ON FUZZY REAL VALUED $\ell(p)^F$ SEQUENCE SPACE

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Abstract. In this article we study different properties of the fuzzy real-valued sequence space $\ell(p)^F$ like convergence free, solidness , sequence algebra and symmetricity .

Key Words. Fuzzy real number, solid space, symmetric space, convergence free, sequence algebra.

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1. INTRODUCTION

The notion of paranormed classical sequence spaces was studied at the initial stage by Nakano [4] and Simons [8] . Later on it was further investigated from sequence space point of view and matrix classes involving paranormed sequence spaces were characterized by Maddox [3] and a large number of workers in the field of sequence spaces.

Throughout the paper $p = (p_k)$ denotes a sequence of bounded strictly positive numbers. For (a_k) and (b_k) two sequences of complex terms , we have the following well known inequality

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\},$$

where $D = \max\{1, 2^{H-1}\}$ and $H = \sup_k p_k$.

Let D denote the set of all closed bounded intervals $X = [a_1, a_2]$ on the real line R. For $X, Y \in D$ we define

$$X \le Y \text{ if and only if } a_1 \le b_1 \text{ and } a_2 \le b_2 .$$

$$d(X , Y) = max(| a_1 - b_1| , | a_2 - b_2 |) ,$$

where X =[a_1 , a_2] and Y =[b_1 , b_2]. It is known that (D, d) is a complete metric space and " $\le$ " is a partial order on D.

A *fuzzy real number X* is a fuzzy set on R *and* is a mapping $X: R \to I$ (=[0,1]) associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* , if $X(t) \ge X(s) \wedge X(r) = min\,(X(s) , X(t))$, where $s < t < r$.

If there exists $t_o \in R$ such that $X(t_0)$=1, then the fuzzy real number X is called *normal*.

The α- level set of a fuzzy real number X, $0 < \alpha \le 1$, denoted by X^{α} is defined as $X^{\alpha} = \{ t \in R : X(t) \ge \alpha \}$.

A fuzzy real number X is said to be *upper semi continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon])$, for all $a \in I$ is open in the usual topology of R .

The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(I)$. The set R of real numbers can be embedded in $R(I)$ if we defined $\overline{r} \in R(I)$ by

$$\overline{r}\,(t) = \begin{cases} 1, & \text{if } t = r , \\ 0, & \text{if } t \ne r . \end{cases}$$

The absolute value of $|X|$ of $X \in R(I)$ is defined by (see for instance Kaleva and Seikkala [2])

$$|X|(t) = \begin{cases} \max\{ X(t) , X(-t) \}, & \text{if } t \ge 0 , \\ 0 , & \text{if } t < 0. \end{cases}$$

Let $\overline{d} : R(I) \times R(I) \to R$ be defined by

$$\overline{d}(X,Y) = \sup_{0 \le \alpha \le 1} d(X^{\alpha}, Y^{\alpha}) .$$

Then $\overline{d}$ defines a metric on $R(I)$. We define $X \le Y$ if and only if $X^{\alpha} \le Y^{\alpha}$ for any $\alpha \in I$.

The additive identity and the multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

2. DEFINITION AND PRELIMINARIES

After the introduction of $R(I)$, different fuzzy real valued sequences were introduced and studied by Nanda [5], Tripathy and Nanda [10], Savas [7], Nuray and Savas [6], Subrahmanyam [9], Das and Choudhury [1] and many others.

Throughout the article w^F , bv^F , c^F , ℓ_1^F denote the classes of *all* , *bounded variation* , *convergent* and *absolutely summable* fuzzy real-valued sequence spaces .

A fuzzy real-valued sequence $\{X_n\}$ is said to be *convergent* to the fuzzy real number X, if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\bar{d}(X_k, X) < \varepsilon$, for all $k \geq n_0$.

