Automorphisms of the Algebra of Fuzzy Truth Values I

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Abstract

The truth values of type-2 fuzzy sets are fuzzy subsets of the unit interval. This set is endowed with some algebraic operations, and the resulting algebra has been studied rather extensively. Here we begin the study of automorphisms of this algebra. The set of automorphisms of an algebra forms a group, and often tells much about the structure and properties of the underlying algebra and identifies subalgebras of particular interest. Of special interest are those subalgebras that are carried onto themselves by automorphisms of the containing algebra, that is, the so-called characteristic subalgebras. Our chief concern here is identifying some of these characteristic subalgebras.

Keywords: Type-2 fuzzy sets, automorphisms

1. The Algebra of Fuzzy Truth Values

Type-2 fuzzy sets—that is, fuzzy sets with fuzzy subsets of $[0,1]$ as truth values were introduced by Zadeh [2], extending the notion of ordinary fuzzy sets.

A **type-2 fuzzy subset** A of a set S is a mapping $A : S \to [0,1]^{[0,1]}$. The truth value set $[0,1]^{[0,1]}$ is the set of all fuzzy subsets of $[0,1]$. The operations on the set of type-2 fuzzy subsets $\left([0,1]^{[0,1]}\right)^S$ of a set S come pointwise from operations on $[0,1]^{[0,1]}$ whose definition involves using operations on both the domain $[0,1]$ and the range $[0,1]$ of a mapping in $[0,1]^{[0,1]}$. The relevant operations follow. Let f and g be in $[0,1]^{[0,1]}$. The elements $f \sqcup g$ and $f \sqcap g$ of $[0,1]^{[0,1]}$ are convolutions of the operations $\vee$ and $\wedge$ on the domain $[0,1]$ with respect to the operations $\vee$ and $\wedge$ on the range $[0,1]$.

$$\begin{aligned}(f \sqcup g)(x) &= \bigvee_{y \vee z = x} (f(y) \wedge g(z)) \\ (f \sqcap g)(x) &= \bigvee_{y \wedge z = x} (f(y) \wedge g(z))\end{aligned}$$

Denote by $\bar{1}$ and $\bar{0}$ the elements of $[0,1]^{[0,1]}$ defined by

$$\bar{1}(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \quad \text{and } \bar{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

The elements $\bar{0}$ and $\bar{1}$ of $[0,1]^{[0,1]}$ are nullary operations, and can also be obtained by convolution of the nullary operations 1 and 0 on the domain $[0,1]$.

The algebra $\mathbb{M} = ([0,1]^{[0,1]}, \sqcup, \sqcap, \bar{0}, \bar{1})$ with the additional operation * obtained by the convolution of the operation $'$ on $[0,1]$ given by $x' = 1 - x$, is the basic algebra for type-2 fuzzy set theory, and is analogous to the algebra $([0,1], \vee, \wedge, ', 0, 1)$ basic for type-1 or ordinary fuzzy set theory. We denote this algebra without its negation operation, that is, $([0,1], \vee, \wedge, 0, 1)$, by $\mathbb{I}$. The algebra $\mathbb{M} = ([0,1]^{[0,1]}, \sqcup, \sqcap, \bar{0}, \bar{1})$ with the additional operation * is also analogous to the algebra which is basic for interval-valued fuzzy set theory. That algebra has underlying set $[0,1]^{[2]} = \left\{(x,y) \in [0,1]^2 : x \leq y\right\}$ with operations $\wedge$ and $\vee$ coming pointwise from coordinatewise operations on $[0,1]^{[2]}$, $(x,y)' = (y', x')$, and bounding constants $(0,0)$ and $(1,1)$. That algebra without its negation operation $'$ is denoted $\mathbb{I}^{[2]}$. Our main purpose in [1] was to study the algebra $(\mathbb{M}, ^*)$ and some of its subalgebras. Here, we study the automorphisms of $\mathbb{M}$ without the negation operation *. Although we are interested in the algebra $\mathbb{M}$, the set $[0,1]^{[0,1]}$ also has the pointwise operations $\wedge$, $\vee$, $\underline{0}$, $\underline{1}$ defined on it coming from operations on the range:

$$\begin{aligned}&(f \wedge g)(x) = f(x) \wedge g(x) \\ &(f \vee g)(x) = f(x) \vee g(x) \\ &\underline{0}(x) = 0 \quad \text{and} \quad \underline{1}(x) = 1\end{aligned}$$

The algebra $\left([0,1]^{[0,1]}, \wedge, \vee, \underline{0}, \underline{1}\right)$ is a bounded, distributive lattice with order given by $f \leq g$ if $f = f \wedge g$, or equivalently, if $g = f \vee g$.

