Arithmetical Operations of Exponential Fuzzy Numbers and Application

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Abstract

Fuzzy numbers with exponential membership function are quite common in real world cases. In this paper, we applied Grade Mean Integration Method to compute the representation of exponential fuzzy numbers. Then Function Principle is applied to generate the arithmetical operations of exponential fuzzy numbers. Finally, an application of the lifetime of the lamps of projectors is proposed.

Keywords: Exponential fuzzy numbers, Function Principle, Graded Mean Integration Representation.

1. Introduction

Fuzzy theory was founded by Zadeh[11] in 1965. Many computation methods[8, 13] of fuzzy numbers have been proposed afterwards. Most of them are designed for fuzzy numbers with triangular and trapezoidal membership function.

In Section 2, the exponential fuzzy number is defined and expressed. The Grade Mean Integration method is applied to compute the representation of exponential fuzzy number in Section 3. In Section 4, Function Principle is applied to generate the arithmetical operations (addition, subtraction, multiplication and division) of exponential fuzzy numbers. An application of the lifetime of the lamps of projectors is proposed in Section 5. Finally, the concluding remarks are presented in the last section.

2. Exponential fuzzy number

In general, a fuzzy number A is described as any fuzzy subset of the real line R, whose membership function μ_A satisfies the following conditions:

(1) μ_A is a continuous mapping from R to the closed interval [0, 1],
(2) $\mu_A(x)=0, -\infty < x \le c$,
(3) $\mu_A(x)=L(x)$ is strictly increasing on [c, a],
(4) $\mu_A(x)=1, a \le x \le b$,
(5) $\mu_A(x)=R(x)$ is strictly decreasing on [b, d],
(6) $\mu_A(x)=0, d \le x < \infty$,

where a, b, c, and d are real numbers. We denote this type of fuzzy number as $A = (c, a, b, d)_{LR}$.

However, these fuzzy numbers always have a fix range as [c, d]. In this paper, we consider an exponential fuzzy number family, which is unlimited range fuzzy number with general form as follows:

$$f_A(x)=\begin{cases} \exp\{-[(a_A-x)/\alpha_A]\}, & x \le a_A; \\ 1, & a_A \le x \le b_A; \\ \exp\{-[(x-b_A)/\beta_A]\}, & b_A \le x; \end{cases} \quad (1)$$

where a_A, b_A are real numbers, and α_A, β_A are positive real numbers. We denote this type of fuzzy number as $A = (a_A , b_A, \alpha_A, \beta_A)_E$. The fuzzy number with exponential membership function is shown in Fig. 1.

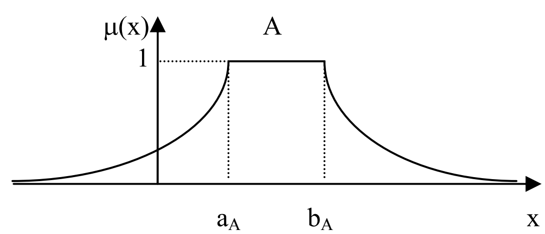

Fig. 1: Exponential membership function.

3. Representation of exponential fuzzy number

The Graded Mean Integration Representation of fuzzy number was introduced by Chen and Hsieh[1] in 1998, and it also had been compared with some other different methods for representation of fuzzy numbers. Then the representation of exponential fuzzy number based on the integral value of graded mean h-levels was defined in [6]. Let the exponential fuzzy number $A= (a_A, b_A, \alpha_A, \beta_A)_E$, where α_A, β_A are positive real numbers, a_A, b_A are real numbers as in formula (1). Now, let two monotonic functions be $L(x) = \exp\{-[(a_A - x)/\alpha_A]\}$, $R(x) = \exp\{-[(x - b_A)/\beta_A]\}$, then the inverse functions of function L and R are L^{-1} and R^{-1} respectively. The h-level graded mean value of exponential fuzzy number $A = (a_A, b_A, \alpha_A, \beta_A)_E$ can be expressed as $h[L^{-1}(h) + R^{-1}(h)]/2$ which is showed in Fig. 2.

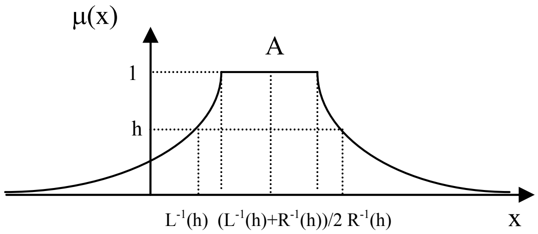

Fig. 2: The h-level graded mean value of an exponential fuzzy number.