The set E^F of sequences taken from $R(I)$ is said to be a sequence space of fuzzy real number if, for $\{X_i\}, \{Y_i\} \in E^F$, $r \in R$, $X_i \in R(I)$, $Y_i \in R(I)$,

$$\{X_i\} + \{Y_i\} = \{X_i + Y_i\} \in E^F$$

and

$$r\{X_i\} = \{rX_i\} \in E^F,$$

where, $r X_i(t) = X_i(r^{-1}t)$ if $r \neq 0$,
$\qquad\qquad\quad = 0$ if $r = 0$.

Let paper $p = (p_k)$ denotes a sequence of bounded strictly positive numbers. Nuray and Savas [6] defined the fuzzy real-valued sequence spaces $\ell(p)^F$ as follows :

$$\ell(p)^F = \left\{ (X_k) \in w^F : \sum_{k=1}^{\infty} \left[\bar{d}(X_k, \bar{0}) \right]^{p_k} < \infty \right\}.$$

Taking $p_k = p$ for all $k \in N$, it represents the fuzzy real-valued sequence space ℓ_p^F , for $1 \leq p < \infty$ introduced and studied by Nanda [5] .

Nuray and Savas [6] have studied some of the properties of the space $\ell(p)^F$ and have shown that it is a complete metric space under the metric defined by

$$\rho(X, Y) = \left[\sum_{k=1}^{\infty} \left\{ \bar{d}(X_k, Y_k) \right\}^{p_k} \right]^{\frac{1}{M}},$$

where $M = \max(1, H)$.

Taking $p_k = 1$, for all $k \in N$, we have the space

$$\ell^F = \left\{ (X_k) \in w^F : \sum_{k=1}^{\infty} \bar{d}(X_k, \bar{0}) < \infty \right\}.$$

A sequence space E^F is said to be *normal* (or *solid*) if $(Y_i) \in E^F$ whenever $|Y_i| \le |X_i|$ for all $i \in N$ for some $\{X_i\} \in E^F$.

Let $K = \{ k_1 < k_2 < \cdots \} \subseteq N$ and E be a sequence space. A *K-step space* of E is a sequence space $\lambda_K^E = \{(x_{k_i}) \in w : (x_n) \in E\}$.

A *canonical preimage* of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, *i.e.*, y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

A sequence space E is said to be *monotone* if it contains the canonical preimage of all its step spaces (*i.e.*, if for all infinite $K \subseteq N$ and $(x_n) \in E$ the sequence $(\alpha_n x_n)$, where $\alpha_n = 1$ for $n \in K$ and $\alpha_n = 0$ otherwise, belongs to E).

Remark . From the above definitions it is follows that " *if a fuzzy real-valued sequence space E^F is solid, then it is monotone* " .

A sequence space E^F is said to be *symmetric* if $(X_{\pi(n)}) \in E^F$, whenever $(X_n) \in E^F$, $\pi(n)$ is a permutation of N.

A sequence space E^F is said to be a *sequence algebra* if $(X_n . Y_n) \in E^F$, whenever $(X_n), (Y_n) \in E^F$.

A sequence space is E^F is said to be *convergence free* if $(Y_n) \in E^F$, whenever $(X_n) \in E^F$ and $X_n = \bar{0}$ implies $Y_n = \bar{0}$.

3. MAIN RESULTS

In this section we prove the results of this article.

Theorem 1. *The sequence space $\ell(p)^F$ is solid, as such is monotone.*

Proof . Let $(X_k) \in \ell(p)^F$ and $(Y_k) \in w^F$ be such that $|Y_k| \le |X_k|$ for all $k \in N$. Then we have the inequality

$$\sum_{k=1}^{\infty} \left\{ \bar{d}\left(Y_k, \bar{0} \right) \right\}^{p_k} \le \sum_{k=1}^{\infty} \left\{ \bar{d}\left(X_k, \bar{0} \right) \right\}^{p_k} < \infty.$$

Thus $(Y_k) \in \ell(p)^F$. Hence the space $\ell(p)^F$ is solid.

The space $\ell(p)^F$ is monotone follows from Remark 1.

Theorem 2. *The space $\ell(p)^F$ is not convergence free.*

Proof . The result follows from the following example .

