ON STRONGLY ALMOST CONVERGENT SEQUENCES OF FUZZY NUMBERS

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Abstract. In this paper, we introduce and study some new sequence spaces of fuzzy numbers generated by an infinite matrix $A = (a_{nk})(n, k = 1, 2, ...)$

1. Introduction

Let $C(\mathbb{R}^n)$ denote the space of all compact and convex subsets of $L(\mathbb{R}^n)$. The Hausdorff distance between the sets A and B of $C(\mathbb{R}^n)$ is defined by

$$\delta_\infty(A,B) = max\{sup_{a\in A} inf_{b\in B}\|a-b\|, \\ sup_{b\in B} inf_{a\in A}\|a-b\|\}.$$

It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is complete (not separable) metric space.

A fuzzy number is a function X from $\mathbb{R}^n$ to $[0,1]$ so that
(i) X is normal, i.e. there exists an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$,
(ii)X is fuzzy convex, i.e. for any $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$,

$$X(\lambda x + (1-\lambda)y) \ge min\{X(x), X(y)\},$$

(iii) X is upper-semi-continuous,
(iv) The closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$ denoted by X^c, is compact.

These properties imply that for each $0 < \alpha \le 1$, the $\alpha-$level set $X^\alpha = \{x \in \mathbb{R}^n : X(x) \ge \alpha\}$ is a non-empty compact convex subset of $\mathbb{R}^n$ with the compact support.

If $\mathbb{R}^n$ is replaced by $\mathbb{R}$, then obviously the set $C(\mathbb{R}^n)$ is reduced to the set D of all closed bounded intervals $A = [\underline{A}, \overline{A}]$ on $\mathbb{R}$, and also

$$\delta_\infty(A,B) = \max\left(|\underline{A} - \underline{B}|,\ |\overline{A} - \overline{B}|\right).$$

Let $L(\mathbb{R})$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R})$ induces addition $X+Y$ and scalar multiplication $\lambda X, (\lambda \in \mathbb{R})$ in terms of α-level sets, by

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda\,[X]^\alpha$$

for each $0 \le \alpha \le 1$.

Define a map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$\overline{d}(X,Y) = \sup_{0<\alpha\le 1} \delta_\infty(X^\alpha, Y^\alpha).$$

For $X, Y \in L(\mathbb{R})$ define $X \le Y$ if and only if $X^\alpha \le Y^\alpha$ for any $\alpha \in [0,1]$. It is known that $\left(L(\mathbb{R}), \overline{d}\right)$ is a complete metric space (see [1])

Recently, S. Nanda [5] studied the spaces of bounded and convergent sequences of fuzzy numbers and showed that they are complete metric spaces.

Quite recently, M. Basarir and Mursaleen [2] defined some new sequence spaces of fuzzy numbers by using the A-transforms and studied some topological properties and inclusion relations for these new sequence spaces.

We will need the following definitions (see [5] and [7]).

Definition 1. *A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $L(\mathbb{R})$. The fuzzy number X_n denotes the value of the function at $n \in N$ and is called the n-th term of the sequence. We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.*

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Definition 2. *A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0, written as $lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer N_0 such that*

$$\bar{d}(X_k, X_0) < \epsilon \quad \text{for} \quad k > N_0.$$

Let $c(F)$ and $c_0(F)$ denote the sets of all convergent and null sequences of fuzzy numbers, respectively.

Definition 3. *A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded. We denote by $\ell_\infty(F)$ the set of all bounded sequences of fuzzy numbers.*

It is straightforward to see that

$$c(F) \subset \ell_\infty(F) \subset w(F).$$

In [5], it was shown that $c(F)$ and $\ell_\infty(F)$ are complete metric spaces with the metric given by

$$\rho(X, Y) = \sup_k \bar{d}(X_k, Y_k)$$

Definition 4. *The sequence $X = (X_k)$ of fuzzy numbers is said to be almost convergent to a fuzzy number L if,*

$$\lim_m \bar{d}(t_{mn}(X), L) = 0, \qquad (1.1)$$

uniformly in n, where

$$t_{mn}(X) = \frac{1}{m+1} \sum\nolimits_{i=0}^{m} X_{n+i}$$

This means that for every $\varepsilon > 0$, there exist a $m_0 \in \mathbb{N}$ such that

$$\bar{d}(t_{mn}(X), L) < \varepsilon$$

whenever $m \geq m_0$ and for all n.

If the limit in (1.1) exist, then we write

$$\hat{c}(F) - \lim X = X_0.$$

Let $\hat{c}(F)$ be the space of all almost convergent sequences of fuzzy numbers. It is evident that

$$\hat{c}_0(F) \subset \hat{c}(F),$$

where $\hat{c}_0(F)$ denotes the space of all almost convergent to zero of fuzzy numbers.

