On the weighted interval approximation of a fuzzy number

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Abstract: The problem of interval approximation of a fuzzy number is discussed. A weighted interval approximation is the best one with respect to our distance measure d between fuzzy numbers and also satisfies some good properties such as being intuitive, simple, natural and continuous. Finally, we give some examples of weighted interval approximations of numbers such as being triangular, trapezoidal, LR type, one sided and general.

Keywords: Fuzzy number, weighted interval approximation

1 Introduction

Fuzzy set theory makes us to process and transform imprecise information more effectively and flexibly. However, sometimes we indeed need to have exact information which replaces a given fuzzy set. For example, people are used to use a crisp set such as an interval number [18, 30] to approximately represent the fuzzy concept "Young People" in age region in order to have an extensively application in the practice. Considered that using a single real number which replaces a fuzzy set, we will lose too much important information. Therefore, a crisp set is considered as a reasonable approximation representation of a fuzzy set. Based on this meaning, a proper interval approximation to represent a fuzzy number becomes an interesting and important topic. Thus, it follows that an interval approximation of a fuzzy number will have many useful applications. The interval approximation of a fuzzy number can be applied in many fields such as fuzzy pattern recognition, fuzzy image processing and fuzzy decision making theory, etc. Especially, it will be applied to a comparison of fuzzy numbers by using the order relations defined on the set of interval numbers. Also, in [1, 2] the $\lambda-$cut sets (with a fixed value of λ) are used for ranking fuzzy numbers, after that the reference [9] points out that the method in [1, 2] has a drawback–the lack of continuity. Next, in [6, 7] fuzzy numbers are ordered through the comparison of its expected intervals. In [4, 5, 8, 9], the interval approximation of a fuzzy number was discussed based on some different distances measure between fuzzy numbers, and interesting results were revealed.

In this paper, we introduce a weighted distance measure d between fuzzy numbers, discuss the problem of interval approximation of fuzzy number and derive a general formulation for weighted interval approximation of a given fuzzy number. We generalize the conclusion of the reference[9], on the other hand, we also point out that the weighted interval approximation of fuzzy number has some good properties such as intuitive, simple, natural and continuous with respect to the distance measure d. Finally, we give some examples of weighted interval approximations of such as triangular, trapezoidal, LR type, one sided and general fuzzy numbers.

2 Fuzzy number

Throughout this paper, we write X to denote the universal set, $\mathbb{R}$ stands for the set of all real number, $\widetilde{\mathbb{R}}$ stands for the set of all fuzzy number in $\mathbb{R}$, $A(x)$ expresses the membership function of fuzzy set A, $\forall x \in X$.

Definition 1 Fuzzy number A is a fuzzy set defined on $\mathbb{R}$ characterized by means of a membership function $A(x)$, $A : \mathbb{R} \to [0, 1]$,

$$A(x) = \begin{cases} 0, & x \le a \\ f_A(x), & a < x \le b \\ 1, & b < x \le c \\ g_A(x), & c < x \le d \\ 0, & d < x \end{cases}$$

where $f_A(x)$ and $g_A(x)$ are continuous functions, f_A is increasing function, g_A is decreasing function. In special cases, it may be $a = -\infty$ and/or $d = +\infty$.

In this paper, we assume that

$$\int_{-\infty}^{+\infty} A(x)dx < +\infty$$

Definition 2 Fuzzy number A is called a fuzzy number of LR type if its membership function $A(x)$ has the following form:

$$A(x) = \begin{cases} L(\dfrac{a-x}{\alpha}), & x \le a \\ 1, & a < x < b \\ R(\dfrac{x-b}{\beta}), & b \le x \end{cases}$$

where L and R are continuous non-increasing functions, defined on $[0, +\infty)$, strictly decreasing to zero in those subintervals of the interval $[0, +\infty)$ in which they are

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*Author to whom all correspondence should be addressed, and this work is part of a project supported by the National Natural Science Foundation of China positive, and fulfilling the condition $L(0) = R(0) = 1, L(1) = R(1) = 0$. The parameters α and β are positive real numbers, and are called left width and right width, respectively. The functions L and R are called shape functions. We use the notation in [3], for a fuzzy number A of LR type, we write, $A = (a, b, \alpha, \beta)_{LR}$.

Definition 3 Fuzzy number A is called a triangular fuzzy number if its membership function $A(x)$ has the following form:

$$A(x) = \begin{cases} 0, & x \leq a-\alpha \\ \dfrac{x-a+\alpha}{\alpha}, & a-\alpha < x < a \\ 1, & x = a \\ \dfrac{a+\beta-x}{\beta}, & a < x \leq a+\beta \\ 0, & a+\beta < x \end{cases}$$

The parameters α and β are positive real numbers, and are called as left width and right width, respectively. We use the notation in [3], for a triangular fuzzy number A, we write, $A = (a, \alpha, \beta)$.