The operations $\sqcup$ and $\sqcap$ can be expressed in terms of the simpler pointwise ones and this was crucial in developing the properties of $\mathbb{M}$. For $f \in \mathbb{M}$, let f^L and f^R be the elements of $\mathbb{M}$ defined by

$$f^L(x) = \bigvee_{y \leq x} f(y) \quad \text{and} \quad f^R(x) = \bigvee_{y \geq x} f(y)$$

Note that f^L is monotone increasing and that f^R is monotone decreasing. The following theorem expresses each of the convolution operations $\sqcup$ and $\sqcap$ directly in terms of pointwise operations in two alternate forms.

Theorem 1: The following hold for all $f, g \in \mathbb{M}$.

$$f \sqcup g = \left(f \wedge g^L\right) \vee \left(f^L \wedge g\right) = (f \vee g) \wedge \left(f^L \wedge g^L\right)$$
$$f \sqcap g = \left(f \wedge g^R\right) \vee \left(f^R \wedge g\right) = (f \vee g) \wedge \left(f^R \wedge g^R\right)$$

There are many relations between the various operations introduced above. We refer to [1] for details. The following for $f, g \in \mathbb{M}$ are crucial:

$$(f \sqcup g)^L = f^L \sqcup g^L \qquad (f \sqcap g)^R = f^R \sqcap g^R$$
$$(f \sqcup g)^R = f^R \sqcup g^R \qquad (f \sqcap g)^L = f^L \sqcap g^L$$

Theorem 2: The basic properties of the algebra $\mathbb{M} = \left([0,1]^{[0,1]}, \sqcup, \sqcap, \bar{0}, \bar{1}\right)$ follow. For $f, g, h \in \mathbb{M}$,

1) $f \sqcup f = f$ and $f \sqcap f = f$ (idempotent)
2) $f \sqcup g = g \sqcup f$ and $f \sqcap g = g \sqcap f$ (commutative)
3) $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$ and $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$ (associative)
4) $\bar{1} \sqcap f = f$ and $\bar{0} \sqcup f = f$ (identities)

Notice that this list does not include the absorption laws or distributive laws. The algebra $\mathbb{M}$ is a bounded partially ordered set, but is not a lattice.

2. Automorphisms

Our interest here is in the automorphisms of the algebra $\mathbb{M} = \left([0,1]^{[0,1]}, \sqcup, \sqcap, \bar{0}, \bar{1}\right)$ and of some of its subalgebras. Automorphisms of an algebra are one-to-one mappings of that algebra onto itself that preserve the operations of that algebra. Thus φ is an automorphism of the algebra $\mathbb{M}$ if φ is a one-to-one mapping of $[0,1]^{[0,1]}$ onto itself such that

$$\varphi(f \sqcup g) = \varphi(f) \sqcup \varphi(g)$$
$$\varphi(f \sqcap g) = \varphi(f) \sqcap \varphi(g)$$
$$\varphi(\bar{0}) = \bar{0} \quad \text{and} \quad \varphi(\bar{1}) = \bar{1}$$

The set of automorphisms of an algebra $\mathbb{A}$ will be denoted $Aut(\mathbb{A})$. The automorphisms of the algebra $\mathbb{I} = ([0,1], \vee, \wedge, 0, 1)$ are the continuous increasing functions $\alpha : [0,1] \to [0,1]$ with $\alpha(0) = 0$ and $\alpha(1) = 1$. For the algebra $\mathbb{M}$, the following proposition shows two different ways to obtain an automorphism of $\mathbb{M}$ from an automorphism of $\mathbb{I}$ via composition of functions.

Proposition 3: For $\alpha \in Aut(\mathbb{I})$, and $f \in [0,1]^{[0,1]}$, let $\alpha_L(f) = \alpha f$ and $\alpha_R(f) = f\alpha$. Then α_L and α_R are automorphisms of $\mathbb{M}$.

Thus $\mathbb{M}$ has many automorphisms, and at this point we know of none that do not come from compositions of these. The automorphisms α_L and α_R have quite nice properties. As elements of the group $Aut(\mathbb{M})$, they satisfy the following:

$$(\alpha\beta)_L = \alpha_L\beta_L \quad \text{and} \quad (\alpha\beta)_R = \beta_R\alpha_R$$
$$\left(\alpha^{-1}\right)_L = (\alpha_L)^{-1} \quad \text{and} \quad \left(\alpha^{-1}\right)_R = (\alpha_R)^{-1}$$
$$\alpha_L\beta_R = \beta_R\alpha_L$$

In particular, the automorphisms of $\mathbb{M}$ of the form $\alpha_L\beta_R$ form a subgroup of $Aut(\mathbb{M})$, and for all we know, may be all of $Aut(\mathbb{M})$.

