Fuzzy Integration and Luroth's Theorem

Shih-Chuan Cheng and John N. Mordeson, Department of Mathematics/Computer Science, Creighton University, Omaha, Nebraska 68178 USA

Introduction.

A *fuzzy subset* of a set X is a function from X into the closed interval $[0,1]$. Let f be a fuzzy subset of X and $\alpha \in [0,1]$. Let $f_\alpha = \{x \in X \mid f(x) \geq \alpha\}$. Then f_α is call a *level* subset of f. f is called *convex* if $\forall \lambda \in [0,1]$ and $\forall x, y \in X$, $f(\lambda x + (1-\lambda)\, y) \geq \min\{f(x), f(y)\}$. It is known that f is convex if and only if f_α is convex $\forall \alpha \in [0,1]$, [3, Proposition 6.1, p.299]. We recall that a monoid is a semigroup with identity. We let $\mathbb{R}$ denote the set of all real numbers.

Fuzzy Numbers and Fuzzy Functions.

Let $\mathbb{C}_p(\mathbb{R})$ denote the set of all piecewise continuous functions of $\mathbb{R}$ into $\mathbb{R}$. We let $\underset{\smile}{[a,b]}$ denote the closed interval where $a, b \in \mathbb{R}$ and $a \leq b$. Let $\widetilde{P}(\mathbb{R})$ denote the set of all fuzzy subsets of $\mathbb{R}$.

Definition 1. [12] Let $\widetilde{A}$ be a fuzzy subset of $\mathbb{R}$. Then $\widetilde{A}$ is called a *fuzzy number* if and only if $\widetilde{A}$ has the following properties:

(1) $\widetilde{A}$ is convex;
(2) there exists unique $x_0 \in \mathbb{R}$ such that $\widetilde{A}(x_0) = 1$;
(3) $\widetilde{A} \in \mathbb{C}_p(\mathbb{R})$.

We let $\mathcal{F}(\mathbb{R})$ denote the set of all fuzzy numbers.

We define $\oplus$ and $\odot$ on $\mathcal{F}(\mathbb{R})$ as follows: $\forall \widetilde{A}, \widetilde{B}$, $\forall y \in \mathbb{R}$,

$$\begin{aligned} & \widetilde{A} \oplus \widetilde{B}(y) \\ = \quad & \sup\left\{\min\left\{\widetilde{A}(w), \widetilde{B}(z)\right\} \mid y = w + z, w, z \in \mathbb{R}\right\} \end{aligned}$$

and

$$\begin{aligned} & \widetilde{A} \odot \widetilde{B}(y) \\ = \quad & \sup\left\{\min\left\{\widetilde{A}(w), \widetilde{B}(z)\right\} \mid y = w \cdot z, w, z \in \mathbb{R}\right\} \end{aligned}$$

[12, p.59].

Theorem 2. Let $\oplus$ and $\odot$ be defined on $\mathcal{F}(\mathbb{R})$ as above. Then $(\mathcal{F}(\mathbb{R}), \oplus)$ and $(\mathcal{F}(\mathbb{R}), \odot)$ are commutative monoids.

Proof. The proof follows from [12, p.60, 61]. $\square$

Let $\widetilde{A} \in \mathcal{F}(\mathbb{R})$ and let $\alpha \in [0,1]$. Since $\widetilde{A}$ is convex, $\widetilde{A}_\alpha$ is convex. Thus $\widetilde{A}_\alpha$ is a connected closed subset of $\mathbb{R}$. If we define the fuzzy subset $\widetilde{B}$ of $\mathbb{R}$ by $\widetilde{B}(0) = 1$ and $\widetilde{B}(y) = 1/2$ for $y \neq 0$, then $\widetilde{B}$ is a fuzzy number. If $\alpha \in [0, 1/2]$, then $\widetilde{B}_\alpha = \mathbb{R}$. Thus the level sets of $\widetilde{B}$ need not be bounded closed intervals.

In the following X denotes a a nonempty set. We now consider functions $\widetilde{f}$ of X into $\widetilde{P}(\mathbb{R})$ such that $\widetilde{f}(x)$ is a fuzzy number for all x in X, [12]. We can consider such an $\widetilde{f}$ to be a function of $X \times \mathbb{R}$ into $[0,1]$ such that $\forall x \in X$, $\widetilde{f}(x, \_)$ is a fuzzy number.

