-3.5	-2	0	2	3.5	4
y	5	0	-3.5	-3.5	-3.5	0	5

3.1. Demonstration

First example is that of a data which given in Table 1. Application of the Hermite cubic spline method (see Section 2) produces the curve in Figure 1. This curve shows noise, which is misguiding. We now apply piecewise cubic of Section 3 to the same data. The Figure 2 is produced by the default settings of the parameters r_i satisfying the convex conditions derived in Section 3. One can see that the convexity nature of the data is preserved in a pleasing way.

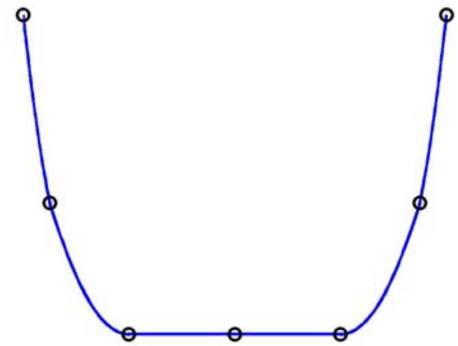

Figure 2: Shape preserving Cubic Spline curve to the data in Table 1.

4. Monotone Interpolation

For given points as in (2.1), let us assume monotonic increasing set of data so that

$$y_1 \leq y_2 \leq \ldots \leq y_n$$

or equivalently

$$\Delta_i \geq 0, \; i = 1, 2, \ldots n-1.$$

In a similar fashion, one can deal with a monotonic decreasing data.

In this section we shall develop necessary and sufficient conditions on piece wise cubics $S(x)$ to assign suitable values to r_i in each interval under which GC^1 monotone interpolation is preserved.

For a monotone interpolant $S(x)$, it is then necessary that derivative parameters should be such that

$d_i \geq 0$, $i = 1, 2, \ldots, n$ for monotonic increasing data,

$d_i \leq 0$, $i = 1, 2, \ldots, n$ for monotonic decreasing data.

Now $S(x)$ is monotonic increasing if and only if

$$S^{(1)}(x) \geq 0, \; x_1 \leq x \leq x_n.$$

The sufficient conditions for monotonic on $[x_1, x_n]$ are $A_{j,i} \geq 0$, $j = 1, 2, 3$, where the necessary conditions $d_i \geq 0$ and $d_{i+1} \geq 0$, are assumed. If $\Delta_i > 0$, then the followings are the sufficient conditions for (3.4):

$$r_i > \frac{d_i + d_{i+1}}{3\Delta_i}. \tag{4.1}$$

Proposition 2. The cubic polynomial (2.7) preserves monotony if and only if (4.1) is satisfied.

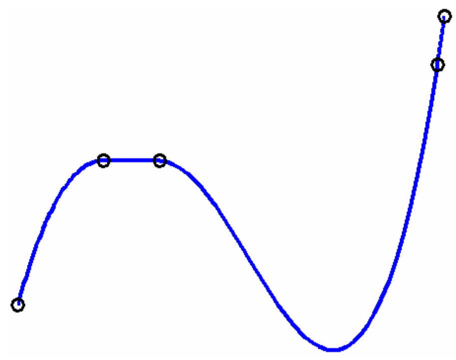

Figure 3: Cubic Hermite Spline curve to the data in Table 3.

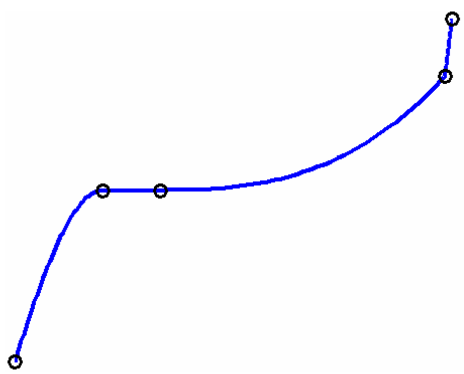

Figure 4: Shape preserving Cubic Spline curve to the data in Table 3.

4.1. Demonstration

Let us take the example of a monotone data as in Table 2. Figure 3 is produced by applying Hermite cubic spline method on this monotone data, which loses the monotonicity. Figure 4 shows the monotone curve through monotone data in Table 2 using monotone cubic function of Section 4.

Table 2

x	0	6	10	29.5	30
y	0	15	15	25	30

5. Conclusions and Suggestions

A piecewise cubic interpolant, in a generalized form, has been utilized to obtain a GC^1 convexity and monotony preserving curve method. Data dependent shape constraints are derived on shape parameters to assure the convex / monotone shape preservation of the data. Choice of the derivative parameters is considered to be the approximations as in [3]. The scheme has been implemented for scalar valued curves whereas the search, for the planar curves, is under the consideration of the author.

Although, the proposed curve scheme is visually enough smooth and presents a reasonably acceptable demonstration of the convex / monotone data. But a higher degree of smoothness, while stitching the pieces of curves, may enhance the visual display. It will be interesting to have a look at this issue in the future work. This curve scheme could also be generalized to the surface case.

6. References

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