Let $\underline{1} : [0,1] \to [0,1] : x \mapsto 1$ be the constant map with value 1. A basic question is whether or not automorphisms of $\mathbb{M}$ fix $\underline{1}$. It is easy to verify the identities $f \sqcup \underline{1} = f^L$ and $f \sqcap \underline{1} = f^R$, and consequently automorphisms that fix $\underline{1}$ commute with the L and R operations. That is, if $\varphi \in Aut(\mathbb{M})$ satisfies $\varphi(\underline{1}) = \underline{1}$, then

$$\varphi\left(f^L\right) = (\varphi(f))^L \quad \text{and} \quad \varphi\left(f^R\right) = (\varphi(f))^R$$

Also, if $\varphi \in Aut(\mathbb{M})$ satisfies $\varphi(\underline{1}) = \underline{1}$, then f is constant if and only if $\varphi(f)$ is constant. The automorphisms α_L and β_R do fix $\underline{1}$. To show that an arbitrary automorphism of $\mathbb{M}$ fixes $\underline{1}$ requires further investigation.

3. Convex Normal Functions

A mapping $f : [0,1] \to [0,1]$ is **normal** if $\bigvee_{x \in [0,1]} f(x) = 1$. The map f is **convex** if for every $x \le y \le z$, $f(y) \ge f(x) \wedge f(z)$. This definition of normal does not require that f actually achieves its maximum. This is sometimes called "nearly normal."

The convex, normal functions in $\mathbb{M}$ form a subalgebra $\mathbb{L}$ which is a bounded, distributive lattice containing in a natural way isomorphic copies of both $\mathbb{I}$ and $\mathbb{I}^{[2]}$. Moreover, $\mathbb{L}$ is a maximal sublattice. For these reasons, it is likely to play a very important role in type-2 fuzzy set theory. The following theorem implies that all automorphisms of $\mathbb{M}$ induce automorphisms of $\mathbb{L}$.

Theorem 4: Let $\varphi \in Aut(\mathbb{M})$ and $f \in [0,1]^{[0,1]}$. If f is normal, then $\varphi(f)$ is normal; and if f is convex then $\varphi(f)$ is convex.

Corollary 5: An automorphism of $\mathbb{M}$ induces an automorphism of $\mathbb{L}$. That is, $\mathbb{L}$ is a characteristic subalgebra of $\mathbb{M}$.

Note that, since $\underline{1}$ is both normal and convex, by the previous two propositions, $\varphi(\underline{1})$ is normal and convex for any automorphism φ of $\mathbb{M}$. The function $\underline{1}$ sits between the monotone decreasing and increasing functions in some sense.

Proposition 6: If f is monotone decreasing and g is monotone increasing, then $f \sqcap \underline{1} = f$ and $g \sqcup \underline{1} = g$. In the notation of [1], this says $f \sqsubseteq \underline{1} \preceq g$.

4. Irreducibles in $\mathbb{M}$

An element $h \in \mathbb{M}$ is **meet irreducible** if $h = f \sqcap g$ implies $f = h$ or $g = h$, **join irreducible** if $h = f \sqcup g$ implies $f = h$ or $g = h$, and **irreducible** if both meet and join irreducible.

The properties join irreducible, meet irreducible, and irreducible are preserved under automorphisms. Identifying these irreducibles provides information about possible automorphisms. For example, the set of irreducible elements of $\mathbb{M}$ must be taken onto itself by every automorphism of $\mathbb{M}$.

A **point function** f is a function that is non-zero at exactly one point a. We say that f is a **point function** of a. A **characteristic function** f is a function that is 1 on some subset S of $[0,1]$ and 0 elsewhere. A characteristic function that is also a point function is called a **singleton**. We identify a subset S of $[0,1]$ with the characteristic function of that set and call these **subsets of** $\mathbb{M}$.

For singletons, we write $\bar{a}$ to denote the characteristic function of the set $\{a\}$. A **constant function** with value $x \in [0,1]$ will be denoted $\underline{x}$. The following theorem identifies all the irreducibles in $\mathbb{M}$.

Theorem 7: A function f in $\mathbb{M}$ is irreducible in $\mathbb{M}$ if and only if f is a singleton, $f = \underline{0}$, or $f = \{0,1\}$.

As we have indicated, the unit interval $[0,1] = \underline{1}$ is of particular interest as an element of $\mathbb{M}$. However, as this theorem asserts, it is not irreducible in $\mathbb{M}$. Also note that the singletons and $\{0,1\}$ are normal, and the singletons and $\underline{0}$ are convex.

Because $\underline{0}$ is the only convex irreducible in $\mathbb{M}$ that is not normal, and $\{0,1\}$ is the only normal irreducible in $\mathbb{M}$ that is not convex, we have the following.