Proposition 3. [7] Let $\widetilde{f}$ be a function of X into $\widetilde{P}(\mathbb{R})$ such that $\widetilde{f}(x)$ is a fuzzy number for all x in X. Suppose that $\forall \alpha \in [0,1]$, $\widetilde{f}(x)_\alpha$ is a bounded closed interval. Then there exist unique functions f^-, f^+ of $[0,1] \times X$ into $\mathbb{R}$ such that

(1) $\forall x \in X$, $f^-(\_, x)$ $(f^+(\_, x))$ is a nondecreasing (nonincreasing) function of α;
(2) $\forall (\alpha, x) \in [0,1] \times X$, $f^-(\alpha, x) \leq f^+(\alpha, x)$;
(3) $\forall (\alpha, x) \in [0,1] \times X$, $\widetilde{f}(x)_\alpha = [f^-(\alpha, x), f^+(\alpha, x)]$;
(4) $\forall x \in X$, $f^-(1, x) = f^+(1, x)$.

Proposition 4. [7] Let g and h be functions of $[0,1] \times X$ into $\mathbb{R}$ such that $\forall x \in X$, $g(\_, x)$ $(h(\_, x))$ is a nondecreasing (nonincreasing) function of α and $\forall (\alpha, x) \in [0,1] \times X$, $g(\alpha, x) \leq h(\alpha, x)$. Let $\widetilde{f}$ be the function of $X \times \mathbb{R}$ into $[0,1]$ defined as follows: $\forall (x, y) \in X \times \mathbb{R}$,

$$\widetilde{f}(x, y) = \sup\{\beta \in [0,1] \mid y \in [g(\beta, x), h(\beta, x)]\}.$$

If $\forall x \in X$, $g(\_, x)$ and $h(\_, x)$ are continuous functions from the left, then

$$\widetilde{f}(x)_\alpha = [g(\alpha, x), h(\alpha, x)] \quad \forall \alpha \in [0,1].$$

Proposition 5. [7] Let g and h be functions of $[0,1] \times X$ into $\mathbb{R}$ such that $\forall x \in X$, $g(\_, x)$ $(h(\_, x))$ is a nondecreasing (nonincreasing) function of α and $\forall (\alpha, x) \in [0,1] \times X$, $g(\alpha, x) \leq h(\alpha, x)$. Let $\widetilde{f}$ be the function of $X \times \mathbb{R}$ into $[0,1]$ such that $\forall \alpha \in [0,1]$, $\widetilde{f}(x)_\alpha = [g(\alpha, x), h(\alpha, x)]$. Then $g(\_, x)$ and $h(\_, x)$ are continuous functions from the left $\forall x \in X$. Furthermore,

$$\widetilde{f}(x)(y) = \sup\{\beta \in [0,1] \mid y \in [g(\beta, x), h(\beta, x)]\}$$

if $y \in [g(0, x), h(0, x)]$ and $\widetilde{f}(x, y) = 0$ otherwise.

Let g and h be functions satisfying properties (1), (2), and (3) of Proposition 3 and which are continuous from the left in $\alpha \in [0,1]$ for each $x \in X$. Then by Proposition 4, there exists a function $\widetilde{f}$ of $X \times \mathbb{R}$ into $[0,1]$ such that $\widetilde{f}(x)_\alpha = [g(\alpha, x), h(\alpha, x)]$ $\forall \alpha \in [0,1]$. Now by Proposition 3, we have $g = f^-$ and $h = f^+$. Thus we see that f^- and f^+ are necessarily continuous from the left in $\alpha \in [0,1]$ for each $x \in X$.

Now suppose we are given a function $\widetilde{f}$ of $X \times \mathbb{R}$ into $[0,1]$ such that $\forall \alpha \in [0,1]$, $\widetilde{f}(x)$ is a bounded closed interval for each $x \in X$. Then there are unique functions f^- and f^+ satisfying properties $(1)-(4)$ of Proposition 4 and which are continuous from the left. By Proposition 4, f^- and f^+ define a function $\widetilde{g}$ of $X \times \mathbb{R}$ into $[0,1]$ such that $\widetilde{g}(x)_\alpha = [f^-(\alpha,x), f^+(\alpha,x)]$ $\forall \alpha \in [0,1]$. Thus $\widetilde{f} = \widetilde{g}$ since the level sets of fuzzy function uniquely determine it.