Definition 1: Let $A = (a_A, b_A, \alpha_A, \beta_A)_E$, be a exponential fuzzy number, then the Graded Mean Integration Representation of A is defined by

$$P(A) = \left(\int_0^1 h\left(\frac{L^{-1}(h) + R^{-1}(h)}{2}\right)dh\right) \Big/ \int_0^1 h dh.$$

Remark: $\frac{L^{-1}(h)+R^{-1}(h)}{2}$ in Definition 1 can be generalized as $kL^{-1}(h)+(1-k)R^{-1}(h)$ for $0 \le k \le 1$.

Property 1: Let $A = (a_A, b_A, \alpha_A, \beta_A)_E$ be a exponential number where α_A, β_A are positive real numbers, a_A, b_A are real numbers., then the graded mean integration representation of A is

$$P(A) = \frac{a_A + b_A}{2} + \frac{\beta_A - \alpha_A}{4}.$$

Proof:

$$L^{-1}(h) = a_A - \alpha_A (\ln \frac{1}{h}),$$

$$R^{-1}(h) = b_A + \beta_A (\ln \frac{1}{h}).$$

$$\therefore P(A) = \frac{1}{2}\int_0^1 h[a_A + b_A + \beta_A (\ln \frac{1}{h}) - \alpha_A (\ln \frac{1}{h})]dh \Big/ \frac{1}{2}$$

$$= \frac{a_A + b_A}{2} + (\beta_A - \alpha_A)\int_0^1 h(\ln \frac{1}{h})dh.$$

$$= \frac{a_A + b_A}{2} + (\beta_A - \alpha_A)[\int_0^1 h \ln(1)dh - \int_0^1 h \ln(h)dh]$$

$$= \frac{a_A + b_A}{2} + (\beta_A - \alpha_A)\int_0^1 h[\ln(1) - \ln(h)]dh$$

$$= \frac{a_A + b_A}{2} + \frac{\beta_A - \alpha_A}{4}$$

Remark: When $\alpha_A = \beta_A$, $P(A) = (a_A + b_A) / 2$.

4. General arithmetical operations of exponential fuzzy numbers

Function Principle was introduced by Chen[6] (1985) as arithmetical operations among generalized fuzzy numbers. In 1998, Chen[2] also illustrated the arithmetical operations of fuzzy number with step form membership function by using Function Principle. Now we use Function Principle as arithmetical operations among generalized exponential fuzzy numbers. The type of membership function of generalized exponential fuzzy numbers does not change after arithmetical operations. It showed that the addition, subtraction, multiplication, and division of finite fuzzy numbers are well defined.

Definition 2: [Function Principle]

Let g be a arithmetical mapping from n-dimension real number R^n into real line R, and f_g is a corresponding mapping from n-dimension fuzzy numbers into fuzzy number. Suppose $A_i = (a_i, b_i, \alpha_i, \beta_i)_E$, $i = 1, 2, \ldots, n$ be n exponential fuzzy numbers, the fuzzy number B on R induced from these n fuzzy numbers A_i through function f_g. That is $f_g(A_1, \ldots\ldots, A_n) = B = (a, b, \alpha, \beta)_E$, where $T = \{g(x_1, \ldots\ldots, x_n) \mid x_i = a_i \text{ or } b_i, i = 1, 2, \ldots, n\}$, $T_1 = \{g(\gamma_1, \ldots\ldots, \gamma_n) \mid \gamma_i = \alpha_i \text{ or } \beta_i, i = 1, 2, \ldots, n\}$, where set T and T_1 have same correspondent order and subscript. Let a = min T and b = max T. Assume the s_{min}th item is the corresponding item of minimum value of T, that is the value of the s_{min}th item of T equal a, we denote it as $(g(x_1, \ldots\ldots, x_n))_{smin} = a$, and Assume the s_{max}th item is the corresponding item of maximum value of T, that is the value of the s_{max}th item of T equal b, we denote it as $(g(x_1, \ldots\ldots, x_n))_{smax} = b$. Now $\alpha = |(g(\gamma_1, \ldots\ldots, \gamma_n))_{smin}|$, $\beta = |(g(\gamma_1, \ldots\ldots\ldots, \gamma_n))_{smax}|$.