Definition 4 Fuzzy number A is called a trapezoidal fuzzy number if its membership function $A(x)$ has the following form:

$$A(x) = \begin{cases} 0, & x \leq a-\alpha \\ \dfrac{x-a+\alpha}{\alpha}, & a-\alpha < x \leq a \\ 1, & a < x \leq b \\ \dfrac{b+\beta-x}{\beta}, & b < x \leq b+\beta \\ 0, & b+\beta < x \end{cases}$$

The parameters α and β are positive real numbers, and are called as left width and right width, respectively. We use the notation in [3], for a trapezoidal fuzzy number A, we write, $A = (a, b, \alpha, \beta)$.

3 Weighted interval approximation

How to approximate a fuzzy number by a interval number is a realistic problem in fuzzy set theory and its application. The common method to deal with fuzzy numbers is using their $\lambda-$cut sets. The $\lambda-$cut set A_λ of a fuzzy number A is a crisp interval defined as $A_\lambda = \{x|A(x) \geq \lambda\}, \lambda \in [0,1]$. The easiest way to substitute a fuzzy number is either by using $supp(A) = A_0 = \{x|A(x) \geq 0\}$ or by using $core(A) = A_1 = \{x|A(x) = 1\}$. However, these two methods can not be recommended to practice because they neglect too much information. Hence, the most popular method in practice is $A_{0.5} = \{x|A(x) \geq 0.5\}$. The interval approximation is a compromise between two extremes A_0 and A_1. However, the simple and natural method has a very unpleasant drawback, the lack of continuity in [9].

After then, many scholars had many contributions in finding a reasonable interval approximation of fuzzy sets, and some different approaches to represent its interval approximation of fuzzy set were introduced in [4, 5, 6, 8, 9].

In this section, we propose another method, weighted interval approximation, to represent its interval approximation of a fuzzy set. Supposed that A is a given fuzzy number and $A_\lambda = [A^-(\lambda), A^+(\lambda)]$ for $\lambda \in (0,1]$ is its $\lambda-$cut set. Our purpose is to find a closed interval number $A^* = [A_L, A_U]$ which is a weighted interval approximation of a fuzzy number A with respect to our distance measure d defined as follows. Hence, we have to minimize

$$\begin{aligned} d^2(A, A^*) &= \int_0^1 f(\lambda)(A^-(\lambda) - A_L)^2 d\lambda \\ &+ \int_0^1 f(\lambda)(A^+(\lambda) - A_U)^2 d\lambda \end{aligned}$$

where the function $f : [0,1] \longrightarrow R$ is called a weighting function if f is non-negative, monotone increasing and satisfies the normalization condition $\int_0^1 f(\lambda) d\lambda = 1$.

In order to minimize $d(A, A^*)$, it suffices to minimize the function $D(A_L, A_U) = (d(A, A^*))^2$. Thus, we can get their partial derivatives:

$$\frac{\partial D(A_L, A_U)}{\partial A_L} = -2\int_0^1 f(\lambda)A^-(\lambda)d\lambda + 2A_L\int_0^1 f(\lambda)d\lambda$$

$$\frac{\partial D(A_L, A_U)}{\partial A_U} = -2\int_0^1 f(\lambda)A^+(\lambda)d\lambda + 2A_U\int_0^1 f(\lambda)d\lambda$$

and then to solve

$$\frac{\partial D(A_L, A_U)}{\partial A_L} = \frac{\partial D(A_L, A_U)}{\partial A_U} = 0$$

Therefore, the solution is

$$A_L = \int_0^1 f(\lambda)A^-(\lambda)d\lambda, \quad A_U = \int_0^1 f(\lambda)A^+(\lambda)d\lambda$$

Moreover, since

$$\det \begin{pmatrix} \dfrac{\partial^2 D(A_L, A_U)}{\partial A_L^2} & \dfrac{\partial^2 D(A_L, A_U)}{\partial A_U \partial A_L} \\ \dfrac{\partial^2 D(A_L, A_U)}{\partial A_L \partial A_U} & \dfrac{\partial^2 D(A_L, A_U)}{\partial A_U^2} \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$$

and $\dfrac{\partial^2 D(A_L, A_U)}{\partial A_L^2} = 2 > 0$, then A_L and A_U given by the above formulation, respectively, actually minimize $D(A_L, A_U)$ and minimize $d(A, A^*)$.

Therefore, we get a closed interval number $A^* = [\int_0^1 f(\lambda)A^-(\lambda)d\lambda, \int_0^1 f(\lambda)A^+(\lambda)d\lambda]$. It is our weighted interval approximation for a given fuzzy number A with respect to our distance measure d.

