# Using parametric functions to solve systems of linear fuzzy equations

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## Abstract

At this moment only one method for solving systems of linear fuzzy equations is known, namely the method proposed by Buckley and Qu. Basically, in their method the solutions of all systems of linear crisp equations formed by the $\alpha$-levels are calculated. We propose in this paper a new method for solving systems of linear fuzzy equations based on a practical algorithm using parametric functions in which the variables are given by the fuzzy coefficients of the system. We show that our algorithm is much more efficient than the method of Buckley and Qu.
**Keywords:** fuzzy numbers, solving systems of fuzzy linear equations

## 1 Introduction

In this paper we search for a solution of the matrix equation: $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ for $\tilde{\mathbf{x}} = [\tilde{x}_k]_{n\times 1}$ where $\tilde{A} = [\tilde{a}_{ij}]_{n\times n}$ is a matrix with fuzzy numbers as entries and $\tilde{\mathbf{b}} = [\tilde{b}_k]_{n\times 1}$ is a vector of fuzzy numbers. Such equations are hard to solve exactly and often the exact solution does not exist or is a vector of fuzzy sets which are not fuzzy numbers. Therefore the search for an alternative solution has a solid ground. Buckley and Qu [1] have already proposed a solution. We follow their line of reasoning although the solution can be adjusted a little bit and a practical algorithm to find this solution is proposed here.

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## 2 Preliminaries

First we recall some definitions concerning fuzzy numbers (see e.g. [4]). Let $A$ be a fuzzy set on $\mathbb{R}$. Then $A$ is called convex if $A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2)))$, for all $x_1$, $x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$. If for $x \in \mathbb{R}$ it holds that $A(x) = 1$, then we call $x$ a modal value of $A$. The support of $A$ is defined as $\operatorname{supp} A = \{x \mid x \in \mathbb{R} \text{ and } A(x) > 0\}$. A mapping $f : \mathbb{R} \to \mathbb{R}$, or in particular $f : \mathbb{R} \to [0,1]$, is called upper-semicontinuous when $f$ is right-continuous where $f$ is increasing and left-continuous where $f$ is decreasing.

**Definition 1** [4] *A fuzzy number is defined as a convex upper-semicontinuous fuzzy set on $\mathbb{R}$ with a unique modal value and bounded support.*

From now on fuzzy numbers will be denoted by a lowercase letter with a tilde, e.g. $\tilde{a}$, and a vector of fuzzy numbers will be denoted as $\tilde{\mathbf{b}}$. Sometimes we will denote the $i$-th component of $\tilde{\mathbf{b}}$ by $(\tilde{\mathbf{b}})_i$. Crisp numbers will be represented by a lowercase letter, e.g. $a$, and vectors of crisp numbers will be denoted as $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$. The notions of support and $\alpha$-level are extended componentwise for vectors or matrices of fuzzy numbers. The arithmetic of fuzzy numbers is based on Zadeh's extension principle and is equivalent to interval arithmetic applied to the $\alpha$-levels.

## 3 Solving systems of linear fuzzy equations

First of all, we require that the matrix $\tilde{A}$ of fuzzy numbers is regular in the sense that the matrix $A^{-1}$ exists for all $a_{ij} \in \mathrm{supp}(\tilde{a}_{ij})$. Buckley and Qu [1] proposed to construct a set of all crisp solutions corresponding to the crisp systems formed by the elements in a certain $\alpha$-level. They define the solution by, for all $\alpha \in ]0,1]$,

$$\begin{aligned}\Omega(\alpha) = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } (\exists A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}) \\ (\exists \mathbf{b} = [b_k]_{n\times 1} \in \mathbb{R}^n)((\forall (i,j,k) \in \{1,2,\ldots,n\}^3) \\ (a_{ij} \in (\tilde{a}_{ij})_\alpha \text{ and } b_k \in (\tilde{b}_k)_\alpha) \text{ and } A\mathbf{x} = \mathbf{b})\}\end{aligned}$$

and for all $\mathbf{x} \in \mathbb{R}^n$, $\tilde{\mathbf{x}}_B(\mathbf{x}) = \sup\{\alpha \mid \alpha \in ]0,1]$ and $\mathbf{x} \in \Omega(\alpha)\}$.