Now suppose that $A_1 = (a_1, b_1, \alpha_1, \beta_1)_E$ and $A_2 = (a_2, b_2, \alpha_2, \beta_2)_E$ are two generalized exponential fuzzy numbers. By using above Function Principle, some arithmetical operations results could be proved as follows:

(1) The addition of A_1 and A_2 is

$$A_1 \oplus A_2 = (a_1+a_2, b_1+b_2, \alpha_1+\alpha_2, \beta_1+\beta_2; w)_E.$$

Note: Following the Function Principle, in this case,

$$T = \{g(x_1, x_2) | x_1 = a_1 \text{ or } b_1, x_2 = a_2 \text{ or } b_2)\}$$
$$= \{a_1+a_2, a_1+b_2, b_1+a_2, b_1+b_2\}.$$

By this sequence,

$$T_1 = \{g_s(y_1, y_2) | y_1 = \alpha_1 \text{ or } \beta_1, y_2 = \alpha_2 \text{ or } \beta_2)\}$$
$$= \{\alpha_1+\alpha_2, \alpha_1+\beta_2, \beta_1+\alpha_2, \beta_1+\beta_2\}.$$

Then $\alpha = |g_{smin}(y_1, y_2)| = \alpha_1+\alpha_2$, $\beta = |g_{smax}(y_1, y_2)| = \beta_1+\beta_2$, where a_1, a_2, b_1, b_2 are all real numbers, and α_1, α_2, β_1, β_2 are positive real numbers.

(2) The multiplication of A_1 and A_2 is

$$A_1 \otimes A_2 = (a, b, \alpha, \beta; w)_E,$$

where $T = g(x_1, x_2)|x_1=a_1 \text{ or } b_1, x_2=a_2 \text{ or } b_2)$
$= \{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}$,
and $a = \min T$, $b = \max T$, that is,
if $a = \min T= a_1b_2$, $b = \max T= b_1a_2$,

$$T_1 = \{g_s(y_1, y_2)|y_1=\alpha_1 \text{ or } \beta_1, y_2=\alpha_2 \text{ or } \beta_2)\}$$
$$=\{\alpha_1\alpha_2, \alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2\},$$

then $\alpha=|g_{smin}(y_1,y_2)|=\alpha_1\beta_2$, $\beta=|g_{smax}(y_1,y_2)|=\beta_1\alpha_2$.

However, when $a_1,a_2, b_1,b_2,\alpha_1,\alpha_2,\beta_1,\beta_2$ are all nonzero positive real numbers, then the fuzzy product of A_1 , A_2 could be as follow:

$$A_1 \otimes A_2 = (a_1a_2, b_1b_2, \alpha_1\alpha_2, \beta_1\beta_2)_E.$$

(3) $-A_2 = (-b_2, -a_2, \beta_2, \alpha_2; w_2)$, then

$$A_1\Theta A_2 = A_1\oplus(-A_2)=(a_1-b_2, b_1-a_2, \alpha_1+\beta_2, \beta_1+\alpha_2 ; w)_E.$$

(4) Let $m\in R^+$, $A = (a, b, \alpha, \beta)_E$, then
$m\otimes A = (ma, mb, m\alpha, m\beta)_E$,
if $m\in R^-$, $A = (a, b, \alpha, \beta)_E$, then
$m\otimes A = (mb, ma, |m|\beta, |m|\alpha)_E$.

(5) Define $1/ A_2=(1/b_2,1/a_2,1/\beta_2,1/\alpha_2)_E$,
we have $A_1/A_2=A_1\otimes(1/ A_2)=(a_1/b_2,b_1/a_2,\alpha_1/\beta_2,\beta_1/\alpha_2)_E$,
where a_1, b_1, a_2, b_2, α_1, α_2, β_1, β_2 are all nonzero positive real numbers.

(6) Suppose $A_i = (a_i, b_i, \alpha_i, \beta_i)_E$, $X_i= (c_i, d_i, \gamma_i, \delta_i)_E$ with a_i, b_i, α_i, β_i, c_i, d_i, γ_i, δ_i, $i=1, 2,\cdots, n$ are all nonzero positive real numbers, then

$$A_1\otimes X_1 \oplus A_2\otimes X_2 \oplus \cdots\oplus A_n\otimes X_n$$
$$= (\sum_{i=1}^{n} a_ic_i, \sum_{i=1}^{n} b_id_i, \sum_{i=1}^{n}\alpha_i \gamma_i, \sum_{i=1}^{n}\beta_i \delta_i)_E.$$

Furthermore, we will prove some properties of graded mean representation of exponential fuzzy number by using the previous properties of the arithmetical operations with Function Principle.