Especially, when the weighting function $f(\lambda) = 1$, then for a fuzzy number A, its weighted interval approximation $A^* = [\int_0^1 A^-(\lambda)d\lambda, \int_0^1 A^+(\lambda)d\lambda]$, is the same result of reference[9]. When the weighting function $f(\lambda) = 2\lambda$, then we will get its weighted interval approximation $A^* = [\int_0^1 2\lambda A^-(\lambda)d\lambda, \int_0^1 2\lambda A^+(\lambda)d\lambda]$ for a fuzzy number A.

Definition 5 An interval approximation A^* of a fuzzy number A is called continuous, if it satisfies the condition:

$$\forall\varepsilon > 0, \exists\delta > 0, d(A, B) < \varepsilon \Longrightarrow d(A^*, B^*) < \delta$$

Hence, we have the following result.

Theorem Weighted interval approximation is continuous interval approximation, where the definition of continuity can be found in reference[9].

Proof For given two fuzzy numbers A and B, their $\lambda-$cut sets are $A_\lambda = [A^-(\lambda), A^+(\lambda)]$ and $B_\lambda = [B^-(\lambda), B^+(\lambda)]$, respectively. Supposed that $A^* = [A_L, A_U]$ and $B^* = [B_L, B_U]$ are weighted interval approximations of fuzzy numbers A and B, respectively. Then we have

$$\begin{aligned}
&d^2(A^*, B^*) = \int_0^1 f(\lambda)(A_L - B_L)^2 d\lambda \\
&+ \int_0^1 f(\lambda)(A_U - B_U)^2 d\lambda \\
&= (A_L - B_L)^2 + (A_U - B_U)^2 \\
&= [\int_0^1 f(\lambda)A^-(\lambda)d\lambda - \int_0^1 f(\lambda)B^-(\lambda)d\lambda]^2 \\
&+ [\int_0^1 f(\lambda)A^+(\lambda)d\lambda - \int_0^1 f(\lambda)B^+(\lambda)d\lambda]^2 \\
&= [\int_0^1 f(\lambda)(A^-(\lambda) - B^-(\lambda))d\lambda]^2 \\
&+ [\int_0^1 f(\lambda)(A^+(\lambda) - B^+(\lambda))d\lambda]^2 \\
&\leq \int_0^1 (\sqrt{f(\lambda)})^2 d\lambda \int_0^1 (\sqrt{f(\lambda)})^2 (A^-(\lambda) - B^-(\lambda))^2 d\lambda \\
&+ \int_0^1 (\sqrt{f(\lambda)})^2 d\lambda \int_0^1 (\sqrt{f(\lambda)})^2 (A^+(\lambda) - B^+(\lambda))^2 d\lambda \\
&= \int_0^1 f(\lambda)(A^-(\lambda) - B^-(\lambda))^2 d\lambda \\
&+ \int_0^1 f(\lambda)(A^+(\lambda) - B^+(\lambda))^2 d\lambda \\
&= d^2(A, B)
\end{aligned}$$

Hence, it shows that if fuzzy numbers A and B are close enough then their weighted interval approximations obtained by above formulation are also close enough. Therefore, the weighted interval approximation is continuous.

4 Examples

In this section, we focus our efforts on the weighted interval approximation of some specific fuzzy numbers. For the sake of convenience, we use the weighting function $f(\lambda) = 2\lambda$.

Example 1(Triangular fuzzy number) Given a triangular fuzzy number $A = (a, \alpha, \beta)$ with a is its center and α and β are left width and right width, respectively. Then we have its $\lambda-$cut set $A_\lambda = [a - (1-\lambda)\alpha, a + (1-\lambda)\beta]$ for $\lambda \in (0, 1]$.

Then, we have

$$A_L = \int_0^1 2\lambda A^-(\lambda)d\lambda = a - \frac{1}{3}\alpha$$

$$A_U = \int_0^1 2\lambda A^+(\lambda)d\lambda = a + \frac{1}{3}\beta$$

Therefore, its weighted interval approximation $A^* = [a - \frac{1}{3}\alpha, a + \frac{1}{3}\beta]$.

Example 2(Trapezoidal fuzzy number) Given a trapezoidal fuzzy number $A = (a, b, \alpha, \beta)$ with α and β are left width and right width, respectively. Then we have its $\lambda-$cut set $A_\lambda = [a - (1-\lambda)\alpha, b + (1-\lambda)\beta]$ for $\lambda \in (0, 1]$.

Then, we have

$$A_L = \int_0^1 2\lambda A^-(\lambda)d\lambda = a - \frac{1}{3}\alpha$$

$$A_U = \int_0^1 2\lambda A^+(\lambda)d\lambda = b + \frac{1}{3}\beta$$

Therefore, its weighted interval approximation $A^* = [a - \frac{1}{3}\alpha, b + \frac{1}{3}\beta]$.