We see that $\tilde{\mathbf{x}}_B$ is defined as a fuzzy set on $\mathbb{R}^n$ and not as a vector of fuzzy numbers. The solution $\tilde{\mathbf{x}}_B(\mathbf{x})$ expresses to what extent the crisp vector $\mathbf{x}$ is a solution of the system of linear fuzzy equations $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. We prefer to define a solution as a vector of fuzzy numbers to avoid information loss. Therefore we give a membership degree to every component of the solution vector and then $(\tilde{\mathbf{x}}_B)_i(x)$ expresses the degree to which $x$ belongs to the fuzzy set $(\tilde{\mathbf{x}}_B)_i$, independent of $(\tilde{\mathbf{x}}_B)_j$, for all $j \neq i$. We thus define for all $x \in \mathbb{R}$ and for all $i \in \{1,2,\ldots,n\}$

$$\begin{aligned}(\tilde{\mathbf{x}}_B)_i(x) = \sup\{\alpha \mid \alpha \in [0,1] \text{ and} \\ (\exists \mathbf{x} \in \Omega(\alpha))(x = x_i)\},\end{aligned} \tag{1}$$

where $x_i$ denotes the $i$-th component of $\mathbf{x}$. This method is purely theoretical: in fact all crisp systems are solved. When all these systems have to be solved, the computation time will be large. In this paper we propose a practical algorithm to compute the solution. Instead of solving all these crisp systems, we determine parametric functions of these solutions.

## 3.1 Systems with one fuzzy coefficient

We first consider the case that we have to solve a system of linear fuzzy equations in which exactly one of the coefficients is a fuzzy number and the other coefficients are crisp. Without loss of generality we may assume that $\tilde{a}_{11}$ is a fuzzy number. In order to obtain the solution $\tilde{\mathbf{x}}_S$ of $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, we have to solve the crisp systems $A(a_{11})\mathbf{x} = \mathbf{b}$, where $\tilde{a}_{11}$ is replaced by $a_{11}$ in the matrix $\tilde{A}$ for all $a_{11} \in [\underline{a}_{11}, \overline{a}_{11}] = \mathrm{supp}(\tilde{a}_{11})$. We can solve all of these systems through Cramer's rule thanks to the non-singularity of the crisp matrix $A(a_{11})$, for all $a_{11} \in \mathrm{supp}(\tilde{a}_{11})$. So we can write the solution for every component as a quotient of two determinants. The determinant of a matrix $A$ is denoted as $|A|$. By expanding the determinants in the numerator and the denominator along the first row, we can write each component of the solution using parameters $c_{1j}$, $c_{2j}$, $c_3$ and $c_4$:

$$x_j = f_j(a_{11}) = \frac{a_{11}c_{1j} + c_{2j}}{a_{11}c_3 + c_4}. \tag{2}$$

Due to this result, every solution can be written using parametric functions with variable $a_{11}$. Note that $c_{1j}$ and $c_{2j}$ are dependent of $j$ due to the fact that the $j$-th column in the numerator contains the components of $\mathbf{b}$. On the other hand, the denominator is the same for all $j \in \{1,\ldots,n\}$, so $c_3$ and $c_4$ are independent of $j$. Thus we propose the following method to solve $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. First we compute the determinants of the matrices $A(\underline{a}_{11})$ and $A(\overline{a}_{11})$. The parameters $c_3$ and $c_4$ are obtained by solving the following system of linear crisp equations:

$$\begin{cases} |A(\underline{a}_{11})| = \underline{a}_{11}c_3 + c_4 \\ |A(\overline{a}_{11})| = \overline{a}_{11}c_3 + c_4. \end{cases} \tag{3}$$

We solve the crisp systems

$$A(\underline{a}_{11})\mathbf{x} = \mathbf{b}, \tag{4}$$

$$A(\overline{a}_{11})\mathbf{x} = \mathbf{b}, \tag{5}$$

and denote by $\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_n)^T$ and $\overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n)^T$ the solutions of (4) and (5) respectively. Then, for all $j \in \{1,\ldots,n\}$, we obtain $c_{1j}$ and $c_{2j}$ by solving the following system of crisp equations:

$$\begin{cases} \underline{x}_j|A(\underline{a}_{11})| = \underline{a}_{11}c_{1j} + c_{2j} \\ \overline{x}_j|A(\overline{a}_{11})| = \overline{a}_{11}c_{1j} + c_{2j}. \end{cases} \tag{6}$$