We now define $[\hat{c}, p]_0(F)$,$[\hat{c}, p](F)$ and $[\hat{c}, p]_\infty(F)$

If p_m is a bounded sequences of strictly positive real numbers. We define,

$$[\hat{c}, p]_0(F) = \{X = (X_k) \in w(F) : \frac{1}{m} \sum_{k=1}^{n} [\bar{d}(X_{k+n}, 0)]^{p_k} \to 0 \text{ as } m \to \infty, \text{ uniformly in } n\}$$

$$[\hat{c}, p](F) = \{X = (X_k) \in w(F) : \frac{1}{m} \sum_{k=1}^{n} [\bar{d}(X_{k+n}, X_0)]^{p_k} \to 0 \text{ as } m \to \infty, \text{ uniformly in } n\}$$

$$[\hat{c}, p]_\infty(F) = \left\{X = (X_k) \in w(F) : \sup_{m,n} \frac{1}{m} \sum_{k=1}^{n} [\bar{d}(X_{k+n}, 0)]^{p_k} < \infty \right\}$$

and call them respectively the spaces of sequences of fuzzy numbers which are strongly almost convergent to zero,strongly almost convergent to X_0 and strongly almost bounded.

Let $A = (a_{n,k})$ be an infinite matrix of fuzzy numbers $a_{n,k}$, then we write

$$A_{mn}(x) = \sum_k a_{mk} [\bar{d}(X_{k+n}, 0)]$$

if the series converges for each m and n.

We define

$$\left[\widehat{A}, p\right]_0(F) = \{X = (X_k) \in w(F) : \sum_k a_{mk} [\bar{d}(X_{k+n}, 0)]^{p_k} \to 0 \text{ as } m \to \infty, \text{ uniformly in } n\}$$

$$\left[\widehat{A}, p\right](F) = \{X = (X_k) \in w(F) : \sum_k a_{mk} [\bar{d}(X_{k+n}, X_0)]^{p_k} \to 0 \text{ as } m \to \infty, \text{ uniformly in } n\}$$

$$\left[\widehat{A},p\right]_\infty(F)=\{X=(X_k)\in w(F):\ \sup_{m,n}\sum_{k=1}^{n}a_{mk}[\bar{d}(X_{k+n},0)]^{p_k}<\infty\Big\}$$

and call them respectively the spaces of strongly almost summable to zero, strongly almost summable to X_0 and strongly almost bounded sequences of fuzzy numbers $X=(X_k)$.

If we take $A=(a_{mk})$ is a Cesáro matrix of order 1, i.e.

$$a_{mk}=\{\begin{array}{l}\frac{1}{m},\ k\le n\\ 0\ ,\ k>n\end{array}$$

then $[\widehat{A},p]_0(F)=[\hat{c},p]_0\,(F)$, $[\widehat{A},p](F)=[\hat{c},p]\,(F)$ and $[\widehat{A},p]_\infty(F)=[\hat{c},p]_\infty\,(F)$, and further on taking $p_k=1$ for all k these spaces are reduce to the following new sequence spaces:

$$[\hat{c}]_0\,(F)=\{X=(X_k)\in w(F):\ \frac{1}{m}\sum_{k=1}^{n}[\bar{d}(X_{k+n},0)]\to 0 \text{ as } m\to\infty,\text{ uniformly in } n\}$$

$$[\hat{c}]\,(F)=\{X=(X_k)\in w(F):\ \frac{1}{m}\sum_{k=1}^{n}[\bar{d}(X_{k+n},X_0)]\to 0 \text{ as } m\to\infty,\text{ uniformly in } n\}$$

$$[\hat{c}]_\infty\,(F)=\{X=(X_k)\in w(F):\ \sup_{m,n}\frac{1}{m}\sum_{k=1}^{n}[\bar{d}(X_{k+n},0)]<\infty\Big\}$$

A metric $\bar{d}$ on $L(\mathbb{R})$ is said to be a translation invariant if

$$\bar{d}(X+Z,Y+Z)=\bar{d}(X,Y)$$

for $X,Y,Z\in L(\mathbb{R})$.

Proposition 1. *If $\bar{d}$ is a translation invariant metric on $L(\mathbb{R})$ then*
(i) $\bar{d}(X+Y,0)\le\bar{d}(X,0)+\bar{d}(Y,0)$
(ii) $\bar{d}(\lambda X,0)\le|\lambda|\ \bar{d}(X,0),\ |\lambda|>1.$

Proof. (i) By the triangle inequality

$$\begin{aligned}\bar{d}(X+Y,0)&\le\bar{d}(X+Y,Y)+\bar{d}(Y,0)\\&=\bar{d}(X+Y,Y+0)+\bar{d}(Y,0)\\&=\bar{d}(X,0)+\bar{d}(Y,0)\end{aligned}$$

since $\bar{d}$ in a translation invariant.
(ii) By (i)

$$\begin{aligned}\bar{d}(2X,0)&\le\bar{d}(X+X,0)\\&\le\bar{d}(X,0)+\bar{d}(X,0)\\&=2\bar{d}(X,0)\end{aligned}$$

Hence by induction we get (ii).