Example 3(Fuzzy number of LR type) Given a fuzzy number of LR type $A = (a, b, \alpha, \beta)_{LR}$ with α and β are left width and right width, respectively. Supposed its membership functions $L(x)$ and $R(x)$ are continuous and strictly monotonic, then we have its $\lambda-$cut set $A_\lambda = [a - \alpha L^{-1}(\lambda), b + \beta R^{-1}(\lambda)]$ for $\lambda \in (0, 1]$.

Then, we have

$$\begin{aligned}
A_L &= \int_0^1 2\lambda A^-(\lambda)d\lambda = 2\int_0^1 \lambda[a - \alpha L^{-1}(\lambda)]d\lambda \\
&= \int_0^1 2\lambda a d\lambda - 2\int_0^1 \lambda\alpha L^{-1}(\lambda)d\lambda \\
&= a - \alpha\int_0^1 2\lambda L^{-1}(\lambda)d\lambda
\end{aligned}$$

Using the well-known formulas for integration by substitution and the integration by parts, we can get $A_L = a - \int_{a-\alpha}^{a} L^2(\frac{a-x}{\alpha})dx$.

With the same method, we can get

$$\begin{aligned}
A_U &= \int_0^1 2\lambda A^+(\lambda)d\lambda = 2\int_0^1 \lambda[b + \beta R^{-1}(\lambda)]d\lambda \\
&= \int_0^1 2\lambda b d\lambda + 2\int_0^1 \lambda\beta R^{-1}(\lambda)d\lambda \\
&= b + \beta\int_0^1 2\lambda R^{-1}(\lambda)d\lambda \\
&= b + \int_b^{b+\beta} R^2(\frac{x-b}{\beta})dx
\end{aligned}$$

Therefore, its weighted interval approximation $A^* = [a - \int_{a-\alpha}^{a} L^2(\frac{a-x}{\alpha})dx, b + \int_{b}^{b+\beta} R^2(\frac{x-b}{\beta})dx]$.

Example 4(One-sided fuzzy numbers) Aimed at some fuzzy numbers such as "rather greater than 100" and "less than 20" and so on. We call this type of fuzzy number A as one-side(left-sided/right-sided) fuzzy number. If it satisfies the following properties:

1) A is normal;
2) A is fuzzy convex;
3) $A(x)$ is upper semicontinuous;
4) $supp(A)$ is bounded only from the left/right side.

It implies that every $\lambda-$cut set of the left-sided and right-sided fuzzy number have this kind of form $[A^-(\lambda), +\infty)$ and $(-\infty, A^+(\lambda)]$, respectively.

Therefore, based on our distance measure d, we can further get that their weighted interval approximations of left-sided and right-sided fuzzy numbers are $[\int_0^1 2\lambda A^-(\lambda)d\lambda, +\infty)$ and $(-\infty, \int_0^1 2\lambda A^+(\lambda)d\lambda]$, respectively.

Example 5(General fuzzy number) Given a fuzzy number A defined by definition 1, and its membership functions $f_A(x)$ and $g_A(x)$ are continuous and strictly monotonic, then we have its $\lambda-$cut set $A_\lambda = [f_A^{-1}(\lambda), g_A^{-1}(\lambda)]$ for $\lambda \in (0, 1]$.

Then, we have

$$\begin{aligned} A_L &= \int_0^1 2\lambda A^-(\lambda)d\lambda = \int_0^1 2\lambda f_A^{-1}(\lambda)d\lambda \\ &= \int_a^b 2xf_A(x)f_A'(x)dx = \int_a^b xdf_A^2(x) \\ &= xf_A^2(x)|_a^b - \int_a^b f_A^2(x)dx \\ &= b - \int_a^b f_A^2(x)dx \end{aligned}$$

$$\begin{aligned} A_U &= \int_0^1 2\lambda A^+(\lambda)d\lambda = \int_0^1 2\lambda g_A^{-1}(\lambda)d\lambda \\ &= \int_d^c 2xg_A(x)g_A'(x)dx = \int_d^c xdg_A^2(x) \\ &= xg_A^2(x)|_d^c - \int_d^c g_A^2(x)dx \\ &= c + \int_c^d g_A^2(x)dx \end{aligned}$$

Therefore, its weighted interval approximation $A^* = [b - \int_a^b f_A^2(x)dx, c + \int_c^d g_A^2(x)dx]$.

5 Conclusion

In this paper, we introduce a new approach to represent the interval approximation of a fuzzy number, weighted interval approximation, based on our distance measure d. We generalize the conclusion of the reference[9] and obtain a more common conclusion, on the other hand, we also point out that our weighted interval approximation of fuzzy number has some good properties such as intuitive, simple, natural and continuous. Some examples of weighted interval approximations of fuzzy numbers such as being triangular, trapezoidal, LR type, one sided and general show that our approach is worth and convenient to apply into practice.

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