Consequently, all possible solutions for the crisp systems $A(a_{11})\mathbf{x} = \mathbf{b}$, for all $a_{11} \in \mathrm{supp}(\tilde{a}_{11})$, can be obtained using (2). We define for all $j \in \{1,\ldots,n\}$ the fuzzy number $\tilde{x}_j$ by

$$\begin{aligned}\tilde{x}_j(x) = \sup\{\tilde{a}_{11}(a_{11}) \mid a_{11} \in \mathrm{supp}(\tilde{a}_{11}) \\ \text{and } x = f_j(a_{11})\},\end{aligned} \tag{7}$$

for all $x \in f_j(\mathrm{supp}(\tilde{a}_{11}))$, and $\tilde{x}_j(x) = 0$ for all $x \in \mathbb{R} \setminus f_j(\mathrm{supp}(\tilde{a}_{11}))$. Finally we define $\tilde{\mathbf{x}}_S = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ and we call $\tilde{\mathbf{x}}_S$ the solution of the system.

## 3.2 Systems with two fuzzy coefficients

Assume we have two fuzzy numbers $\tilde{a}_{11}$ and $\tilde{a}_{12}$, and that the other coefficients are crisp. In

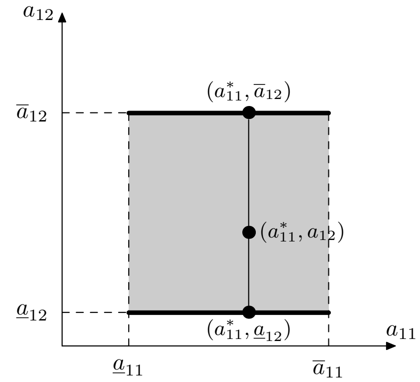


Figure 1: Solving systems with two fuzzy coefficients

Figure 1 the grey area is the set $\mathrm{supp}(\tilde{a}_{11}) \times \mathrm{supp}(\tilde{a}_{12})$.

If we fix $a_{12}$, e.g. $a_{12} = \underline{a}_{12}$, then we have a system with only one fuzzy number $\tilde{a}_{11}$. So for any $a_{11} \in \mathrm{supp}(\tilde{a}_{11})$, the solution of the crisp system $A(a_{11}, \underline{a}_{12})\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = (f_1(a_{11}), f_2(a_{11}), \ldots, f_n(a_{11}))^T$ where

$$f_j(a_{11}) = \frac{a_{11}c_{1j} + c_{2j}}{a_{11}c_3 + c_4},$$

where $c_{1j}$, $c_{2j}$, $c_3$ and $c_4$ are calculated in a similar way as described in subsection 3.1. Note that we have found the solution of the crisp systems corresponding to the points on the lower thick line in Figure 1. Similarly, we obtain for all $a_{11} \in \mathrm{supp}(\tilde{a}_{11})$, the solution of the system $A(a_{11}, \overline{a}_{12})\mathbf{x} = \mathbf{b}$ by constructing functions $f'_j$ (with parameters $c'_{1j}$, $c'_{2j}$, $c'_3$ and $c'_4$), for $j \in \{1, \ldots, n\}$, and by calculating $\mathbf{x} = (f'_1(a_{11}), f'_2(a_{11}), \ldots, f'_n(a_{11}))^T$