Definition 6. Let $\widetilde{f}$ be a function from X into $\widetilde{P}(\mathbb{R})$. Then $\widetilde{f}$ is called a *fuzzy function* of X into $\mathbb{R}$ if and only if $\widetilde{f}(x)$ is a fuzzy number for all $x \in X$ such that $\widetilde{f}(x)_\alpha$ is a bounded closed interval.

Luroth's Theorem for Fuzzy Intermediate Fields.

Luroth's Theorem states that if F/K is a simple pure transcendental field extension and E is an intermediate field of F/K with $E \supset K$, then E/K is also a simple pure transcendental field extension.

Let F/K be a pure transcendental field extension, say $F = K(x)$, where x is transcendental over K. Let $\mathcal{L} = \{L_\alpha \mid \alpha \in \Omega\}$ be a chain of intermediate field of F/K, i.e., $\forall L_\alpha, L_\beta \in \mathcal{L}$ with $\alpha \neq \beta$, either $L_\alpha \supset L_\beta$ or $L_\beta \supset L_\alpha$. Then $\Omega' \subseteq \Omega$, $\{L_\alpha \mid \alpha \in \Omega'\}$ has a first element with respect to $\supset$ since $[F:L] < \infty$ $\forall$ intermediate fields L of F/K such that $L \supset K$. Thus $\supset$ well orders $\mathcal{L}$, [10]. If $|\Omega| = \infty$, $\left[F : \bigcap_{\alpha\in\Omega} L_\alpha\right] = \infty$ since $\mathcal{L}$ is a chain. Thus, $\bigcap_{\alpha\in\Omega} L_\alpha = K$ when $|\Omega| = \infty$.

Suppose that $|\Omega| = \infty$. Since for any such $\mathcal{L}$, $\bigcap_{\alpha\in\Omega} L_\alpha = K$ and $\supset$ well orders $\mathcal{L}$, we have that there exists a one-to-one function of Ω onto $\mathbb{N}$ such that $\forall \alpha, \beta \in \Omega$, $L_\alpha \supset L_\beta$ if and only if $f(\alpha) < f(\beta)$. Hence $\mathcal{L} = \{L_1, \cdots, L_i, \cdots\}$ where $L_i \supset L_{i+1}$, $i = 1, 2, \cdots$.

A fuzzy subset μ of F is called a *fuzzy intermediate field* of F/K if μ is a constant on K, i.e., $\forall k \in K$, $\mu(k) = \alpha$ for some $\alpha \in (0,1]$, and $\alpha \geq \sup\{\mu(x) \mid x \in F\backslash K\}$. Let μ be a fuzzy immediate field of F/K. Then, by the previous paragraph, either $\mu(F) = \{\alpha_1, \cdots, \alpha_n\}$ for some $n \in \mathbb{N}$ or $\mu(F) = \{\alpha_1, \cdots, \alpha_n, \cdots\}$ since every level subset of μ is an intermediate field of F/K. By Luroth's Theorem each μ_{α_i}/K is a pure transcendental extension, say $\mu_{\alpha_i} = K(z_i)$. Now $z_i = f_i(x)/g_i(x)$ where $f_i(x)$ and $g_i(x)$ are polynomials in x over K with no common factor of positive degree in x. Let $n = \max\{\deg f, \deg g\}$. Then x is algebraic over $K(z_i)$ and $[F : K(z)] = n$. Moreover $h(t,z) = f(t) - zg(t)$ is irreducible in $K(z)[t]$.

Lemma 7. Let $F = K(x)$ be a pure transcendental extension of K. Let μ be a fuzzy intermediate field of F/K. If $F = K(y)$, then $\mu(y) = \mu(x)$.