Note that if $A=(c,1)$ and $p_k=1$ for all k, then

$$\begin{aligned}t_{mn}(X)&=\frac{1}{m}\sum_{k=1}^{m}[\bar{d}(X_{k+n},0)]\\&=\frac{1}{m}\sum_{k=n+1}^{n+m}[\bar{d}(X_k,0)].\end{aligned}$$

Now

$$\begin{aligned}\sup_{m,n}t_{mn}(X)&\le\sup_{m}\frac{\sup_k[\bar{d}(X_k,0)]}{m}\sum_{k=n+1}^{n+m}1\\&=\sup_k[\bar{d}(X_k,0)]\end{aligned}\tag{1.2}$$

and

$$\begin{aligned}\sup_{m,n}t_{mn}(X)&\ge\sup_{1,n}t_{1,n}(X)\\&=\sup_n[\bar{d}(X_{n+1},0)].\end{aligned}\tag{1.3}$$

From (1.2) and (1.3) we get $[\hat{c}]\,(F)=\ell_\infty(F)$.

Proposition 2. *Let (p_k) be a bounded sequence of strictly positive real numbers. Then,*

$$\left[\widehat{A},p\right]_0(F),\ \left[\widehat{A},p\right](F)\ \text{and}\ \left[\widehat{A},p\right]_\infty(F)$$

are linear spaces over the complex field $\mathbb{C}$, if d is a translation invariant metric.

Proof. This can be proved by using the techniques similar to those used in

Proposition 2.2. of Basarir and Mursaleen [2]

If $X = (X_k)$ is strongly almost summable to X_0, then we write

$$X_k \to X_0([\widehat{A}, p](F)).$$

A pair (A, p) will be called strongly almost regular if $X_k \to X_0$ implies that $X_k \to X_0([\widehat{A}, p](F))$.

Theorem 1. *Suppose that $A \in (c_0(F), \hat{c}_0(F))$ and $p = (p_k)$ converges to a positive limit.*

Then $X_k \to X_0([\widehat{A}, p](F))$, $X_k \to Y_0([\widehat{A}, p](F))$ imply $X_0 = Y_0$ if

$$\sum_k a_{mk} \nrightarrow 0 (m \to \infty). \qquad (1.4)$$

Proof. Suppose that condition (1.4) holds, $A \in (c_0(F), \hat{c}_0(F))$ and $p_k \to r > 0$. Further assume that $X_k \to X_0$ implies

$$X_k \to X_0([\widehat{A}, p](F))$$

and $X_k \to Y_0([\widehat{A}, p](F))$, where $\overline{d}(X_0, Y_0) = b > 0$. Then we get

$$\lim_{m\to\infty} \sum_k a_{mk} u_{k,n} = 0, \qquad (1.5)$$

uniformly in n, where

$$u_{k,n} = \overline{d}(X_{k+n}, X_0)^{p_k} + \overline{d}(X_{k+n}, Y_0)^{p_k}.$$

By the assumption $u_{k,n} \to b^r$. Since $A \in (c_0(F), \hat{c}_0(F))$, $u_{k,n} \to b^r$ implies that

$$\sum_k a_{mk} \overline{d}(u_{k,n}, b^r) \to 0, \qquad (1.6)$$

as $m \to \infty$, uniformly in n.

But we have

$$b^r \sum_k a_{mk} \leq \sum_k a_{mk} u_{k,n} + \sum_k a_{mk} \overline{d}(u_{k,n}, b^r). \qquad (1.7)$$

By (1.5) , (1.6) and (1.7) it follows that

$$\lim_{m\to\infty} \sum_k a_{mk} = 0.$$

Since this contradicts (1.4), we must have $X_0 = Y_0$, and this completes the proof.

Finally we conclude this paper by stating the following theorem. We omit the proof since it involves ideas similar to those used in proving theorem 3.4. of Mursaleen and Basarir [3]

Theorem 2. *Let m_1 and m_2 be constants such that $0 < m_1 \leq p_k \leq m_2$. Then (A, p) is strongly almost regular if and only if $A = (a_{mk})$ transforms null sequences into almost null sequences, i.e. $A \in (c_0(F), \hat{c}_0(F))$.*

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