Property 2. Suppose $A = (a_A, b_A, \alpha_A, \beta_A)_E$ and $B = (a_B, b_B, \alpha_B, \beta_B)_E$ are two exponential fuzzy numbers, and $P(A)$ and $P(B)$ are the representations of A and B respectively. Then

(i) $P(m\otimes A)=mP(A)$, if $m>0$;

(ii) $P (A \oplus B) = P (A) + P (B)$;

(iii) $P(A\Theta B) = P(A)-P(B)$,

where $\oplus$ and Θ are the fuzzy addition and subtraction operation of Function Principle.

Proof: To prove (i), since $m\otimes A = (ma_A, mb_A, m\alpha_A, m\beta_A)_E$,

$$P(m\otimes A) = (ma_A + mb_A) / 2 + (m\beta_A - m\alpha_A) / 4$$
$$=m P (A).$$

Similarly to prove (ii), we have

$$A \oplus B = (a_A + a_B, b_A + b_B,\alpha_A +\alpha_B,\beta_A +\beta_B)_E.$$

Then

$$P(A\oplus B) = (a_A+a_B+b_A+b_B)/2 +(\beta_A +\beta_B -\alpha_A-\alpha_B) / 4$$
$$= P (A) + P (B).$$

In terms of

$$A\Theta B = (a_A - a_B, b_A - b_B, \alpha_A +\beta_B, \beta_A+\alpha_B)_E.$$

Then

$$P(A\Theta B) = (a_A-a_B+b_A-b_B) / 2 +(\beta_A+\alpha_B -\alpha_A -\beta_B) / 4$$
$$=(a_A+b_A)/2+(\beta_A-\alpha_A)/4-(a_B+b_B)/2-(\beta_B-\alpha_B)/4$$
$$=P(A) – P(B).$$

This proves (iii).

5. Application

In general, the lifetime of the lamps of projectors could be expressed as exponential fuzzy number. After surveying a few brands of lamps of projectors, the lifetime is around 2000 to 3000 hours. Now we choose two types of lamps:

(1) lamp A
Lifetime is around 3000 hours. $\alpha= 0.1.\beta=0.2$.
Then we can let A=(2900,3100,0.1,0.2).

(2) lamp B
Lifetime is around 2000 hours. $\alpha= 0.2.\beta=0.3$.
Then we can let B=(1900,2100,0.2,0.3).

The first case is to suppose that a projector is using lamp A normally and lamp B for backup. Then the exponential fuzzy number of the lifetime of the two lamps of the projector can be gained (see Fig. 3 next page):

$$A\oplus B=(2900+1900, 3100+2100, 0.1+0.2, 0.2+0.3)$$
$$=(4800, 5200, 0.3, 0.5).$$

The second case is to suppose that a project is using both lamp A for normal use and backup. Then the exponential fuzzy number of the lifetime of the two lamps of the projector can be gained:

$$A\oplus A=(2900+2900, 3100+3100, 0.1+0.1, 0.2+0.2)$$
$$=(5800, 6200, 0.2, 0.4);$$

the result is the same as

$$2\otimes A=(2\times 2900, 2\times 3100, 2\times 0.1, 2\times 0.2)$$
$$=(5800, 6200, 0.2, 0.4).$$

Then we can use our method to calculate the lifetime of any combinations of lamps.

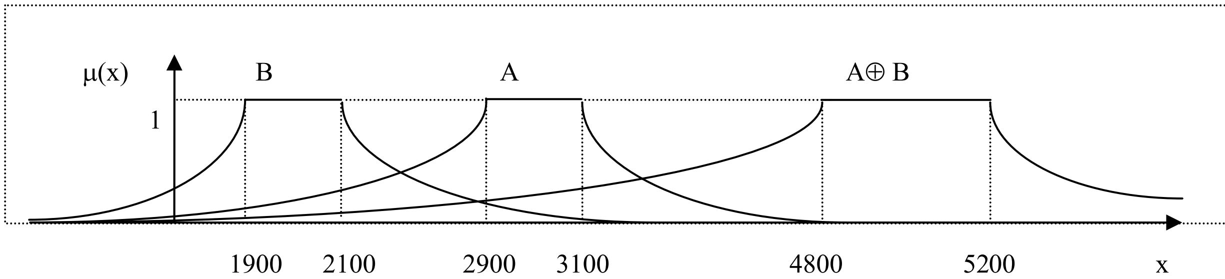

Fig. 3: Addition of the lifetime of two lamps of a projector.

6. Concluding Remarks

In this paper, we have found an example of the calculation of the lifetime of the lamps of projectors and we can see that this example is very close to the real world phenomenon. Actually, our method can be used in any kinds of calculations for exponential fuzzy numbers.

7. References

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