Proof. By [2], there exists $a, b, c, d \in K$ such that $y = (ax+b)/(cx+d)$ where $ad - bc \neq 0$. Hence

$$\begin{aligned} \mu(y) &\geq \min\{\mu(ax+b), \mu(cx+d)\} \\ &\geq \min\{\mu(a), \mu(b), \mu(c), \mu(d), \mu(x)\} \\ &= \mu(x). \end{aligned}$$

Similarly, $\mu(x) \geq \mu(y)$. $\square$

Lemma 8. Let $F = K(x)$ be a pure transcendental extension of K. Let μ be a fuzzy intermediate field of F/K. Let $\alpha \in \mu(F)$ be such that $\mu_\alpha \supset K$. Then there exists $z \in F$ such that $\mu_\alpha = K(z)$ and $\mu(z) = \alpha$.

Proof. That z exists such that $\mu_\alpha = K(z)$ follows from Luroth's Theorem. Since $z \in \mu_\alpha$, $\mu(z) \geq \alpha$. Since $\alpha \in \mu(F)$, there exits $y \in \mu_\alpha$ such that $\mu(y) = \alpha$. Now $y = f(z)/g(z)$ for some polynomials $f(z)$ and $g(z)$ in z over K. Thus,

$$\alpha = \mu(y) \geq \min\{\mu(f(z)), \mu(g(z))\} \geq \mu(z) \geq \alpha.$$

Hence, $\mu(z) = \alpha$. $\square$

Definition 9. Let S be a set of fuzzy singletons in F such that if $x_\alpha, x_\beta \in S$, then $\alpha = \beta$. Let K be a subfield of F and μ a fuzzy intermediate field of F/K. Then S is said to be a *weak minimal generating set* of μ over K if $\mu = \mu|_K(S)$ and $\mu \supset \mu|_K(S\backslash\{x_\alpha\})$ $\forall x_\alpha \in S$.

Theorem 10. Let $F = K(x)$ be a pure transcendental field extension of K. Let μ be a fuzzy intermediate field of F/K. Then the following assertions hold:

(1) Either $\mu(F) = \{\alpha_1, \cdots, \alpha_n\}$ for some $n \in \mathbb{N}$ or $\mu(F) = \{\alpha_1, \cdots, \alpha_n, \cdots\}$.

(2) For each $\alpha_i \in \mu(F)$, $\mu_{\alpha_i} = K(z_i)$ for some $z_i \in F$ and $\mu(z_i) = \alpha_i$.

(3) $\{(z_i)_{\alpha_i} \mid i = 1, 2, \cdots\}$ is a weak minimal generating set of μ over K.

Proof. That (1) and (2) hold follows from previous discussions and the Lemmas. We now show that (3) holds. Let $Z = \{(z_i)_{\alpha_i} \mid i = 1, 2, \cdots\}$. Let $Z_j = Z\backslash\{(z_j)_{\alpha_j}\}$, $j = 1, 2, \cdots$. Then

$$\begin{aligned} &\mu(Z_j)(z_j) \\ = \;& \sup\Big\{\sum k_{i_1\cdots i_m}(((z_1)_{\alpha_1})^{i_1}\cdots del\left(((z_j)_{\alpha_j})^{i_j}\right) \\ &\cdots((z_m)_{\alpha_m})^{i_m}(z_j)\big) \mid k_{i_1\cdots i_m} \in K, \mu(k_{i_1\cdots i_m}) \\ &= u_{i_1\cdots i_m}, m \in \mathbb{N}\} \\ = \;& \alpha_{j+1} < \alpha_j = \mu(Z)(z_j). \end{aligned}$$

Hence Z is a weak minimal generating set for μ over K.

Integration of a Fuzzy Function over a Crisp Curve.

In the remainder of the paper, X denotes the Cartesian cross-product $[a,b] \times [c,d]$.

We let $C : f(x,y) = 0$ denote a crisp curve. Let $\widetilde{f}$ be a *fuzzy function* of $[a,b] \times [c,d]$ into $\mathbb{R}$. If $(C)\int f^-(\alpha,x,y)dx$ and $(C)\int f^+(\alpha,x,y)dx$ exist $\forall \alpha \in [0,1]$, then we say that

$\widetilde{f}$ is *integrable* over C.