Now we fix arbitrarily $a^*_{11} \in \mathrm{supp}(\tilde{a}_{11})$ and let $a_{12} \in \mathrm{supp}(\tilde{a}_{12})$ vary. So, again, we obtain a system with only one fuzzy number, but this time the fuzzy number is $\tilde{a}_{12}$. Thus we are looking for the solution of the crisp systems corresponding to the points on the vertical thin line in Figure 1. Similarly as we did before for $\tilde{a}_{11}$, we can obtain the solution of the crisp system $A(a^*_{11}, a_{12})\mathbf{x} = \mathbf{b}$ as

$$x_j = f_j^{a^*_{11}}(a_{12}) = \frac{a_{12}c_{1j}^{a^*_{11}} + c_{2j}^{a^*_{11}}}{a_{12}c_3^{a^*_{11}} + c_4^{a^*_{11}}} \tag{8}$$

for all $j \in \{1, \ldots, n\}$. We find the parameters $c_3^{a^*_{11}}$ and $c_4^{a^*_{11}}$ by solving the system

$$\begin{cases} a^*_{11}c_3 + c_4 = \underline{a}_{12}c_3^{a^*_{11}} + c_4^{a^*_{11}} \\ a^*_{11}c'_3 + c'_4 = \overline{a}_{12}c_3^{a^*_{11}} + c_4^{a^*_{11}}. \end{cases} \tag{9}$$

We have seen above that the solutions of the crisp systems

$$A(a^*_{11}, \underline{a}_{12})\mathbf{x} = \mathbf{b},$$
$$A(a^*_{11}, \overline{a}_{12})\mathbf{x} = \mathbf{b},$$

are given by $\underline{\mathbf{x}}^{a^*_{11}} = (f_1(a^*_{11}), \ldots, f_n(a^*_{11}))^T$ and $\overline{\mathbf{x}}^{a^*_{11}} = (f'_1(a^*_{11}), \ldots, f'_n(a^*_{11}))^T$ respectively. Then, for all $j \in \{1, \ldots, n\}$, we obtain $c_{1j}^{a^*_{11}}$ and $c_{2j}^{a^*_{11}}$ by solving the following system:

$$\begin{cases} a^*_{11}c_{1j} + c_{2j} = \underline{a}_{12}c_{1j}^{a^*_{11}} + c_{2j}^{a^*_{11}} \\ a^*_{11}c'_{1j} + c'_{2j} = \overline{a}_{12}c_{1j}^{a^*_{11}} + c_{2j}^{a^*_{11}}. \end{cases}$$

Consequently, all possible solutions for the crisp systems $A(a^*_{11}, a_{12})\mathbf{x} = \mathbf{b}$, for all $a_{12} \in \mathrm{supp}(\tilde{a}_{12})$, can be obtained using (8).

We now introduce for all $j \in \{1, \ldots, n\}$ a function $f_j$ on $\mathrm{supp}(\tilde{a}_{11}) \times \mathrm{supp}(\tilde{a}_{12})$ by, for $(a_{11}, a_{12}) \in \mathrm{supp}(\tilde{a}_{11}) \times \mathrm{supp}(\tilde{a}_{12})$,

$$f_j(a_{11}, a_{12}) = \begin{cases} f_j(a_{11}), & \text{if } a_{12} = \underline{a}_{12}, \\ f'_j(a_{11}), & \text{if } a_{12} = \overline{a}_{12}, \\ f_j^{a_{11}}(a_{12}), & \text{else.} \end{cases}$$

We define for all $j \in \{1, \ldots, n\}$ the fuzzy set $\tilde{x}_j$ on $\mathbb{R}$ by

$$\begin{aligned} \tilde{x}_j(x) = \sup\{\mathrm{aggr}(\tilde{a}_{11}(a_{11}), \tilde{a}_{12}(a_{12})) \mid \\ (a_{11}, a_{12}) \in \mathrm{supp}(\tilde{a}_{11}) \times \mathrm{supp}(\tilde{a}_{12}) \\ \text{and } x = f_j(a_{11}, a_{12})\}, \end{aligned} \tag{10}$$

for all $x \in f_j(\mathrm{supp}(\tilde{a}_{11}), \mathrm{supp}(\tilde{a}_{12}))$, where aggr denotes an arbitrary idempotent aggregation operator, and $\tilde{x}_j(x) = 0$, for all $x \in \mathbb{R} \setminus f_j(\mathrm{supp}(\tilde{a}_{11}), \mathrm{supp}(\tilde{a}_{12}))$. Finally, we define $\tilde{\mathbf{x}}_S = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ and we call $\tilde{\mathbf{x}}_S$ the solution of the system.