Corollary 8: Every automorphism of $\mathbb{M}$ fixes both $\underline{0}$ and $\{0,1\}$.

The map $\mathbb{I} \to \mathbb{M} : a \mapsto \bar{a}$ is a monomorphism. The image of this map is a subalgebra of $\mathbb{M}$ isomorphic to $\mathbb{I}$. We will identify $\mathbb{I}$ with its image and say $\mathbb{I} \subset \mathbb{M}$. The map $\mathbb{I}^{[2]} \to \mathbb{M} : (a,b) \mapsto \bar{a}^L \wedge \bar{b}^R$ is also a monomorphism, and identifying $\mathbb{I}^{[2]}$ with its image, we have $\mathbb{I} \subset \mathbb{I}^{[2]} \subset \mathbb{L} \subset \mathbb{M}$.

The fact that the singletons are the only convex normal irreducibles in $\mathbb{M}$, together with Theorem 4, yields the following.

Corollary 9: Every automorphism of $\mathbb{M}$ induces an automorphism of $\mathbb{I}$ by its action on singletons.

In other words, $\mathbb{I}$ is a characteristic subalgebra of $\mathbb{M}$. So in the sequence $\mathbb{I} \subset \mathbb{I}^{[2]} \subset \mathbb{L} \subset \mathbb{M}$, the subalgebras $\mathbb{I}$ and $\mathbb{L}$ are characteristic. The subalgebra $\mathbb{I}^{[2]}$ will later be shown also to be characteristic.

We now focus on the subalgebra $\mathbb{L}$, which is a bounded, distributive lattice.

5. Irreducibles in $\mathbb{L}$

An element $h \in \mathbb{L}$ is **meet irreducible in** $\mathbb{L}$ if $h = f \sqcap g$ with $f, g \in \mathbb{L}$ implies $f = h$ or $g = h$, **join irreducible in** $\mathbb{L}$ if $h = f \sqcup g$ with $f, g \in \mathbb{L}$ implies $f = h$ or $g = h$, and **irreducible in** $\mathbb{L}$ if it is both meet and join irreducible in $\mathbb{L}$. Note that elements of $\mathbb{L}$ that are meet or join irreducible in $\mathbb{M}$ have the same property in $\mathbb{L}$, but the converse need not hold.

Proposition 10: In the subalgebra $\mathbb{L}$, intervals $[0,x]$ and $[0,x)$ are join irreducible and the intervals $(x,1]$ and $[x,1]$ are meet irreducible.

Corollary 11: The interval $[0,1] = \underline{1}$ is irreducible in the subalgebra $\mathbb{L}$.

The following theorem describes all the irreducible elements of $\mathbb{L}$.

Theorem 12: The irreducibles in $\mathbb{L}$ are the singletons $\bar{a}$ and the functions of the form $\underline{x} \vee \bar{0}$ and $\underline{x} \vee \bar{1}$ for $0 \leq x \leq 1$. The functions of the latter two types form a chain. If $x \leq y$, then

$$\bar{0} \sqsubseteq \underline{x} \vee \bar{0} \sqsubseteq \underline{y} \vee \bar{0} \sqsubseteq \underline{1} \sqsubseteq \underline{y} \vee \bar{1} \sqsubseteq \underline{x} \vee \bar{1} \sqsubseteq \bar{1}$$

where $\sqsubseteq$ is the order in the lattice $\mathbb{L}$. This result enables us to prove the following theorem. The proof depends heavily on results in [1].

Theorem 13: Every automorphism of $\mathbb{M}$ fixes $\underline{1}$.

Corollary 14: Automorphisms of $\mathbb{M}$ commute with R and L.

6. Comments and Questions

This is just the beginning of an investigation of automorphisms of $\mathbb{M}$. There are a number of group theoretic aspects not touched upon here. We have succeeded in identifying the irreducibles in $\mathbb{M}$ and in $\mathbb{L}$, which is a major step. Details will appear in a paper under preparation, and are quite computational and tedious. Here are some specific questions.

- Does an automorphism of $\mathbb{I}$ extend uniquely to an automorphism of $\mathbb{M}$?
- Does an automorphism of $\mathbb{L}$ extend uniquely to an automorphism of $\mathbb{M}$?
- Is every automorphism of $\mathbb{L}$ induced by an automorphism of $\mathbb{M}$?
- Are all automorphisms generated by the automorphisms α_L and α_R?

References

[1] Walker, C., and Walker, E., The Algebra of Fuzzy Truth Values, *Fuzzy Sets and Systems*, 149(2) (2005), 309-347.

[2] Zadeh, L., The concept of a linguistic variable and its application to approximate reasoning, *Inform Sci.* 8 (1975) 199-249.