Definition 11. Let $\widetilde{f}$ be a *fuzzy function* of $[a, b] \times [c, d]$ into $\mathbb{R}$ which is integrable over C. Then the integral of $\widetilde{f}$ over C, written $(C) \int \widetilde{f}(x, y)dx$, is defined to be the fuzzy subset $\widetilde{F}$ of $\mathbb{R}$ such that $\forall r \in \mathbb{R}$,

$$\begin{aligned}\widetilde{F}(r) \quad &= \quad \sup\{\beta \in [0,1] \mid r \in \Big[(C)\int f^{-}(\beta, x, y)dx, \\ &\qquad (C)\int f^{+}(\beta, x, y)dx\Big]\end{aligned}$$

if $y \in \left[(C)\int f^{-}(0,x,y)dx, (C)\int f^{+}(0,x,y)dx\right]$ and $\widetilde{F}(r) = 0$ otherwise.

A parametrization of C will yield the integral of $\widetilde{f}$ in the sense of [8].

Problem. Is $\widetilde{F}$ a "linear transformation" as in [8], etc.?

Problem. Use the fuzzy version of Luroth's theorem to get a fuzzy parametrization of C. Parametrization at levels.

Integration of a Crisp Function over a Fuzzy Curve.

Definition 12. A *fuzzy ordered pair* is a function $\widetilde{A} : \mathbb{R} \times \mathbb{R} \to [0, 1]$ such that the following properties hold:

(1) $\widetilde{A}$ is convex;
(2) there exists unique $(r_0, r_1) \in \mathbb{R} \times \mathbb{R}$ such that $\widetilde{A}(r_0, r_1) = 1$;
(3) $\widetilde{A}$ is continuous.

Definition 13. A *fuzzy curve* $\widetilde{C}$ is a set of fuzzy ordered pairs such that

$$C = \left\{(r_0, r_1) \in \mathbb{R} \times \mathbb{R} \mid \exists \widetilde{A} \in \widetilde{C}, \widetilde{A}(r_0, r_1) = 1\right\}$$

is a curve.

Definition 14. Let g be a function of $[a, b] \times [c, d]$ into $\mathbb{R}$. Let $\widetilde{C}$ be a fuzzy curve such that $C \subseteq [a, b] \times [c, d]$. Let (e_0, e_1) and (e_2, e_3). Let $\widetilde{A}, \widetilde{B} \in \widetilde{C}$ be such that $\widetilde{A}(e_0, e_1) = 1$ and $\widetilde{B}(e_2, e_3) = 1$. Define the function $\widetilde{G}$ of $\mathbb{R}$ into $[0, 1]$ as follows:

$$\begin{aligned}\widetilde{G} = \sup\Big\{\sup\Big\{\min\Big\{\widetilde{A}(d_0, d_1), \widetilde{B}(d_2, d_3)\Big\} \ \mid \\ (d_0, d_1), (d_2, d_3) \in \widetilde{D}\Big\} \ \mid \widetilde{D} \in D\Big\}\end{aligned}$$

where $z = (D) \int g(x, y)dx$ and $\widetilde{D} \in D$.

We note that this definition reduces to the one given in [12] for integrals of real-valued functions of one variable over an interval. To see this, project the plane onto the x-axis and so $c = d$ in the above definition. Then $D = \left\{\widetilde{C}\right\}$. Hence $\widetilde{G} = \sup\left\{\min\left\{\widetilde{A}(d_0, d_1), \widetilde{B}(d_2, d_3)\right\} \mid (d_0, d_1), (d_2, d_3) \in \widetilde{C}\right\} = \sup\left\{\min\left\{\widetilde{A}(d_0), \widetilde{B}(d_2)\right\} \mid d_0, d_2 \in [e_0, e_2]\right\}$ where $z = \int g(x, y)dx$, the integral being taken over the interval $[d_0, d_2]$.

Problem. Show that $\widetilde{G}$ is a fuzzy number.

Problem. Use the fuzzy version of Luroth's theorem to get a fuzzy parametrization of C. Parametrization at levels.

Integration of a Crisp Function over a Fuzzy Curve.

Problem. Use the fuzzy version of Luroth's theorem to get a fuzzy parametrization of C. Parametrization at levels.