### 3.3 Systems with more than two fuzzy coefficients

Clearly, the procedure proposed in Subsection 3.2 can be extended to systems with more than two fuzzy coefficients.

**Theorem 1** *If* aggr = min*, then the solution $\tilde{\mathbf{x}}_S$ obtained by the method described above is the same as the solution $\tilde{\mathbf{x}}_B$ obtained in* (1).

**Theorem 2** *Let $\tilde{\mathbf{x}}_S = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ be the solution obtained by the above method. Then, for all $j \in \{1, \ldots, n\}$, $\tilde{x}_j$ is a fuzzy number,*

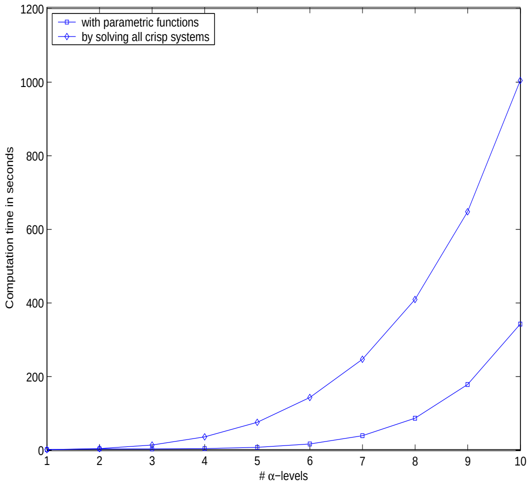


Figure 2: A $500 \times 500$ system with three fuzzy numbers. Computation time in function of the number of $\alpha$-levels

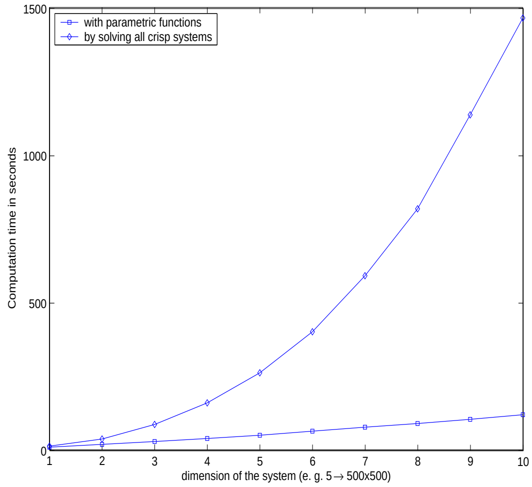


Figure 3: Computation time in function of the dimension of the system. The system contains four fuzzy numbers and four $\alpha$-levels are considered.

*given that the aggregation operator is continuous, idempotent and* $\mathrm{aggr}(x_1, \ldots, x_n) = 1 \iff x_1 = \ldots = x_n = 1$*, for all* $(x_1, \ldots, x_n) \in [0,1]^n$*, for all* $n \in \mathbb{N}^*$*, and given that all the fuzzy numbers in* $\tilde{A}$ *and* $\tilde{\mathbf{b}}$ *are continuous.*

In general, if we have $K$ fuzzy coefficients and we want to approximate the solution using $m$ intermediate points in the support of each fuzzy number, then the total operation count for the method described above is $\frac{2^{K+1}}{3}n^3 + 2^{K-1}3n^2 - \frac{7}{3}2^{K-1}n + \frac{m^K - 2^K}{m-2}(4n+3m-2)$. The total operation count for the method of Buckley and Qu, since they need to solve $m^K$ crisp $n \times n$ systems is $\frac{m^K(n^3+3n^2-n)}{3}$. It is easy to see that for large $n$, $K$ and $m$ the method described above needs less computation time than the method of Buckley and Qu. The performance of both methods is compared in Figure 2 and Figure 3.

# 4 Conclusion

In this paper we have proposed a new method for solving $n \times n$ systems in which some (or all) coefficients are fuzzy. While in the method of Buckley and Qu for every element in the support of each fuzzy number the corresponding crisp $n \times n$ system is solved, in our method only the crisp $n \times n$ systems corresponding to the bounds of each support must be solved, and the solution for the intermediate elements in the support is obtained using parametric functions.

Our method performs much better when a lot of $\alpha$-levels are considered and when $n$ is large.

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