Example 1. Let the sequence (p_k) be defined by $p_k = 2$ for all k odd and $p_k = 3$ for all k even. Consider the sequence (X_k) defined by $X_k = \overline{k^{-1}}$, for all $k \in N$ and $Y_k = \overline{k}$ for all $k \in N$. Then $(X_k) \in \ell(p)^F$, but $(Y_k) \notin \ell(p)^F$. Hence the space $\ell(p)^F$ is not convergence free.

Theorem 3. *The space $\ell(p)^F$ is not symmetric.*

Proof . The result follows from the following example .

Example 2. Let the sequence (p_k) be defined by $p_k = 1$ for all k odd and $p_k = 2$ for all k even. Consider the sequence (X_k) defined by $X_k = \overline{k^{-2}}$, for all k odd and $X_k = \overline{k^{-1}}$ for all k even. Then $(X_k) \in \ell(p)^F$. Consider the rearrangement (Y_k) of (X_k) defined as

$$(Y_k) = (X_2, X_1, X_4, X_3, X_6, X_5, X_8, X_7, X_{10}, X_9, \ldots\ldots).$$

Then $(Y_k) \notin \ell(p)^F$. Hence the space $\ell(p)^F$ is not symmetric.

Theorem 4. *The space $\ell(p)^F$ is a sequence algebra.*

Proof . The result follows from the following inequality

$$\sum_{k=1}^{\infty}\left\{\bar{d}\left(X_k \otimes Y_k, \bar{0}\right)\right\}^{p_k} \leq \left(\sum_{k=1}^{\infty}\left\{\bar{d}\left(X_k, \bar{0}\right)\right\}^{p_k}\right)\left(\sum_{k=1}^{\infty}\left\{\bar{d}\left(Y_k, \bar{0}\right)\right\}^{p_k}\right).$$

Theorem 5. *Let $p_k > 0$ and $q_k > 0$ for all $k \in N$. Then $\ell(q)^F \subset \ell(p)^F$ if and only if*

$$\underset{k}{\underline{\lim}} \inf \frac{p_k}{q_k} > 0 \qquad \text{- - - - - - - - (1)}$$

Proof. Let (1) holds, then there exists $\beta > 0$ such that $p_k > \beta q_k$, for sufficiently large k. Hence the inclusion follows from the following inequality :

For large values of k, we have

$$\left\{\overline{d}\left(X_k, \overline{0}\right)\right\}^{p_k} \le \left(\left\{\overline{d}\left(X_k, \overline{0}\right)\right\}^{q_k}\right)^{\beta}.$$

Conversely, suppose that the inclusion holds, but (1) fails. Then there exists a sequence (k_i) of naturals such that $k_i < k_{i+1}$ for all $i \in N$ and $ip_{k_i} < q_{k_i}$. Define the sequence (X_k) defined by $X_{k_i} = i^{-\frac{2}{q_{k_i}}}$ for all $i \in N$ and $X_k = \overline{0}$, otherwise. Then $X = (X_k) \in \ell(q)^F$. We have

$$\left\{\overline{d}\left(X_{k_i}, \overline{0}\right)\right\}^{p_{k_i}} > \left\{\exp\left(-\frac{\log i}{i}\right)\right\}^2$$

$$> \left\{\exp\left(-i^{-1}\right)\right\}^2$$

$$\nrightarrow 0, \text{ as } i \to \infty.$$

Hence $(X_k) \notin \ell(p)^F$.

The following result follows from the above result.

Corollary 6. *Let* $p_k > 0$ *and* $q_k > 0$ *for all* $k \in N$. *Then* $\ell(q)^F = \ell(p)^F$ *if and only if*

$$\liminf_k \frac{p_k}{q_k} > 0 \text{ and } \liminf_k \frac{q_k}{p_k} > 0.$$

We have the following result in view of the above result.

Proposition 7. *Let* $0 < \inf p_k \le p_k \le \sup p_k < \infty$. *Then* $\ell(p)^F = \ell^F$.

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