Integration of a Crisp Function over a Fuzzy Curve. Another Possibility

The condition, $\forall (x, y) \in [a, b] \times [c, d]$, $f^{-}(1, x, y) = f^{+}(1, x, y)$, will allow us to define a fuzzy curve in such a way that a crisp curve can be associated with it in a natural way by defining a function f of $[a, b] \times [c, d]$ into $\mathbb{R}$ as follows: $f(x, y) = f^{-}(1, x, y) = f^{+}(1, x, y) \ \forall (x, y) \in [a, b] \times [c, d]$. The equation $\widetilde{C} : \widetilde{f}(x, y) = \widetilde{0}$ stands for the curve $C^{-}_{\alpha} : f^{-}(\alpha, x, y) = 0$ and $C^{+}_{\alpha} : f^{+}(\alpha, x, y) = 0 \ \forall \alpha \in [0, 1]$.

Definition 15. Let $\widetilde{f}$ be a *fuzzy function* of $[a, b] \times [c, d]$ into $\mathbb{R}$. Then a fuzzy plane algebraic curve is $\widetilde{C} : \widetilde{f}(x, y) = \widetilde{0}$, where $f(x, y)$ is a polynomial.

Definition 16. Let g be a function of $[a, b] \times [c, d]$ into $\mathbb{R}$ which contains the fuzzy curve $\widetilde{C} : \widetilde{f}(x, y) = \widetilde{0}$. Then the integral of g over $\widetilde{C}$, written $(\widetilde{C}) \int g(x, y)dx$, is defined to be the fuzzy subset $\widetilde{G}$ of $\mathbb{R}$ such that $\forall r \in \mathbb{R}$,

$$\begin{aligned}&\widetilde{G}(r) = \sup\{\beta \in [0,1] \mid r \in \\ &\left[(C^{-}_{\beta})\int g(x,y)dx, (C^{+}_{\beta})\int g(x,y)dx\right] \\ &\cup \left[(C^{+}_{\beta})\int g(x,y)dx, (C^{-}_{\beta})\int g(x,y)dx\right]\Big\}\end{aligned}$$

if

$$\begin{aligned}r \quad \in \quad &\left[(C^{-}_{\beta})\int g(x,y)dx, (C^{+}_{\beta})\int g(x,y)dx\right] \\ &\cup \left[(C^{+}_{\beta})\int g(x,y)dx, (C^{-}_{\beta})\int g(x,y)dx\right]\end{aligned}$$

for some β and $\widetilde{G}(y) = 0$ otherwise.

Problem. Use the fuzzy version of Luroth's theorem to get a fuzzy parametrization of C. Parametrization at levels.

Suppose there exists an isomorphism of f of F into $K(t)$. Then $f(\mu)$ is a fuzzy intermediate of F/K. This would seem to give us a fuzzy parametrization.

Let $\alpha \in \mu(F)$. If $\mu_{\alpha} = K(u)$, then μ has been characterized above. We now consider when $\mu_{\alpha} = K(u, v)$.

References

1. Dubois, D. and Prade, H., *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, Inc. 1980.

2. Jacobson, N., *Lectures in Abstract Algebra, Vol. III, Theory of Fields and Galois Theory*, D. Van Nostrand Company, Inc., Princeton, New Jersey 1964.

3. Lowen, R., Convex fuzzy sets, *Fuzzy Sets and Systems* 3 (1980) 291-310.

4. Mordeson, J. N., Fuzzy algebraic field extensions, *Fuzzy Sets and Systems* 45 (1992) 359-365.

5. Mordeson, J. N., Fuzzy field extensions, *Fuzzy Sets and Systems* 47 (1992) 253-264.

6. Mordeson, J. N., Fuzzy transcendental field extensions, *Chinese J. Math*. 20 (1992) 55-68.

7. Mordeson, J. N., Differentiation of fuzzy functions, *J. Fuzzy Math*. 11 (2003) 967-985.

8. Mordeson, J. N. and Newman, W. G., Fuzzy integral equations, *Inf. Sci.* 87 (1995) 215-229.

9. Shafarevich, I. R., *Basic Algebraic Geometry*, Speinger-Verlag, New York - Heidelberg - Berlin, 1974.

10. Suppes, P., *Axiomatic Set Theory*, D. Van Nostrand Co., Inc. 1960.

11. Zadeh, L. A., Fuzzy Sets, *Inform. Control* 8 (1965) 338-353.

12. Zimmermann, H. J., *Fuzzy Set Theory and Its Applications*, Kluwar Academic Publisher, Second Revised Edition, 1991.